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On the total restrained domination number of a graph

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Abstract

For a graph $G = (V, E)$, a set $D \subseteq V(G)$ is a *total restrained dominating set* if it is dominating and both $\langle D \rangle$ and $\langle V(G) - D \rangle$ are isolate free. The cardinality of a minimum total restrained dominating set in G is the *total restrained domination number* and is denoted by $\gamma_r^t(G)$. We investigate several properties of total restrained dominating sets and give some bounds on the total restrained domination number.

1 Introduction

Let $G = (V, E)$ be a simple graph with $|V(G)| = n(G)$ and $|E(G)| = m(G)$. The *neighbourhood* $N_G(v)$ of a vertex v is the set of all vertices adjacent to v in G. The *degree* $d_G(v)$ of a vertex v is the number of edges incident to v in G, that is $d_G(v) = |N_G(v)|$. The *minimum* and *maximum degree* among vertices of $V(G)$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. If $d_G(x) = 0$, then x is *isolate* in G. Let $\Omega(G)$ be the set of all leaves of G, that is the set of vertices degree 1, and let $n_1(G)$ be the cardinality of $\Omega(G)$. A vertex which is a neighbour of a leaf is called a *support*. Let $S(G)$ be the set of supports in G and let $n_S(G)$ be the cardinality of $S(G)$. For notational convenience, we denote $\Omega(G) \cup S(G)$ by $J(G)$.

A set $D \subseteq V(G)$ is a *dominating set of* G if for every vertex $v \in V(G) - D$, there exists a vertex $u \in D$ such that v and u are adjacent. The minimum cardinality of a dominating set in G is the *domination number* denoted $\gamma(G)$.

A set $D \subseteq V(G)$ is a *total dominating set* if each vertex of $V(G)$ has a neighbour in D. The cardinality of a minimum total dominating set in G is the *total domination number* and is denoted by $\gamma_t(G)$. Total domination in graphs is now well studied in graph theory (see for example [1, 6]).

A set $D \subseteq V(G)$ is a *restrained dominating set of* G if each vertex of $V(G) - D$ has a neighbour in D as well as another neighbour in $V(G) - D$. The cardinality of a minimum restrained dominating set in G is the *restrained domination number* and is denoted by $\gamma_r(G)$. The concept of restrained domination was introduced by Telle and Proskurowski [8], albeit indirectly, as a vertex partitioning problem. Restrained domination was studied further for example by Domke et al. [3, 4].

In this paper we study the total restrained domination number of a graph defined by De-Xiang Ma, Xue-Gang Chen and Liang Sun in [7]. A set $D \subseteq V(G)$ is a *total restrained dominating set* if it is dominating and the induced subgraphs $\langle D \rangle$ and $\langle V(G) - D \rangle$ do not contain isolate vertices. The cardinality of a minimum total restrained dominating set in G is the *total restrained domination number* and is denoted by $\gamma_r^t(G)$. We note that every graph without isolates has a total restrained dominating set, since $D = V(G)$ is such a set.

For unexplained terms and symbols see [5].

2 Bounds on the γ_r^t

In this section we first illustrate the total restrained domination number by presenting the value of $\gamma_r^t(G)$ for some classes of graphs. We begin with some basic properties of total restrained dominating sets.

Proposition 1 *Let D be a total restrained dominating set of a connected graph G on $n \geq 3$ vertices. Then*

(i) *every leaf is in D;*

(ii) *every support is in D;*

(iii) $\gamma_r^t(G) \geq n_1(G) + n_S(G) \geq 2n_S(G)$.

Proof.

(i) Suppose x is a leaf such that $x \notin D$ and let $y \in N_G(x)$. Then $y \in D$, because D is dominating. But then x is isolate in $\langle V(G) - D \rangle$, a contradiction.

(ii) Suppose x is a leaf and let $y \in N_G(x)$ be a support such that $y \notin D$. Then $x \in D$, because D is dominating. But then x is isolate in $\langle D \rangle$, a contradiction.

(iii) By (i) and (ii) and simply observation that $n_1(G) \geq n_S(G)$ the inequalities follows.

■

In [7] we find the following result.

Proposition 2 *Let G be a connected graph on $n \geq 2$ vertices. Then*

(i) $\gamma_r^t(K_n) = \begin{cases} 3, & n = 3, \\ 2, & \text{otherwise;} \end{cases}$

(ii) $\gamma_r^t(K_{p,q}) = \begin{cases} p+q, & \min\{p,q\}=1, \\ 2, & \text{otherwise}; \end{cases}$

(iii) $\gamma_r^t(P_n) = n - 2\lfloor \frac{n-2}{4} \rfloor$;

(iv) $\gamma_r^t(C_n) = n - 2\lfloor \frac{n}{4} \rfloor$.

A *caterpillar* is a tree with the property that the removal of its leaves results in a path, called the *spine* of the caterpillar. The *code* C of a caterpillar T with spine $v_1, v_2, \ldots, v_s$ is the sequence of nonnegative integers $(t_1, t_2, \ldots, t_s)$ where t_i is the number of leaves adjacent to v_i in T. The substrings of consecutive zeros in C are called the *zero strings* of C and are labeled from 1 to k. For $i = 1, 2, \ldots, k$ the number of zeros in string i is denoted by z_i. For example, the caterpillar with code $(2, 0, 0, 1, 0, 4, 3, 0, 0, 0, 1)$ has $z_1 = 2$, $z_2 = 1$ and $z_3 = 3$.

Proposition 3 *For any caterpillar T with code C is $\gamma_r^t(T) = n(T) - 2\sum_{i=1}^{k} \lfloor \frac{z_i+2}{4} \rfloor$.*

Proof. Of course each leaf of T belongs to the total restrained dominating set D. Moreover, if $t_i \neq 0$, then v_i is a support and thus $v_i \in D$. Let $v_j, v_{j+1}, \ldots, v_{j+z_i-1}$ be the vertices of zero string and let x and y be leaves adjacent to v_{j-1} and v_{j+z_i}, respectively. Then $x, v_{j-1}, v_j, v_j+1, \ldots, v_{j+z_i-1}, v_{j+z_i}, y$ induces a path P on z_i+4 vertices in T and by Proposition 2, $\gamma_r^t(P) = z_i+4-2\lfloor \frac{z_i+2}{4} \rfloor$. As $\{x, v_{j-1}, v_{j+z_i}, y\} \subseteq D$, we conclude that exactly $2\lfloor \frac{z_i+2}{4} \rfloor$ vertices of the zero string j do not belong to D. Consequently, $|D| = n(T) - 2\sum_{i=1}^{k} \lfloor \frac{z_i+2}{4} \rfloor$. ■

Now we find some bounds on the total restrained domination number. For this purpose, denote by $\mathcal{A}$ the family of connected graphs such that G belongs to $\mathcal{A}$ if and only if every edge of G is incident to a support or if G is a cycle on three vertices.

Theorem 4 *If a graph G is connected, then $\gamma_r^t(G) = n(G)$ if and only if G belongs to the family $\mathcal{A}$.*

Proof. For $n(G) = 2$ and $n(G) = 3$ the result is obvious. Thus assume $n(G) \geq 4$. Let $G \in \mathcal{A}$. By Proposition 1, $J(G)$ is a subset of every total restrained dominating set D. Moreover, no two vertices of $V(G) - J(G)$ are adjacent, which implies that $D = V(G)$ is a minimum total restrained dominating set of G.

Now we prove that if $G \notin \mathcal{A}$, then $\gamma_r^t(G) \leq n(G)-2$. If $G \notin \mathcal{A}$, then there exists an edge uv where neither u nor v is a support. Of course $d_G(u) > 1$ and $d_G(v) > 1$. Let $D = V(G) - \{u, v\}$. Obviously D is a dominating set and $\langle V(G) - D \rangle$ is connected. If $\langle D \rangle$ is without isolates, then D is a total restrained dominating set of G of cardinality $n(G) - 2$. If it is not the case, then there exists a vertex $x \in V(G) - J(G)$ such that $N_G(x) = \{u, v\}$. As G is not a cycle on 3 vertices, we conclude that u or v has degree 3 or more. Without loss of generality we assume that $d_G(u) \geq 3$ and let $D' = V(G) - \{v, x\}$. This time $\langle V(G) - D' \rangle$ is connected, D' is dominating and $\langle D' \rangle$ is without isolates. As $|D'| = n(G) - 2$, we obtain that $\gamma_r^t(G) \leq n(G) - 2$. ■

Corollary 5 *If G is a connected graph with $n(G) \geq 4$ and $\delta(G) \geq 2$, then $\gamma_r^t(G) \leq n(G) - 2$.*

Theorem 6 *If G is a connected graph on $n \geq 4$ vertices and $2 \leq \delta(G) \leq n-2$, then $\gamma_r^t(G) \leq n - \delta(G)$.*

Proof. Let x be a vertex with $d_G(x) = \delta(G)$. We consider two cases:
Case 1. There exists a vertex $y \in N_G(x)$ such that $N_G[y] \neq N_G[x]$. Let $A = N_G[x] - \{y\}$ and $D = V(G) - A$. As $|A| = \delta(G)$, each vertex of A has a neighbour in D, so D is dominating. Moreover, $\langle A \rangle$ is isolate-free. Suppose now that u is isolate in $\langle D \rangle$. Then obviously $u \neq y$ and $N_G(u) = A$. Hence u is a neighbour of x. But then $d_G(x) > \delta(G)$, a contradiction. We conclude that D is a total restrained dominating set of G and thus $\gamma_r^t(G) \leq |D| = n - \delta(G)$.
Case 2. $N_G[y] = N_G[x]$ for every vertex $y \in N_G(x)$. Then the connectivity of G implies that G is complete. But then $\delta(G) = n - 1$, a contradiction. ■

Proposition 7 *Let G be a connected graph on $n(G) \geq 3$. Then $\gamma_r^t(G) = n_1(G) + n_S(G)$ if and only if $\langle V(G) - J(G) \rangle$ has no isolates and each vertex of $V(G) - J(G)$ has a neighbour in $J(G)$.*

Proof. Assume first that G is connected, each vertex of $V(G) - J(G)$ has a neighbour in $J(G)$ and $\langle V(G) - J(G) \rangle$ has no isolates. Then by Proposition 1, $\gamma_r^t(G) \geq n_1(G) + n_S(G)$. We claim that $J(G)$ is a total restrained dominating set of G. It is clear that $\langle J(G) \rangle$ does not contain isolates. Moreover, we know that $\langle V(G) - J(G) \rangle$ has no isolates and each vertex of $V(G) - J(G)$ has a neighbour in $J(G)$. Hence, $\gamma_r^t(G) = |J(G)| = n_1(G) + n_S(G)$.

Conversely, if $\gamma_r^t(G) = n_1(G) + n_S(G)$, then $J(G)$ is a minimum total restrained dominating set of G. Thus, $\langle V(G) - J(G) \rangle$ has no isolates and each vertex of $V(G) - J(G)$ is dominated by a vertex of $J(G)$, so each vertex of $V(G) - J(G)$ has a neighbour in $J(G)$. ■

In order to present our next bound on the total restrained domination number, we denote by $\mathcal{B}$ the family of all connected graphs G in which there exists a set $D \subseteq V(G)$ with property that

- the induced subgraphs $\langle D \rangle$ and $\langle V(G) - D \rangle$ are isolate free;
- each vertex of D has exactly $\Delta(G) - 1$ neighbours in $V(G) - D$;
- each vertex of $V(G)$ has exactly one neighbour in D.

Theorem 8 *If G is a connected graph without isolates on n vertices and with m edges, then*

$$\frac{n}{\Delta(G)} \leq \gamma_r^t(G) \leq 2m - n + 2,$$

with equality for the lower bound if and only if G belongs to the family $\mathcal{B}$ and equality for the upper bound if and only if G is a tree belonging to the family $\mathcal{A}$.

Proof. Since $\frac{n}{\Delta(G)} \leq \gamma_t(G) \leq \gamma_r^t(G)$ the lower bound follows. If $G \in \mathcal{B}$, then D is a total restrained dominating set of G and hence $|D| \geq \gamma_r^t(G) \geq \frac{n}{\Delta(G)}$. On the other hand, $n(G) = |D| + (\Delta(G)-1) \cdot |D| = \Delta(G) \cdot |D|$, so $|D| = \frac{n}{\Delta(G)}$. Thus $\gamma_r^t(G) = \frac{n}{\Delta(G)}$.

Conversely, assume that $\gamma_r^t(G) = \frac{n}{\Delta(G)}$ and let D be a minimum total restrained dominating set of G. Then $\langle D \rangle$ and $\langle V(G) - D \rangle$ are isolate free and each vertex of $V(G)$ has at least one neighbour in D. We prove that each vertex of D has exactly $\Delta(G) - 1$ neighbours in $V(G) - D$. If not, then there exists a vertex $u \in D$ such that u has at most $\Delta(G) - 2$ neighbours in $V(G) - D$. For this reason

$$n \leq \gamma_r^t(G) + (\Delta(G) - 1) \cdot (\gamma_r^t(G) - 1) + \Delta(G) - 2 < \Delta(G) \cdot \gamma_r^t(G)$$

and hence $\gamma_r^t(G) > \frac{n}{\Delta(G)}$. Further, since D is a total restrained dominating set of G, it follows that each vertex of D has exactly one neighbour in D.

If there exists a vertex $u \in V(G)$ such that u has at least two neighbors in D, then $u \notin D$ and similarly

$$n \leq \gamma_r^t(G) + (\Delta(G) - 1) \cdot (\gamma_r^t(G) - 1) + \Delta(G) - 2 < \Delta(G) \cdot \gamma_r^t(G)$$

and hence $\gamma_r^t(G) > \frac{n}{\Delta(G)}$.

Now we prove the upper bound. Obviously, $\gamma_r^t(G) \leq n = 2(n-1) - n + 2$ and since G is connected, $m \geq n - 1$. Thus, $\gamma_r^t(G) \leq 2m - n + 2$. Now we show that $\gamma_r^t(G) = 2m - n + 2$ if and only if G is a tree belonging to the family $\mathcal{A}$. If G is a tree belonging to the family $\mathcal{A}$, then $m = n - 1$ and $\gamma_r^t(G) = n = 2m - n + 2$. Conversely, let $\gamma_r^t(G) = 2m - n + 2$. Then $2m - n + 2 \leq n$, which implies that $m \leq n - 1$. Since G is connected, $m \geq n - 1$. We conclude that $m = n - 1$, so G is a tree with $\gamma_r^t(G) = n$ and by Theorem 4, G belongs to the family $\mathcal{A}$. ■

In order to present our next bound on the total restrained domination number, we denote by $\mathcal{C}$ a family of all connected graphs such that G belongs to $\mathcal{C}$ if and only if there exists a set $A \subseteq V(G)$ such that $\langle A \rangle = pK_2$ and $\langle V(G) - A \rangle = qK_2$ for some non-negative integers p and q, and each vertex of $V(G) - A$ has exactly one neighbour in A.

Theorem 9 *If G is a graph without isolates on n vertices and with m edges, then $\gamma_r^t(G) \geq \frac{3}{2}n - m$ with equality for the bound if and only if G belongs to the family $\mathcal{C}$.*

Proof. Let D be a minimum total restrained dominating set in G. Since $\langle D \rangle$ and $\langle V - D \rangle$ are isolate free and D is dominating, we have the following inequalities:

$$\begin{aligned} m(\langle D \rangle) &\geq \tfrac{\gamma_r^t(G)}{2}, \\ m(\langle V(G) - D \rangle) &\geq \tfrac{n - \gamma_r^t(G)}{2}, \\ m_{\gamma_r^t} &\geq n - \gamma_r^t(G), \end{aligned} \tag{1}$$

where $m_{\gamma_r^t}$ is the number of the edges connecting vertices of $V(G) - D$ to vertices of D. By summing the inequalities we obtain

$$m = m(\langle D \rangle) + m(\langle V - D \rangle) + m_{\gamma_r^t} \geq \frac{3}{2}n - \gamma_r^t(G),$$

and thus $\gamma_r^t(G) \geq \frac{3}{2}n - m$.

If G belongs to the family $\mathcal{C}$, then $n = 2p + 2q$, $m = p + 3q$ and A is a total restrained dominating set of G. As $|A| \geq \gamma_r^t(T) \geq \frac{3}{2}n - m$ we conclude that $|A| = 2p \geq \frac{3}{2}n - m = 2p$. This implies that A is a minimum total restrained dominating set and $\gamma_r^t(G) = \frac{3}{2}n - m$.

Conversely, if $\gamma_r^t(G) = \frac{3}{2}n - m$ and D is a minimum total restrained dominating set of G, then we have equalities in (1). This implies that $\langle D \rangle = pK_2$, $\langle V(G) - D \rangle = qK_2$ and each vertex of $V(G) - D$ has exactly one neighbour in D. Hence G belongs to the family $\mathcal{C}$. ■

Corollary 10 *If G is a graph without isolates and does not contain a perfect matching, then $\gamma_r^t(G) \geq \frac{3}{2}n - m + 1$.*

Corollary 11 *If G is a graph without isolates and with odd number of vertices, then $\gamma_r^t(G) \geq \frac{3(n+1)}{2} - m$.*

Proposition 12 *If G is hamiltonian, then $\gamma_r^t(G) \leq n - 2\lfloor \frac{n}{4} \rfloor$.*

Proof. Let G be hamiltonian and let C be a hamiltonian cycle of G. By Proposition 2, $\gamma_r^t(C) = n - 2\lfloor \frac{n}{4} \rfloor$. As adding an edge does not increase the total restrained domination number of G, we obtain the result. ■

3 Edge subdivision and vertex removal

In this section we study the influence of edge subdivision and vertex removal on the total restrained domination number of a graph. An *edge subdivision* in a nonempty graph G is an operation of removal of an edge $e = uv$ and the addition of a new vertex w and edges uw and vw. A graph obtained from G by subdividing the edge $e = uv$ is denoted by $G \oplus w_{uv}$.

Theorem 13 *For every integer k, $k \geq 2$, there exists a graph G such that $\gamma_r^t(G \oplus w_{uv}) - \gamma_r^t(G) = k$.*

Proof. We construct graphs G and $G \oplus w_{uv}$ as follows. We begin with two stars $K_{1,k-1}$ and denote its centers by u and v. Next we add a vertex x and edges joining x with all vertices of the stars. Finally we add an edge uv and a pendant edge xx' (see Fig. 1). It is easy to observe that $\{x, x'\}$ is a minimum total restrained dominating set of G.

For the graph $G \oplus w_{uv}$ notice, that the set $N_G[v] \cup \{x, x'\}$ is a minimum total restrained dominating set of cardinality $k + 2$. Thus $\gamma_r^t(G \oplus w_{uv}) - \gamma_r^t(G) = k$. ■

Theorem 14 *For every integer k, $k \geq 2$, there exists a graph H such that $\gamma_r^t(H) - \gamma_r^t(H \oplus z_{uw}) = k - 2$.*

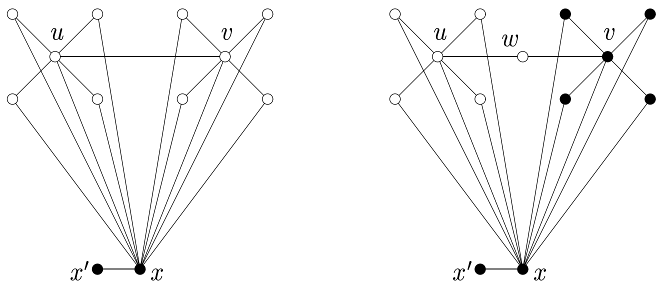

Figure 1: Graphs G and $G \oplus w_{uv}$ for k=5

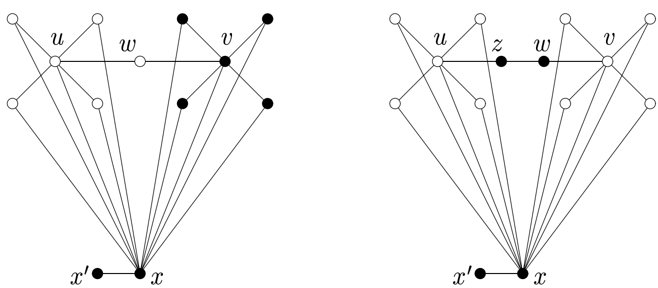

Figure 2: Graphs H and $H \oplus z_{uw}$ for k=5

Proof. Let H be the graph $G \oplus w_{uv}$ constructed as in the proof of Theorem 13. Then $\gamma_r^t(H) = k + 2$ (see Fig. 2).

For the graph $H \oplus z_{uw}$ notice, that the set $\{x, x', w, z\}$ is a minimum total restrained dominating set of cardinality 4. Thus $\gamma_r^t(H) - \gamma_r^t(H \oplus z_{uw}) = k+2-4 = k-2$. ∎

Now we investigate how removal of a vertex influences the total restrained domination number. Let us denote by $G - x$ the graph received from G by removing the vertex $x \in V(G)$ with all edges incident to x.

Theorem 15 *For every integer k, $k \geq 1$, there exists a graph G such that $\gamma_r^t(G) - \gamma_r^t(G - x) = k + 2$.*

Proof. Let G be the graph obtained from the join $K_{1,k} + K_1$ by adding to each of the two vertices of degree $k + 1$ a leaf (see Fig. 3). It is easy to observe that $V(G)$ is a minimum total restrained dominating set of G. Thus, $\gamma_r^t(G) = k + 4$.

Denote by x a leaf in G and consider the graph $G - x$. Obviously $\gamma_r^t(G - x) = 2$ and finally $\gamma_r^t(G) - \gamma_r^t(G - x) = k + 2$. ∎

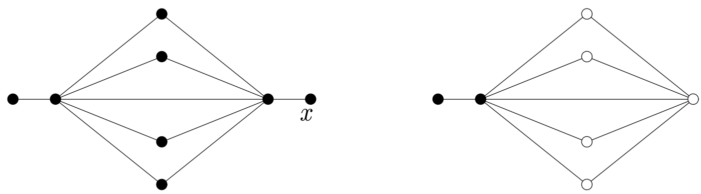

Figure 3: Graphs G and $G-x$ for $k=4$

Theorem 16 *For every integer k, $k \geq 2$, there exists a graph G such that $\gamma_r^t(G-x)-\gamma_r^t(G)=k-1$.*

Proof. Let G be a join $K_{1,k}+K_1$. Then $\gamma_r^t(G)=2$. Let x be a vertex of degree $k+1$. As $G-x$ is a star, $\gamma_r^t(G-x)=k+1$.

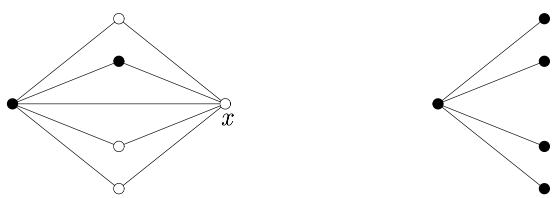

Figure 4: Graphs G and $G-x$ for $k=4$

Thus $\gamma_r^t(G-x)-\gamma_r^t(G)=k-1$. ■

4 Comparison γ_r^t to other types of domination numbers

In this section we investigate relations between total restrained domination number and other types of domination numbers.

A set $D \subseteq V(G)$ is a *connected dominating set* if it is dominating and the induced subgraph $\langle D\rangle$ is connected. The cardinality of a minimum connected dominating set in G is the *connected domination number* and is denoted by $\gamma_c(G)$.

Numbers $\gamma_r^t(G)$ and $\gamma_c(G)$ are incomparable. Consider, for example, any star and a path on more than 10 vertices.

A set $D \subseteq V(G)$ is a *doubly connected dominating set* if it is dominating and the induced subgraphs $\langle D\rangle$ and $\langle V(G)-D\rangle$ are connected. The cardinality of a minimum doubly connected dominating set in G is the *doubly connected domination number* and is denoted by $\gamma_{cc}(G)$. Properties of the doubly connected domination number are studied in [2]. Here we note that for any connected graph G, $1 \leq \gamma_{cc}(G) \leq n(G)-1$ and $\gamma_{cc}(G) = 1$ if and only if G has a non-cut vertex of degree $n(G)-1$.

Theorem 17 *For an arbitrary connected graph G with $n(G) > 1$,*

(i) $\gamma_r^t(G) \leq \gamma_{cc}(G) + 2$;

(ii) $\gamma_r^t(G) = \gamma_{cc}(G) + 2$ *if and only if G is the cycle C_3;*

(iii) *For any integer k, where $k \geq -2$, there exists a graph G such that $\gamma_{cc}(G) - \gamma_r^t(G) = k$.*

Proof.

(i) If $2 \leq \gamma_{cc}(G) \leq n-2$, then every doubly connected dominating set is a total restrained dominating set. That is, if $2 \leq \gamma_{cc}(G) \leq n-2$, then $\gamma_r^t(G) \leq \gamma_{cc}(G)$. Hence, $\gamma_r^t(G) \leq \gamma_{cc}(G) + 2$.

(ii) If G is the cycle C_3, the result is obvious. Thus assume $\gamma_r^t(G) = \gamma_{cc}(G)+2$. By the proof of (i), $\gamma_{cc}(G) = 1$ and hence $\gamma_r^t(G) = 3$. This implies that there exists a vertex $u \in V(G)$ of degree $n(G)-1$ such that $\langle V(G) - \{u\}\rangle$ is connected, and so $\delta(G) \geq 2$. Moreover, for each $v \in V(G) - \{u\}$ the set $\{u, v\}$ is not a total restrained dominating set, although $\{u, v\}$ is dominating and connected. Hence we conclude that for each $v \in V(G) - \{u\}$ the induced subgraph $\langle V(G) - \{u, v\}\rangle$ has isolates. Denote by x an isolate vertex of $\langle V(G) - \{u, v\}\rangle$. Of course $N_G(x) = \{u, v\}$. Similarly, v must be the only isolate in $\langle V(G) - \{u, x\}\rangle$ and the connectivity of $\langle V(G) - \{u\}\rangle$ implies that $n(G) = 3$. Consequently, $G = C_3$.

(iii) For $k = -2$ let $G = C_3$, for $k = -1$ let $G = P_4$ and for $k = 0$ let $G = C_5$. For $k \geq 1$ let G be the graph as in Fig. 5.

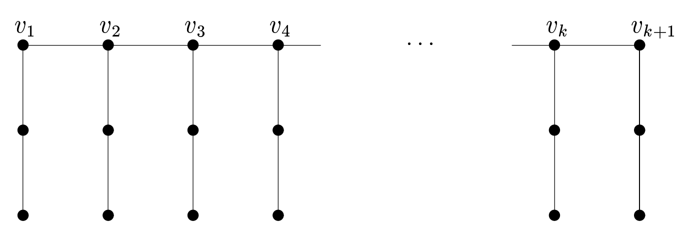

Figure 5: Graph G with $\gamma_{cc}(G) - \gamma_r^t(G) = k$ for $k \geq 1$.

It is left to the reader as an easy exercise to check that $\gamma_{cc}(G) - \gamma_r^t(G) = k$.

■

For an arbitrary graph G which is isolate free, we have the following inequalities: $\gamma(G) \leq \gamma_r^t(G)$, $\gamma_t(G) \leq \gamma_r^t(G)$ and $\gamma_r(G) \leq \gamma_r^t(G)$. However, each of these differences may be arbitrarily large.

Proposition 18 *For each positive integer k there exists graphs G_1, G_2, G_3 such that $\gamma_r^t(G_1) - \gamma(G_1) = k$, $\gamma_r^t(G_2) - \gamma_t(G_2) = k$ and $\gamma_r^t(G_3) - \gamma_r(G_3) = k$.*

Proof. Let G_1 be the star $K_{1,k}$, let G_2 be the star $K_{1,k+1}$ and let G_3 be the subdivided star $K^*_{1,k}$ for $k \geq 2$ and let G_3 be a graph obtained from the cycle C_4 by adding a pendant edge for $k = 1$. ■

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