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Bounding approach to parameter estimation without prior knowledge on modeling error and application to quality modeling in drinking water distribution systems

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Parameter estimation of an autoregressive with moving average and exogenous variable (ARMAX) model is discussed in this paper by using bounding approach. Bounds on the model structure error are assumed unknown, or known but too conservative. To reduce this conservatism, a point-parametric model concept is proposed, where there exist a set of model parameters and modeling error corresponding to each input. Feasible parameter sets are defined for point-parametric model. Bounded values on the model parameters and modeling error can then be computed jointly by tightening the feasible set using observations under deliberately designed input excitations. Finally, a constantly bounded parameter model is established, which can be used for robust output prediction and control.

Key words: modeling errors, bounding method, parameter estimation, uncertain dynamic systems, robust control

1. Introduction

The following discrete time model with delayed inputs that describes a linear SISO time-varying and continuously-delayed system is considered:

$$y(t) = \sum_{s=1}^{N_s} a_s(t)y(t-s) + \sum_{d\in D} b_d(t)u(t-d) + e(t) \tag{1}$$

where $t \in [t_0, t_0 + T_M] = \Xi_M$ is a time step, T_M is the considered modeling horizon, N_s describes the dynamics of the system, $y(t)$ and $u(t)$ are the system output and input, respectively, $a_s(t)$ and $b_d(t)$ are time-varying model parameters, D is the time delay set, and $e(t)$ is the modeling error that is input dependent. This model can be written compactly as:

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$$
\begin{aligned}
&y(t) = \phi(t)^T \theta(t) + e(t),\\
&\phi(t)^T = \left[y(t-1),\ldots,y(t-s),\ldots,y(t-N_s),u(t-d^{min}),\ldots,u(t-d),\ldots,u(t-d^{max})\right],\\
&\theta(t)^T = [a_1(t),\ldots,a_s(t),\ldots,a_{N_s}(t),b_{d^{min}}(t),\ldots,b_d(t),\ldots,b_{d^{max}}(t)],\\
&\dim\phi = \dim\theta = N_R,
\end{aligned}
\tag{2}
$$

where $\left\{d^{min},\ldots,d,\ldots,d^{max}\right\} = D$ – ordered set of time delays. With the observation data available, the parameter bounds on $\theta(t)$ can be computed using bounding approach as described in [4] and [6]. Bounds on the modeling error $e(t)$ also are needed as prior system knowledge. However, in practice it is not trivial to obtain the tightest bounds. This implies that otherwise a more conservative assumption on $e(t)$ is to be applied or the prior knowledge about $e(t)$ could be wrong.

In this paper, instead of trying to obtain the prior bounds on $e(t)$ in advance, it is to be handled together with model parameters during estimation. It can be shown that there is an internal link between the uncertainty in the parameters and the uncertainty in the modelling error part of the model:

$$y(t) = \phi(t)^T \theta(u,t) + e(u,t) \tag{3}$$

where u denotes input function of time for simplified notation, $\phi(t)$ is the same as in (2).

In the above formulation, θ and e are input dependent. Notice, that the meaning of θ and e in (3) is different from the meaning of θ and e in (2). Actually, they are different parameters. Only the model structure described by (2) is remained. For the notation simplicity, we still use θ and e in further considerations. Given an input $u(\cdot)$, there exists at least one θ and e that satisfy (3). As the parameter values and the modeling error explaining the system response depend on the input the relation (3) define so-called point-parametric model of the system.

Uncertainty in the parameters can now be linked to the modeling error uncertainty directly. It is possible now to trade off the uncertainty distribution between the parameter part and the modeling error part and to estimate them jointly. Hence, a less conservative uncertainty model is conceivable. However, a dedicated input excitation design is needed in order to get sufficiently rich information in the system outputs. It means that with this information, it will be possible to find such parameter - error set in the parameter and modeling error space that for any output there exist the parameter and error values in this set that the model with these parameter - error values, can produce the plant output. Of course, it is impossible to try all system input signals. It will be shown that it is necessary and sufficient to excite the plant by a finite number of inputs that are specially designed.

The uncertainties in the system may be located in the parameters and in the modeling error part of the model. This results in two types of possible model structures differing in uncertainty allocation. The first type of the model allocates all of the uncertainties in the system into the model parameters, resulting in:

$$y(t) = \phi(t)^T \theta(u,t) \tag{4}$$

where the parameters must be time-varying in order to make the model capable of explaining the system output.

Alternatively, the uncertainties in the system can be explained by a constant parameter and a time-varying modeling error part together:

$$y(t) = \phi(t)^T \theta(u) + e(u,t). \tag{5}$$

Notice that u denotes function of time and $\theta(u)$ is the corresponding constant parameter value. In both cases, the parameters and the modelling error part are input dependent. In the paper, without lost of generality, only the model described by equation (5) is considered.

The paper is organized as follows, in Section 2 the point-parametric model is defined and an algorithm is derived in order to estimate its parameters and the modeling error. As the model is point-parametric, the parameter and modeling error estimation problem is formulated accordingly. An experiment design for parameter estimation in open - loop is discussed in Section 3. The set - bounded model of uncertainty implies the set - bounded structure of the model based predictions. An uncertainty radius of the bounds on the predicted system output (robust prediction) is defined in Section 4 in order to assess the model uncertainty. In Section 5 the proposed methodology and algorithms are applied to the quality modelling in the drinking water distribution system. Finally, Section 6 concludes the paper.

2. Point-parametric model and the parameter estimation algorithm

2.1. Point-parametric model

We shall start with defining the point-parametric model in which the parameters are constant but input dependent and the modeling error is time varying.

Definition 1: *The model described by the equation (5) is said point-parametric iff for any input scenario $u(\cdot)$ and the corresponding system output $y(\cdot)$ there exist a pair $\{\theta(u(\cdot)), e(u(\cdot),\cdot)\}$ such that equation (5) recursively generates the sequence of the model outputs $y^M(t), t \in \Xi_M$ that are equal to the system output $y(t)$, for any $t \in \Xi_M$, that is the following holds*

$$\forall t \in \Xi_M \ : \ y(t) = y^M(t). \tag{6}$$

Notice, that the parameters and modeling error trajectory $\{\theta(u(\cdot)), e(u(\cdot),\cdot)\}$ satisfying (6) is perfectly capable to explain the system output. Hence, we shall further call these parameters and modeling error trajectory as *input consistent*. Typically, for given input scenario $u(\cdot)$ there is more than one consistent pair $\{\theta(u(\cdot)),\ e(u(\cdot),\ \cdot)\}$. A set of all consistent pairs is called a *feasible* set of the parameters and modeling error trajectories corresponding to the input scenario $u(\cdot)$. Let us denote the feasible set as $\Theta_f(\theta(u(\cdot)), e(u(\cdot),\cdot))$.

In order to exactly predict based on the point-parametric model a system response to any input from a certain class $\mathcal{U}$ the consistent pair needs to be known for any $u(\cdot) \in \mathcal{U}$. It would be an immense task to determine all such pairs. It is assumed that the class $\mathcal{U}$ contains only the scenarios that are valued on a compact set U, that is $u(t) \in U$, $t \in \Xi_M$. It will be shown in Section 3 that an orthotopic outer approximation of the set $\bigcup_{u(\cdot)\in\mathcal{U}}\{\theta(u(\cdot)), e(u(\cdot),\cdot)\}$ can be produced by knowing the system responses only to finite number of input scenarios $u^j(\cdot) \in \mathcal{U}$, $j = \overline{1,N_E}$. Moreover, the orthotopic outer approximation achieves a good trade off between the experimental effort, in a case of an open - loop estimation or between information base, in a case of closed - loop estimation, and the estimation accuracy, hence the system output prediction accuracy. Being the orthotope (see Fig. 1) the outer approximation is determined by the lower and the upper bounds $[\theta^l_{out,i} \;\; \theta^u_{out,i}]$, $i = \overline{1,N_R}$ on the parameters and by the lower and upper bounds $[e^l_{out,i} \;\; e^u_{out,i}]$ on the modelling error. Let

$$\theta^l = [\theta^l_1, \ldots, \theta^l_i, \ldots, \theta^l_{N_R}]^T,$$

$$\theta^u = [\theta^u_1, \ldots, \theta^u_i, \ldots, \theta^u_{N_R}]^T.$$

Let us denote for notational simplicity

$$\theta^j \triangleq \theta(u^j(\cdot)),$$

$$e^j(t) \triangleq e(u^j(t), t).$$

Hence, $\{\theta^j, e^j(t)\}$ is the consistent pair with the input scenario $u^j(\cdot)$. Given vectors of the parameter bounds θ^l, θ^u and the modeling error bounds e^l, e^u, $\Theta(\theta^l, \theta^u, e^l, e^u)$ denotes an orthotope determined by the bounds $\theta^l, \theta^u, e^l, e^u$. It is straightforward to show that the following Lemma 1 holds.

Lemma 1: *The orthotope* $\Theta(\theta^l, \theta^u, e^l, e^u)$ *is an outer approximation of the set* $\bigcup_{j=\overline{1,N_E}}\{\theta^j, e^j(\cdot)\}$ *iff*

$$\left(\bigcup_{j=\overline{1,N_E}}\{\theta^j, e^j(\cdot)\}\right) \subseteq \Theta(\theta^l, \theta^u, e^l, e^u).$$

The orthotope $\Theta(\theta^l, \theta^u, e^l, e^u)$, which is an outer approximation, is denoted $\Theta_{out}(\theta^l_{out}, \theta^u_{out}, e^l_{out}, e^u_{out})$.

Let us notice that every input-output experiment with the input $u^j(\cdot)$, $j = \overline{1,N_E}$ brings new consistent pair $\{\theta^j, e^j(t)\}$ that must be added to the set of consistent pairs that have been produced so far. It is the fundamental difference between classic bounding approaches used in the parametric model parameter estimation and the parameter bounding in the point-parametric models. Hence, for the point-parametric models an approach and the corresponding algorithms must be derived. The outer approximation

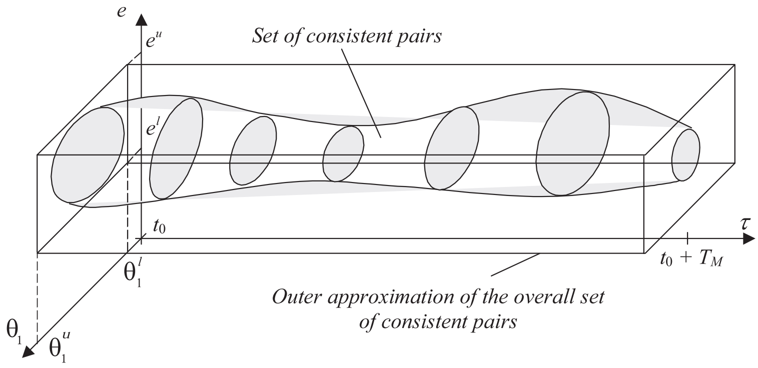

Figure 1. An outer approximation

may be more or less conservative. Among all the outer approximations we shall distinguish the one that is least conservative and call *minimal outer approximation.*

Definition 2: *Let be given the experiment set* $u^j(\cdot),\ j=\overline{1,N_E}$. *A minimal volume outer approximation* Θ_{out}^{min} *of a set of all consistent pairs* $\{\theta^j, e^j(t), j=\overline{1,N_E}\}$ *is called minimal outer approximation.*

The vector $(\theta^l, \theta^u; e^l, e^u)$ that determines the minimal outer approximation will be denoted $(\theta_{out}^{l,min}, \theta_{out}^{u,min}; e_{out}^{l,min}, e_{out}^{u,min})$. Hence

$$\Theta_{out}^{min} = \Theta(\theta_{out}^{l,min}, \theta_{out}^{u,min}; e_{out}^{l,min}, e_{out}^{u,min})$$

The set Θ_{out}^{min} is illustrated in Fig. 2. Clearly, the input - consistent pair link is not mapped in the set Θ_{out}^{min} that is plain. However, this is sufficient for the robust prediction that produces not exact system output, but the lower and upper envelopes that bound this output. All we can achieve with the prior and measurement information at hand is to minimise the bounding conservatism that is to make the bounding envelopes as close as possible. Although the minimal approximation is appealing it still can be improved with regard to the resulting robust prediction conservatism. Indeed according to Definition 1, for selected input scenario there exist many values of the consistent pairs. It is enough for the robust prediction to guarantee that for each input $u^j(\cdot),\ j=\overline{1,N_E}$ there exists at least one consistent pair that belongs to the set-estimate Θ_{est}. An example selection of such set Θ_{est} is illustrated in Fig. 3 where the minimum volume Θ_{est}^{min} is also shown. The sets having impact on the prediction conservatism are illustrated in Fig. 3 in dark grey. The benefit of using the set Θ_{est} or Θ_{est}^{min} instead of the minimal outer approximation set Θ_{out}^{min} is now clearly seen.

The least conservative robust parameter estimation problem (LCRPEP) can now be formulated as follows. Let the sets of the input scenarios and the corresponding system

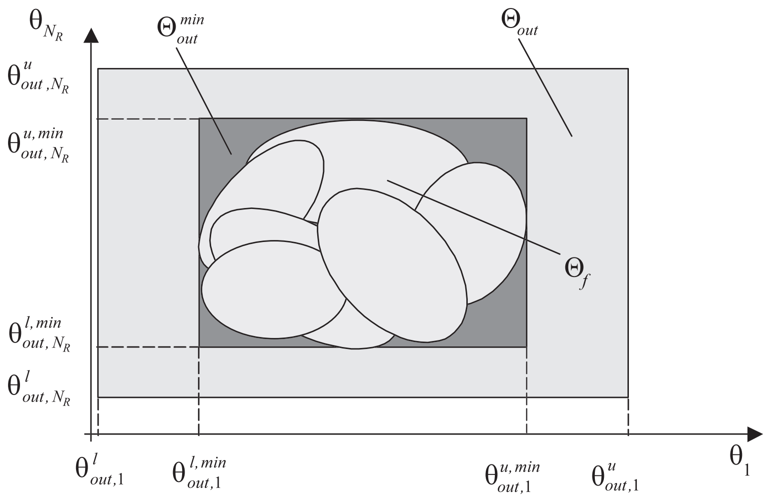

Figure 2. Minimal outer approximation

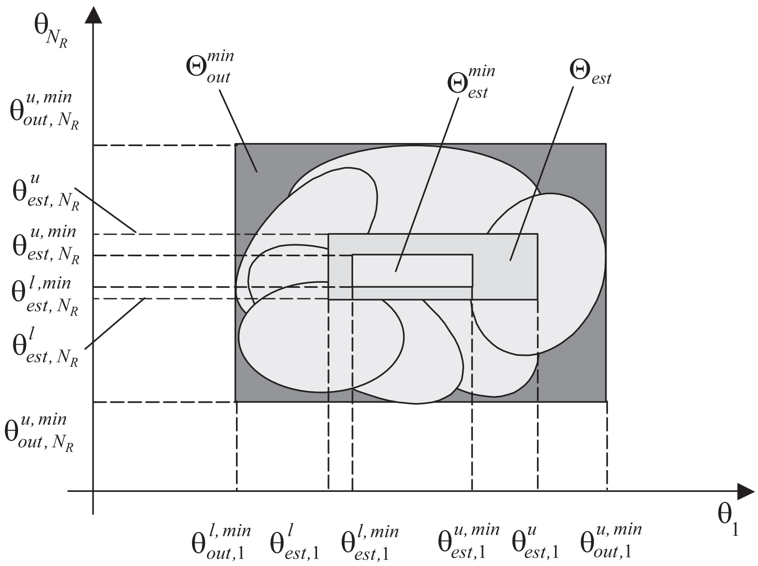

Figure 3. Parameter set – estimates Θ_{est} and Θ_{est}^{min}

output trajectories $u^j(\cdot),\ j=\overline{1,N_E}$ and $y^j(\cdot),\ j=\overline{1,N_E}$ be given.

(LCRPEP_1):

$$\text{minimize}\ \{\ \text{Volume of}\ \ \Theta_{est}\}. \tag{7}$$

It is pointed out again that the set $\Theta_{est}^{min} = \Theta_{est}^{min}(\theta_{est}^{l,min}, \theta_{est}^{u,min}; e_{est}^{l,min}, e_{est}^{u,min})$ is not an outer approximation set but the following holds

$$\Theta_{est}^{min} = \Theta_{est}^{min}(\theta_{est}^{l,min}, \theta_{est}^{u,min}; e_{est}^{l,min}, e_{est}^{u,min}) \subseteq \Theta_{out}^{min} = \Theta_{out}^{min}(\theta_{out}^{l,min}, \theta_{out}^{u,min}; e_{out}^{l,min}, e_{out}^{u,min}). \quad (8)$$

2.2. Parameter estimation algorithm

We shall now derive an algorithm for determining an optimal robust model parameter and modeling error set-estimate Θ_{est}^{min}. In order to obtain the efficient and reliable algorithm we shall assume an orthotopic shape of the parameters-error-estimate set. Minimizing the volume of the Θ_{est} in (7) can be performed by minimizing the orthotope diagonal that can be expressed as

$$(\Theta_{est}^{u} - \Theta_{est}^{l})^T(\Theta_{est}^{u} - \Theta_{est}^{l}) + (e_{est}^{u} - e_{est}^{l})^2. \quad (9)$$

In order to provide for a mechanism to tune the uncertainty distribution between the modelling error and the parameter values so that uncertainty impact on the set-estimate conservatism can be minimised the weighting matrix P and scalar Q are introduced into (9) as follows

$$(\Theta_{est}^{u} - \Theta_{est}^{l})^T P(\Theta_{est}^{u} - \Theta_{est}^{l}) + Q(e_{est}^{u} - e_{est}^{l})^2. \quad (10)$$

The estimation algorithm can now be written as

(LCRPEP_2):

$$[\theta_{est}^{l,min}, \theta_{est}^{u,min}; e_{est}^{l,min}, e_{est}^{u,min}] = \underset{\theta_{est}^{l}, \theta_{est}^{u}, \theta^{j}; e_{est}^{l}, e_{est}^{u}, e^{j}(\cdot)}{\arg\min} J(\theta_{est}^{l}, \theta_{est}^{u}; e_{est}^{l}, e_{est}^{u}), \quad (11a)$$

$$\text{satysfying} \;\; : \;\; [\theta^j, e^j(\cdot)] \in \Theta_{est}(\theta_{est}^{l}, \theta_{est}^{u}; e_{est}^{l}, e_{est}^{u}), \quad j = \overline{1, N_R}, \quad (11b)$$

$$\begin{aligned} \theta_{est}^{l} \geqslant \theta_{out}^{l}, \quad \theta_{est}^{u} \leqslant \theta_{out}^{u}, \\ e_{est}^{l} \geqslant e_{out}^{l}, \quad e_{est}^{u} \leqslant e_{out}^{u}, \end{aligned} \quad (11c)$$

where $J(\theta_{est}^{l}, \theta_{est}^{u}; e_{est}^{l}, e_{est}^{u}) = (\theta_{est}^{u} - \theta_{est}^{l})^T P(\theta_{est}^{u} - \theta_{est}^{l}) + Q(e_{est}^{u} - e_{est}^{l})^2$. Notice that the left hand sides of the inclusions (11b) are just the pair that are consistent with the input scenarios $u^j(\cdot)$. Hence the variables θ^j; $e^j(t), t \in \Xi_M$ must satisfy the consistency condition (6) and the constraints (11b) can be written in a form that is explicit with respect to the search variables θ^j; $e^j(t), t \in \Xi_M$, $j = \overline{1, N_E}$. Finally the parameters-error estimation problem reads

(LCRPEP_3):

$$\underset{\theta_{est}^{l}, \theta_{est}^{u}, \theta^{j}; e_{est}^{l}, e_{est}^{u}, e^{j}(\cdot),\; j=\overline{1,N_E},\; t \in \Xi_M}{\text{minimize}} J(\theta_{est}^{l}, \theta_{est}^{u}; e_{est}^{l}, e_{est}^{u}), \quad (12a)$$

$$\text{with respect to} \;\; : \;\; y^j(t) = \phi^j(t)^T \theta^j + e^j(t), \quad t \in \Xi_M, \quad j = \overline{1, N_E}, \quad (12b)$$

$$\begin{aligned} &\theta_{est}^{l} \leqslant \theta^{j} \leqslant \theta_{est}^{u}, \quad j=\overline{1,N_E}, \\ &e_{est}^{l} \leqslant e^{j}(t) \leqslant e_{est}^{u}, \quad t \in \Xi_M, \quad j=\overline{1,N_E}. \end{aligned} \tag{12c}$$

Notice, that the LCRPEP given above is a strictly convex linear - quadratic optimization problem. Hence, an interior-point algorithm [5] can be used as the efficient solver. Apparently, a result of LCRPEP solving depends on the experiments. The experiment design will be discussed in the subsequent section.

Finally, a point-parametric model with constant bounded parameters and time-varying bounded modeling error can be obtained as:

$$\begin{aligned} &y(t)=\phi(t)^T\theta+e(t) \\ &[\theta;e(t)]\in\Theta_{est}^{min}, \quad \Theta_{est}^{min}=\left\{ \begin{array}{ll} (\theta;e(t))\in\Re^{N_R+1} : & \theta_{est}^{l,min}\leqslant\theta\leqslant\theta_{est}^{u,min}; \\ & e_{est}^{l,min}\leqslant e(t)\leqslant e_{est}^{u,min} \end{array} \right\}. \end{aligned} \tag{13}$$

The least conservative constant bounded parameters and time-varying bounded modeling error model (13) is sufficient for robust control design [1].

2.3. Remarks on parameter estimation under measurement errors

Under measurement errors and input noises, outputs and inputs in regressor $\phi(t)$ become unknown values. Thus, parameter estimation problem becomes more complex, which is often referred to as ‘error-in-variables’ in set-bounded estimation whereas estimation problem without measurement errors is referred to as ‘error-in-equation’ estimation problem when the inputs and outputs are exactly known [4].

The definition of the feasible parameter set under point-parametric model in the error-in-variables case become more complicated. A constructive formulation of the estimation problem is under current research.

3. Experiment design

3.1. Representation of inputs

The system input in practice is usually restricted by actuator performance and operational limits. The input constraints can be expressed as:

$$0 \leqslant u^{min} \leqslant u(t) \leqslant u^{max}, \quad \text{for any } t. \tag{14}$$

Also, it is assumed that a system response to any admissible input scenario satisfies

$$y(t) \geqslant 0, \quad \text{for } t \in \Xi_M. \tag{15}$$

Then, any input scenario $u=[u(1),\ldots,u(N_t)]$ over time horizon $t=\overline{1,N_t}$ can be viewed as a point in N_t dimension Euclidean space bounded by a cube with r vertices, and

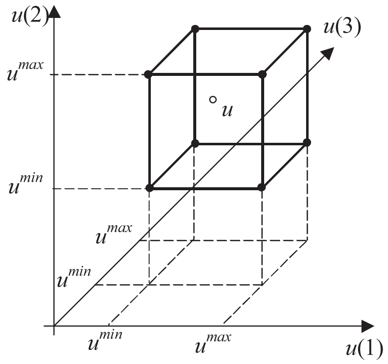

Figure 4. Input scenario vectors in Euclidean space

$r = 2^{N_t}$. The example of such input vector in 3-dimension case, where $N_t = 3$ is illustrated in Fig. 4. For parameter - error estimation problem over time interval Ξ_M the following information is needed:

- output trajectory over time interval $\Xi_{OUT} = \Xi_M = [t_0, t_0 + T_M]$,
- input scenario over time interval $\Xi_{IN} = [t_0 - d^{max}, t_0 + T_M - d^{min}]$,
- past system output values over time interval $\Xi_p = [t_0 - 1, t_0 - N_s]$.

The above constitute the input – output information that determines the system output over the modeling horizon T_M.

3.2. Experiment design

Given $t \in \Xi_M$ let us consider jth vertex of the orthotope in Fig. 4. Let us define an input scenario $u^j(\cdot)$ as

$$u^j(t) = u^{min} \quad \text{or} \quad u^{max}, \quad t \in \Xi_M. \tag{16}$$

It means the input scenario defined by (16) is built up from limit values of the input variable. Hence, it will be called jth vertex input scenario and $j = \overline{1, N_E}$. Hence, any admissible input scenario $u(\cdot)$ can be expressed as a convex combination of the orthotope vertices that means as a convex combination of the vertex input scenarios [3]. It means that considering the time horizon T_M, any admissible input at the time instant $t \in \Xi_M$ can be expressed

$$u(t) = \sum_{j=1}^{N_E} \lambda_j u^j(t), \quad t \in \Xi_M \tag{17}$$

where $\sum_{j=1}^{N_E} \lambda_j = 1$ and $\lambda_j \geqslant 0,\ j = \overline{1, N_E}$.

Let $y^j(t)$, $t \in \Xi_M$ be the output response corresponding to the input $u^j(t)$, $t \in \Xi_M$. Clearly,

$$y^j(t) = \phi^j(t)^T \theta^j + e^j(t), \quad t \in \Xi_M \tag{18}$$

where $\{\theta^j, e^j(\cdot)\}$ is consistent with the jth vertex input scenario $u^j(\cdot)$. Let

$$\forall j = \overline{1, N_E} \; : \; \exists \{\theta^j, e^j(\cdot)\} \in \Theta_{ver}(\theta^l_{ver}, \theta^u_{ver}; e^l_{ver}, e^u_{ver})$$

where $\Theta_{ver}(\theta^l_{ver}, \theta^u_{ver}; e^l_{ver}, e^u_{ver})$ is bounded orthotope with finite θ^l_{ver}, θ^u_{ver}, e^l_{ver} and e^u_{ver}. Selecting the input excitations for the parameters-error estimation purposes can be based on the following theorem.

Theorem 1: *For any admissible input scenario $u(\cdot)$ there exists $\{\theta, e(\cdot)\} \in \Theta_{ver}(\theta^l_{ver}, \theta^u_{ver}; e^l_{ver}, e^u_{ver})$ that is a pair consistent with the input scenario $u(\cdot)$.*

A proof of the Theorem 1 is given in Appendix.

Indeed, solving the least conservative robust parameter-error estimation problem (LCRPEP) for the input scenario set $\{u^j(\cdot), j = \overline{1, N_E}\}$ yields the parameter-error set – estimate Θ^{min}_{est} (see Fig. 3). Clearly, by a definition of Θ^{min}_{est} for any input scenario $u^j(\cdot), j = \overline{1, N_E}$ there exists the consistent pair $\{\theta^j, e^j(\cdot)\}$ that belongs to Θ^{min}_{est}. Hence, according to the Theorem 1 for any admissible input scenario $u(\cdot)$ there exists in the set Θ^{min}_{est} a consistent with this input pair that explains the corresponding system response. In other words the input scenarios $u^j(\cdot), j = \overline{1, N_E}$ are sufficient to excite the system in order to gather the output information so that corresponding LCRPEP yields the parameter - error set - estimate containing the parameter - error pairs that can explain any admissible input.

3.3. Summary and remarks

The following remark is given for the parameter estimation algorithm based on point-parametric model:

- The number of the needed experiments, $N_E = 2^{N_t}$, increases with the modeling horizon. Thus, with larger N_t, the computing task in solving the optimisation problem dramatically augments.

- The computing efficiency can be improved in practice by reducing the experiment number in the following way:

  1. Replace the vertex-base inputs by a number of random-base inputs. After obtaining the bounded model, it can be checked whether the estimation results are consistent with the responses to the vertex-base inputs. Increase the number of random inputs until satisfactory model is obtained.
  2. Find a polytope with less vertices that can bound the cube space containing the inputs. Use the vertices of the new polytope as the experiment inputs.

- An input range $u^{min} \leqslant u(t) \leqslant u^{max}$ is required in the experiment design. However, larger input range results in larger uncertainties in the obtained model parameters, which is the price to be paid.

4. Assessing the model uncertainty

Output prediction performed based on the obtained set-bounded models described by (13) is under uncertainty as well. Given the input scenario, a robust prediction of the corresponding outputs over a prediction horizon H_p can be calculated that are consistent with the model uncertainty. The upper and lower envelopes that bound the output trajectory set determine a set in the output space in which a real output is contained, provided that the uncertainty model is convincing. This is illustrated in Fig. 5, where the robust prediction is carried out at time instant t. The bounding envelopes are $y^l(t+k|t)$,

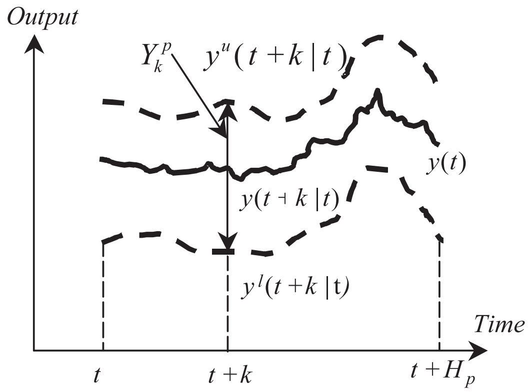

Figure 5. Robust output prediction

$y^u(t+k|t)$, and they can be viewed as producing at time t the worst case k step output prediction, which belongs to a output set Y_k^p that is composed of all possible output trajectories:

$$Y_k^p \triangleq \left\{ \begin{array}{l} y(t+k|t) \in \Re \ : \\ y(t+1|t) = \phi(t+1)^T\theta + e(t+1) \\ \quad \vdots \\ y(t+k|t) = \phi(t+k)^T\theta + e(t+k) \end{array} \right\} \quad \text{for} \quad k = \overline{1, H_p} \tag{19}$$

where $\theta^l \leqslant \theta \leqslant \theta^u$ and $e^l \leqslant e(t+i) \leqslant e^u$ for $i = \overline{1,k}$.

The above formulation is a batch type as in defining the output set at time instant $t+k$ all the previous output equations from $t+1$ are contained, which are constraints on $y(t+k|t)$. In order to achieve sufficient accuracy and at the same time to reduce computation complexity, formulations of recursive algorithms with moving information window have been discussed in [2].

Let the bounds on θ and $e(t)$ be from the model obtained in the previous section. The robust output prediction is defined as:

$$\begin{aligned} y_p^l(t+k|t) = & \underset{\substack{y(t+1|t),\ldots,y(t+k|t),\\ \theta,\\ e(t+1),\ldots,e(t+k)}}{\arg\min} y(t+k|t), \\ \text{with respect to} \quad : \quad & y(t+k|t) \in Y_k^p \end{aligned} \tag{20}$$

and

$$\begin{aligned} y_p^u(t+k|t) = & \underset{\substack{y(t+1|t),\ldots,y(t+k|t),\\ \theta,\\ e(t+1),\ldots,e(t+k)}}{\arg\max} y(t+k|t) \\ \text{with respect to} \quad : \quad & y(t+k|t) \in Y_k^p \end{aligned} \tag{21}$$

There are product terms of variables in (19), hence, solving (20) and (21) is a non-linear programming problem with continuous and discrete variables. The problem is solved by first applying linearization and next by using efficient Mixed Integer Linear Programming (MILP) solver [2]. An output prediction envelope can be generated over prediction horizon H_p as:

$$Y_p^l = [y_p^l(t+1|t),\ldots,y_p^l(t+k|t)], \quad k = \overline{1,H_p},$$

$$Y_p^u = [y_p^u(t+1|t),\ldots,y_p^u(t+k|t)], \quad k = \overline{1,H_p}.$$

In order to assess model uncertainty impact on the robust output prediction accuracy so-called an uncertainty radius is defined as:

$$W = \max\{Y_p^u - Y_p^l\}, \quad \text{under} \quad u(t) = 1(t) \tag{22}$$

where $1(t)$ defines a unit step input. In order to extend the feasible input set that the controller operates on, a least conservative output prediction is wanted. Requirements on desired model accuracy for robust controller design can be expressed by an upper limit on W. Through W the model estimation is related to the controller design directly and the model design and the controller design can be performed in an integrated way.

5. Application to quality modeling in drinking water distribution systems

The point-parametric model has been used for quality modeling in Drinking Water Distribution System (DWDS). Drinking water is usually taken from sources such as rivers, lakes, or underground wells. Raw water is treated in the water treatment plant

(WTP) to remove unwanted substances and to kill pathogens. The treated drinkable water from the water plant is then transported to demand nodes by drinking water distribution systems. DWDS is large-scale network systems composed of pumps, valves, storage tanks and demand nodes connected together by pipes. Maintaining disinfectant concentration over the DWDS within prescribed limits is called quality control. A number of chemical disinfectants can be used in the DWDS, e.g. chlorine, chlorine dioxide, chloramines and ozone. The most common disinfectant is chlorine, as it is inexpensive and effective. From the viewpoint of control technology, we are interested in the dynamic properties of the disinfectant concentration when added to water. The methodology developed for a particular chemical can be used for other chemicals without many difficulties. The modeling of chlorine concentration in DWDS was considered as an application of the point-parametric model. The developed methodology can be extended to other disinfectant application in DWDS.

Three main aspects govern chlorine concentration in DWDS: transportation, mixing and reaction kinetics. The first two aspects depend on the hydraulics of the DWDS. The water demand is time varying and results in time-varying hydraulics. The predicted water demand is assumed to be constant over hydraulic control step. The prediction error implies the uncertain hydraulic information. Chlorine reaction kinetics is a complicated chemical process that is still under research. Only first- and second-order mathematical models are generally applied at present. Uncertainty in the modeling error resulting from the above factors need to be appropriately embedded into the system model.

Transportation delay is the main feature in the chlorine concentration modeling in DWDS. Driven by varying hydraulics, the time-delay is continuous and time varying. As it is still difficult to process continuous time-varying delays in model estimation and control design, the continuous time delay is discretized over certain time horizon. However, the discretization leads to modeling errors, and it is difficult to analytically evaluate the error so that the less conservative prior error bounds would be produced. In this case, applying the point-parametric model concept helps to better evaluate the modeling error.

There is no explicit relationship between chlorine injections in the injection nodes (system control inputs) and chlorine concentrations in the monitored nodes (controlled outputs). In the controller design and operation an explicit relationship between input and output is usually wanted. First explicit input-output (IO) model was proposed by [7]. A backward tracking algorithm was proposed in order to obtain structure of the input-output model. This algorithm is called path analysis of chlorine transportation throughout the DWDS. The transportation time-delay in this input-output model is continuous in time and time varying. The range of the continuous time-delay over the modeling horizon is then discretized and approximated by a series of time delay numbers. The existence of storage facilities enlarges the detention time from the input node to the output node and introduces dynamics in the model. Thus, more model parameters are added. The final model with tanks in the DWDS has an autoregressive with moving average and exogenous variable (ARMAX) format given by (3).

The conceptual DWDS in Figure 6 is taken from EPANET package (Rossman, 2000). It is a simple small DWDS of the SISO type with the chlorine injection at node 11 and the chlorine concentration monitoring at the node 32. Modelling of chlorine dynamics of this DWDS over a 24-hour horizon was considered. Path analysis was performed first to obtain the parameter structure of the model. The discretisation time step is 10 minutes. Thus, the 24-hour horizon is discretised into 144 time steps. There are 8 water

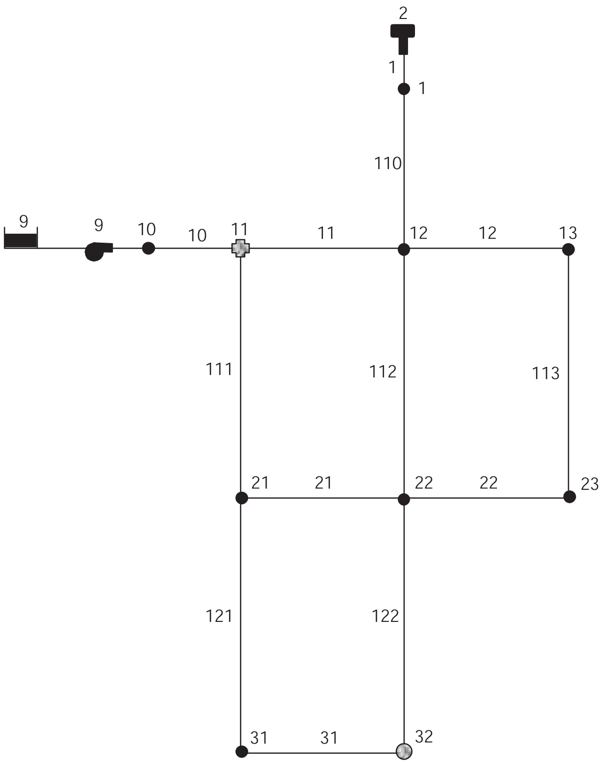

Figure 6. A sample DWDS network

consumption nodes in this DWDS linked with 10 pipes. The predicted water demand pattern at different nodes is assumed the same and the pattern time step is 2 hours. An example of the water demand at node 32 over the 24-hour modeling horizon is shown in Fig. 7. The water is pumped from the source (node 9) and it is also supplied from the tank (node 2).

As the examples, the upper and lower bounds of values of the parameter denoted b_9 over the 24-hour horizon is shown in Fig. 8, while the bounds of the parameter $a_{D,10}$ and $a_{F,2}$ values are illustrated in Fig. 9 and Fig. 10 (the subscripts $_D$ and $_F$ are related to a draining and filling cycle of the switching tank operation). It can be seen from the figures that the IO quality model parameter bounds are piece-wise constant over the time horizon. The real parameter trajectory over the time horizon is time-varying and could

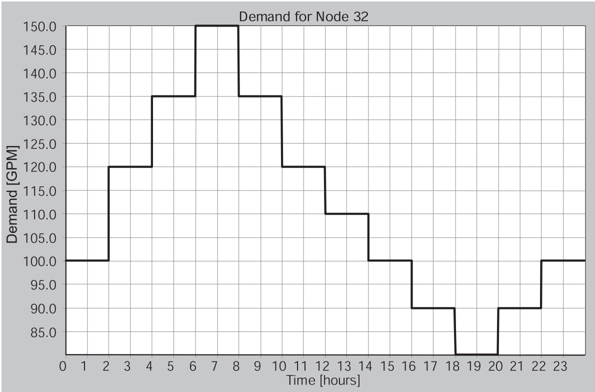

Figure 7. Water demand predictions at node 32

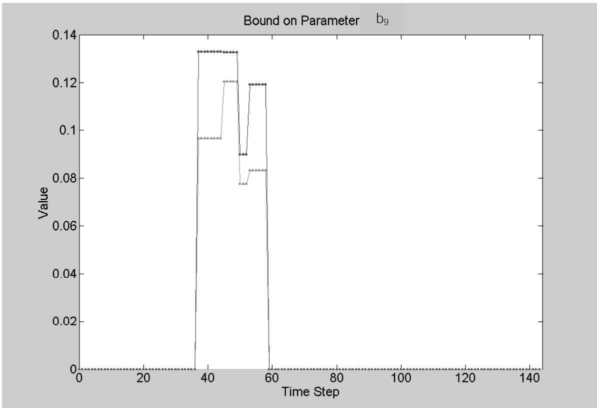

Figure 8. Parameter bounds of b_9

be any trajectory within the upper-lower bounded envelopes. Taking the centre of the bounds as the nominal value of the parameter value, a nominal model can be obtained. The model performance is illustrated in Figure 11 by applying chlorine injection as the step input of amplitude 0.25[mg/l]. By changing the weight of the P and Q in (10) the uncertainty allocation in the parameters part and the modelling error part can be changed. Using uncertainty radius W as the index of estimated model uncertainty, the output predictions are compared and assessed for difference weight setting.

There are 8 samples and 6 experiments, $N_t = 8$, $N_E = 6$. The parameters correspond to time delay number 12 to 16, thus the parameters are b_{12}, b_{13}, b_{14}, b_{15}, b_{16}. The output prediction is calculated assuming a step input $u(t) = 1.8 \times 1(t)$. The following uncer-

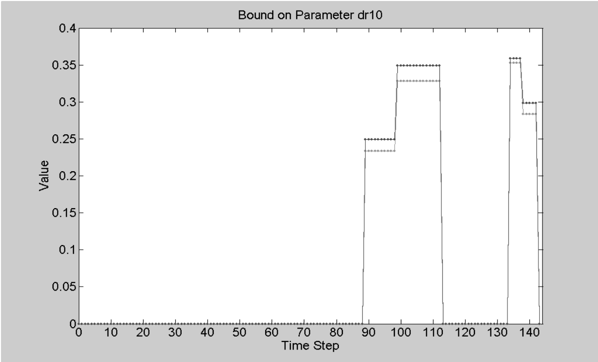

Figure 9. Parameter bounds of $a_{D,10}$

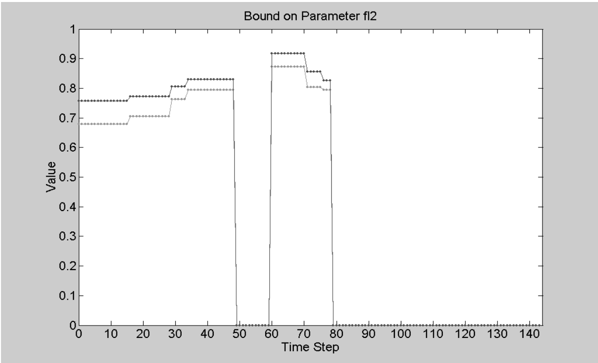

Figure 10. Parameter bounds of $a_{F,2}$

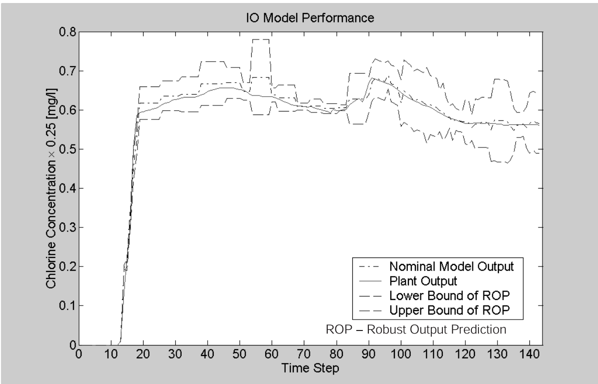

Figure 11. Step response of the IO model

tainty radius have been found with different combination of P and Q:

$$\begin{array}{lll} P=1, & Q=10, & W=0.3615 \\ P=1, & Q=5, & W=0.2742 \\ P=1, & Q=1, & W=0.1603 \\ P=1, & Q=0.5, & W=0.1462 \\ P=1, & Q=0.1, & W=0.1576 \\ P=1, & Q=0.05, & W=0.1611 \end{array}$$

It can be found that the uncertainty radius W is minimum at $P=1$, $Q=0.5$ according to the simulation data. However, how to find the optimal value of P and Q that make W global minimum is still an open problem.

6. Conclusion

Bounding approach to parameter estimation based on a point-parametric model has been developed to reduce the conservatism caused by inaccurate prior knowledge of the modeling error. For a robust MPC a bounded-parameter model is sufficient for controller design and operation. In this case, it is not necessary that nominal model outputs must exactly fit plant outputs. If in the robust output prediction, the upper and lower envelopes, can bound the plant output with a desired accuracy, then the model is considered sufficiently accurate for the controller design and operation. This has been illustrated in Fig. 11 for IO quality model. Simulation results have showed that less uncertainty can be achieved by tuning weighting matrices in the parameter estimation problem.

Appendix. Proof of the theorem 1

Let $y^j(t), t \in \Xi_M$ be the output response corresponding to the admissible input $u^j(t), t \in \Xi_M$. Due to the definition of $\{\theta^j, e^j(\cdot)\}$, $j=\overline{1,N_E}$ and (18) for each vertex input scenario the following hold:

$$\begin{array}{lllll} (\phi^j(t_1))^T\theta^j & +e^j(t_1) & & & =y^j(t_1) \\ \vdots & & \ddots & & \vdots \\ (\phi^j(t))^T\theta^j & & +e^j(t) & & =y^j(t) \\ \vdots & & & \ddots & \vdots \\ (\phi^j(t_{N_t}))^T\theta^j & & & +e^j(t_{N_t}) & =y^j(t_{N_t}) \end{array}$$

and

$$[\theta^j, e^j(\cdot)] \in \Theta_{ver}(\theta^l_{ver}, \theta^u_{ver}; e^l_{ver}, e^u_{ver}); \;\; t \in \Xi_M.$$

Due to (17)

$$u(t)=\sum_{j=1}^{N_E}\lambda_j u^j(t),\quad \lambda_j\geqslant 0,\quad \sum_{j=1}^{N_E}\lambda_j=1,\quad t\in\Xi_M.$$

Hence, the corresponding output satisfies

$$y(t)=\sum_{j=1}^{N_E}\lambda_j y^j(t),\quad \lambda_j\geqslant 0,\quad \sum_{j=1}^{N_E}\lambda_j=1,\quad t\in\Xi_M$$

and (see (2))

$$\phi(t)=\sum_{j=1}^{N_E}\lambda_j \phi^j(t),\quad \lambda_j\geqslant 0,\quad \sum_{j=1}^{N_E}\lambda_j=1,\quad t\in\Xi_M.$$

Let us define

$$\mu_k^j(t)=\frac{\lambda_j\varphi_k^j(t)}{\sum_{j=1}^{N_E}\lambda_j\varphi_k^j(t)};\quad j=\overline{1,N_E},\quad k=\overline{1,N_R},$$

$$\theta_k=\sum_{j=1}^{N_E}\mu_k^j(t)\theta_k^j, \tag{23}$$

$$e(t)=\sum_{j=1}^{N_E}\lambda_j e^j(t),$$

where φ_k^j and θ_k^j are the kth components of the vectors ϕ^j and θ^j, $j=\overline{1,N_E}$, respectively. We shall now show that the pair $\{\theta,e(\cdot)\}$ defined by (23) is consistent with the input $u(\cdot)$. Indeed the following hold:

$$\begin{aligned}
&\phi(t)^T\theta+e(t)=\\
&=\left[\sum_{j=1}^{N_E}\lambda_j\phi^j(t)\right]^T\theta+e(t)=\sum_{k=1}^{N_R}\left[\theta_k\sum_{j=1}^{N_E}\lambda_j\varphi_k^j(t)\right]+e(t)\\
&=\sum_{k=1}^{N_R}\left[\sum_{j=1}^{N_E}\mu_k^j(t)\theta_k^j\sum_{j=1}^{N_E}\lambda_j\varphi_k^j(t)\right]+\sum_{j=1}^{N_E}\lambda_j e^j(t)\\
&=\sum_{k=1}^{N_R}\left[\sum_{j=1}^{N_E}\frac{\lambda_j\varphi_k^j(t)}{\sum_{j=1}^{N_E}\lambda_j\varphi_k^j(t)}\theta_k^j\sum_{j=1}^{N_E}\lambda^j\varphi_k^j(t)\right]+\sum_{j=1}^{N_E}\lambda_j e^j(t)\\
&=\sum_{j=1}^{N_E}\sum_{k=1}^{N_R}\lambda_j\varphi_k^j(t)\theta_k^j+\sum_{j=1}^{N_E}\lambda_j e^j(t)\\
&=\sum_{j=1}^{N_E}\left(\sum_{k=1}^{N_R}\lambda_j y^j(t)+e^j(t)\right)=\sum_{j=1}^{N_R}\lambda_j y^j(t)=y(t).
\end{aligned}$$

In order to complete the proof it is left to show that the pair $\{\theta, e(\cdot)\}$ belongs to the set $\Theta_{ver}(\theta^l_{ver}, \theta^u_{ver}; e^l_{ver}, e^u_{ver})$. Let us denote

$$\begin{aligned} \theta_k^{min} &= \min\{\theta_k^j;\, j = \overline{1,N_E}\},\\ \theta_k^{max} &= \max\{\theta_k^j;\, j = \overline{1,N_E}\}. \end{aligned} \tag{24}$$

Clearly,

$$\theta^l_{ver,k} \leqslant \theta_k^{min} \leqslant \theta_k^{max} \leqslant \theta^u_{ver,k};\quad k = \overline{1,N_R}.$$

As $u^j(\cdot) \geqslant 0$ and $y^j(\cdot) \geqslant 0$ (see (14) and (15)), hence by the definition of $\mu_k^j(t)$ and from $\varphi_k^j(t_n) \geqslant 0;\ j = \overline{1,N_E},\ k = \overline{1,N_R}$ and $\lambda_j \geqslant 0,\ j = \overline{1,N_E}$ it follows that

$$\mu_k^j(t) \geqslant 0,\quad \sum_{j=1}^{N_E} \mu_k^j(t) = 1;\quad k = \overline{1,N_R}.$$

Hence, the value θ_k, $k = \overline{1,N_R}$ given by (23) is a convex combination of θ_k^j, $j = \overline{1,N_E}$ and due to (24)

$$\theta^l_{ver,k} \leqslant \theta_k^{min} \leqslant \theta_k \leqslant \theta_k^{max} \leqslant \theta^u_{ver,k};\quad k = \overline{1,N_R}.$$

Similarly for any $t \in \Xi_M$ the following holds:

$$e^l_{ver} \leqslant e^{min}(t) \leqslant e(t) \leqslant e^{max}(t) \leqslant e^u_{ver}$$

where

$$\begin{aligned} e^{min}(t) &= \min\{e^j;\ j = \overline{1,N_E}\},\\ e^{max}(t) &= \max\{e^j;\ j = \overline{1,N_E}\}. \end{aligned} \tag{25}$$

Therefore,

$$[\theta, e(t_n)] \in \Theta_{ver}(\theta^l_{ver}, \theta^u_{ver}; e^l_{ver}, e^u_{ver}).$$

The theorem proof has now been completed.

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