

# Lower bound on the paired domination number of a tree

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## Abstract

We prove that the paired domination number  $\gamma_p(T)$  of a tree  $T$  on  $n > 1$  vertices and with  $n_1$  end-vertices satisfies the inequality  $\gamma_p(T) \geq (n + 2 - n_1)/2$  and we characterize the trees for which  $\gamma_p(T) = (n + 2 - n_1)/2$ .

## 1 Introduction

In this paper, all graphs considered will be finite and without multiple loops or edges. A set  $D \subseteq V(G)$  is a *dominating set* of a graph  $G$  if every vertex in  $V(G) - D$  is adjacent to least one vertex in  $D$ . A set  $D \subseteq V(G)$  is a *paired dominating set* of  $G$  if it is dominating and the induced subgraph  $\langle D \rangle$  has a perfect matching. The *paired domination number*  $\gamma_p(G)$  is the cardinality of a smallest paired dominating set  $D$  in  $G$ . This type of domination was introduced by Haynes and Slater in [4, 5] and is studied, for example, in [1, 7, 8, 9].

Let  $n(G)$  be the cardinality of the vertex set  $V(G)$ . The *open neighbourhood* of a vertex  $x \in V(G)$ , denoted by  $N_G(x)$ , is the set  $\{v \in V(G) : d_G(v, x) = 1\}$ , where  $d_G(v, x)$  is the distance between  $v$  and  $x$  in  $G$ . The set  $N_G[x] = N_G(x) \cup \{x\}$  is called the *closed neighbourhood* of  $x$  in  $G$ . For a set  $X \subseteq V(G)$ , the *closed neighbourhood*  $N_G[X]$  is defined to be  $\bigcup_{x \in X} N_G[x]$ . The *private neighbourhood of a vertex  $x$  with respect to a set  $D \subseteq V(G)$*  is the set  $PN_G[x, D] = N_G[x] - N_G[D - \{x\}]$ . Let  $\Omega(G)$  be the set of all end-vertices of  $G$ , that is the set of vertices degree 1, and let  $n_1(G)$  be the cardinality of  $\Omega(G)$ . A vertex  $v$  is called a *support* if  $v$  is a neighbour of an end-vertex. The *diameter*  $\text{diam}(G)$  of a connected graph  $G$  is the number  $\max_{u, v \in V(G)} d_G(u, v)$ . A *double star*  $S(p, r)$ , where  $p$  and  $r$  are positive integers, is the tree obtained from stars  $K_{1,p}$  and  $K_{1,r}$  by adding the edge joining one central vertex of  $K_{1,p}$  with one central vertex of  $K_{1,r}$ .

For unexplained terms and symbols see [2, 3].

Lemańska [6] has given a lower bound on the domination number of a tree  $T$  in terms of  $n(T)$  and  $n_1(T)$ . In this paper we present a similar lower bound on the

paired domination number of a tree. We have two aims in this paper: to prove that the paired domination number  $\gamma_p(T)$  of a tree  $T$  on  $n(T) > 1$  vertices satisfies inequality  $\gamma_p(T) \geq (n(T) + 2 - n_1(T))/2$  and to give a constructive characterization of the trees for which  $\gamma_p(T) = (n(T) + 2 - n_1(T))/2$ .

## 2 Results

We begin with a basic property of a paired dominating set.

**Observation 1** *If  $v$  is a support in  $G$ , then  $v$  is in every paired dominating set of  $G$ .* ■

Let  $D$  be a minimum paired dominating set of a tree  $T$ . By  $\Omega_l(T)$  we denote the set of all end-vertices which belong to any longest path in  $T$ . We say that  $D$  has property  $\mathcal{F}$  if the number  $|\Omega_l(T) \cap D|$  is as small as possible.

**Lemma 1** *If  $T$  is a tree with  $\gamma_p(T) > 2$ , then there exists an edge  $e \in E(T)$  such that  $\gamma_p(T) = \gamma_p(T_1) + \gamma_p(T_2)$ , where  $T_1$  and  $T_2$  are the components of  $T - e$ .*

**Proof.** Let  $T$  be a tree with  $\gamma_p(T) > 2$  and let  $D$  be a minimum paired dominating set with property  $\mathcal{F}$  in  $T$ . Then  $\text{diam}(T) > 3$  and we consider two cases:

*Case 1.* If  $\Omega_l(T) \cap D \neq \emptyset$ , then there exists a longest path  $S = (s_0, s_1, \dots, s_l)$  in  $T$  such that  $s_0$  and  $s_1$  belong to  $D$ . In this case  $s_2$  also belongs to  $D$ , as otherwise  $D' = D - \{s_0\} \cup \{s_2\}$  would be a minimum paired dominating set of  $T$  with  $|\Omega_l(T) \cap D'| < |\Omega_l(T) \cap D|$ , a contradiction. Now it is easy to observe that if  $T_1$  and  $T_2$  are the components of  $T - s_1s_2$  containing  $s_1$  and  $s_2$  respectively, then  $\{s_0, s_1\}$  and  $D - \{s_0, s_1\}$  are minimum paired dominating sets in  $T_1$  and  $T_2$  respectively, and therefore  $\gamma_p(T_1) = 2$ , while  $\gamma_p(T_2) = \gamma_p(T) - 2$ .

*Case 2.* Assume now that  $\Omega_l(T) \cap D = \emptyset$ , and let  $S = (s_0, s_1, \dots, s_l)$  be a longest path in  $T$ . In this case  $s_0 \notin D$ ,  $s_1, s_2 \in D$ , and  $s_1s_2$  is an edge of a perfect matching of  $\langle D \rangle$ . We claim that  $d_T(v) = 1$  for each vertex  $v \in N_T(s_2) - V(S)$ . Suppose on the contrary, that there exists  $v \in N_T(s_2) - V(S)$  with  $d_T(v) > 1$ . Thus, since  $S$  is a longest path in  $T$ , every vertex belonging to  $N_T(v) - \{s_2\}$  has degree 1. Therefore,  $v$  is a support and from Observation 1,  $v \in D$ . Since  $v \in D$  and  $\Omega_l(T) \cap D = \emptyset$ , the edge  $vs_2$  belongs to a perfect matching of  $\langle D \rangle$ , which is impossible as the edge  $s_1s_2$  already belongs to the same perfect matching. This proves the claim. We consider two subcases:  $s_3 \in PN_T[s_2, D]$  and  $s_3 \notin PN_T[s_2, D]$ .

*Subcase 2.1.* If  $s_3 \in PN_T[s_2, D]$ , then it is easy to observe that  $d_T(s_3) = 2$ . In addition, if  $T_1$  and  $T_2$  are the components of  $T - s_3s_4$  containing  $s_3$  and  $s_4$  respectively, then  $\gamma_p(T_1) = 2$  and  $\gamma_p(T_2) = \gamma_p(T) - 2$ .

*Subcase 2.2.* If  $s_3 \notin PN_T[s_2, D]$  and if  $T_1$  and  $T_2$  are the components of  $T - s_2s_3$  containing  $s_2$  and  $s_3$  respectively, then  $\gamma_p(T_1) = 2$  and  $\gamma_p(T_2) = \gamma_p(T) - 2$ .



Thus, in any event the statement holds. ■

**Theorem 2** *If  $T$  is a tree on  $n(T) > 1$  vertices, then*

$$n_1(T) \geq n(T) + 2 - 2\gamma_p(T).$$

**Proof.** We proceed by induction on  $\gamma_p(T)$ . If  $T$  is a tree with  $\gamma_p(T) = 2$ , then  $T$  is a star or a double star, and it is easy to observe that  $n_1(T) \geq n(T) - 2 = n(T) + 2 - 2\gamma_p(T)$ .

Assume now that the result is true for all trees  $T'$  with  $2 \leq \gamma_p(T') \leq j$  and let  $T$  be a tree with  $\gamma_p(T) = j + 2$ . Let  $D$  be a minimum paired dominating set of  $T$ . In this case  $\text{diam}(T) > 3$  and by Lemma 1, there exists an edge  $e \in E(T)$  such that  $\gamma_p(T) = \gamma_p(T_1) + \gamma_p(T_2)$ , where  $T_1$  and  $T_2$  are the components of  $T - e$ . It is immediate that  $n(T_1) + n(T_2) = n(T)$  and  $n_1(T_1) + n_1(T_2) \leq n_1(T) + 2$ . By induction hypothesis,  $n_1(T_1) \geq n(T_1) + 2 - 2\gamma_p(T_1)$  and  $n_1(T_2) \geq n(T_2) + 2 - 2\gamma_p(T_2)$ . Therefore

$$\begin{aligned} n_1(T) + 2 &\geq n_1(T_1) + n_1(T_2) &\geq (n(T_1) + 2 - 2\gamma_p(T_1)) + (n(T_2) + 2 - 2\gamma_p(T_2)) \\ &= (n(T_1) + n(T_2)) + 2 - 2(\gamma_p(T_1) + \gamma_p(T_2)) + 2 \\ &= n(T) + 2 - 2\gamma_p(T) + 2 \end{aligned}$$

and consequently,

$$n_1(T) \geq n(T) + 2 - 2\gamma_p(T).$$

■

We are now in a position to provide a constructive characterization of the trees  $T$  for which  $n_1(T) = n(T) + 2 - 2\gamma_p(T)$ . For this purpose, we introduce the following operation: if  $T_1$  and  $T_2$  are vertex disjoint trees, then by  $T_1 \oplus T_2$  we denote a tree obtained from  $T_1$  and  $T_2$  by adding an edge joining an end-vertex of  $T_1$  with an end-vertex of  $T_2$ .

Let  $\mathcal{R}_p$  denote the family of trees such that:

- (i) Every double star  $S(p, r)$  belongs to  $\mathcal{R}_p$ ;
- (ii)  $T_1 \oplus T_2$  belongs to  $\mathcal{R}_p$  if only  $T_1$  and  $T_2$  belong to  $\mathcal{R}_p$ .

**Observation 2** *If  $T$  is a tree belonging to the family  $\mathcal{R}_p$ , then either  $T$  is a double star or there are double stars  $S_1, \dots, S_j$  ( $j \geq 2$ ) such that  $T = (\dots(S_1 \oplus S_2) \oplus \dots \oplus S_{j-1}) \oplus S_j$ .*

**Lemma 3** *If  $T$  is a tree belonging to the family  $\mathcal{R}_p$ , then*

$$n_1(T) = n(T) + 2 - 2\gamma_p(T).$$

**Proof.** If  $T$  is a double star, then  $\gamma_p(T) = 2$ ,  $n_1(T) = n(T) - 2$  and certainly  $n_1(T) = n(T) + 2 - 2\gamma_p(T)$ . Otherwise, if  $T$  is a tree obtained from  $j$  double stars  $S_1, \dots, S_j$  ( $j \geq 2$ ), then it is easily seen that  $\gamma_p(T) = 2j$ . Moreover,

$$n(T) = \sum_{i=1}^j n(S_i) = \sum_{i=1}^j (n_1(S_i) + 2),$$



and

$$n_1(T) = \sum_{i=1}^j n_1(S_i) - 2(j - 1).$$

It is easy to check that the equality  $n_1(T) = n(T) + 2 - 2\gamma_p(T)$  holds. ■

**Lemma 4** *If  $T$  is a tree with  $n(T) > 1$  and  $n_1(T) = n(T) + 2 - 2\gamma_p(T)$ , then  $T$  belongs to the family  $\mathcal{R}_p$ .*

**Proof.** We proceed by induction on  $\gamma_p(T)$ . If  $\gamma_p(T) = 2$  then  $\text{diam}(T) \leq 3$  and  $n_1(T) = n(T) + 2 - 2\gamma_p(T) = n(T) - 2$ . Hence  $n(T) \geq 4$  and there are exactly two supports in  $T$ . Therefore  $T$  is a double star.

Assume now that the result is true for all trees  $T'$  with  $2 \leq \gamma_p(T') \leq j$ , and let  $T$  be a tree with  $\gamma_p(T) = j + 2$  and such that  $n_1(T) = n(T) + 2 - 2\gamma_p(T)$ .

Lemma 1 implies that there exists an edge  $e \in E(T)$  such that  $\gamma_p(T) = \gamma_p(T_1) + \gamma_p(T_2)$ , where  $T_1$  and  $T_2$  are the components of  $T - e$ . It is immediate that  $n(T_1) + n(T_2) = n(T)$ . Moreover,  $n_1(T_1) + n_1(T_2) \leq n_1(T) + 2$ . By Theorem 2,  $n_1(T_1) \geq n(T_1) + 2 - 2\gamma_p(T_1)$  and  $n_1(T_2) \geq n(T_2) + 2 - 2\gamma_p(T_2)$ . Therefore,

$$n_1(T) \geq n_1(T_1) + n_1(T_2) - 2 \geq n(T) + 2 - 2\gamma_p(T).$$

As  $n_1(T) = n(T) + 2 - 2\gamma_p(T)$  we conclude that

$$n_1(T) = n_1(T_1) + n_1(T_2) - 2 = n(T) + 2 - 2\gamma_p(T),$$

which implies that

$$\begin{aligned} n_1(T_1) + n_1(T_2) &= n_1(T) + 2 \\ n_1(T_1) &= n(T_1) + 2 - 2\gamma_p(T_1) \\ n_1(T_2) &= n(T_2) + 2 - 2\gamma_p(T_2). \end{aligned}$$

Thus, by induction  $T_1$  and  $T_2$  belong to the family  $\mathcal{R}_p$  and, if  $e = uv$  was the edge we removed from  $T$  to obtain  $T_1$  and  $T_2$ , then  $d_{T_1}(u) = d_{T_2}(v) = 1$ , that is  $u$  and  $v$  are end-vertices in  $T_1$  and  $T_2$  respectively. Therefore,  $T = T_1 \oplus T_2$  and we conclude that  $T \in \mathcal{R}_p$ . ■

The following result is obvious from Lemmas 3 and 4.

**Theorem 5** *If  $T$  is a tree on  $n(T) > 1$  vertices, then*

$$n_1(T) = n(T) + 2 - 2\gamma_p(T)$$

*if and only if  $T$  belongs to the family  $\mathcal{R}_p$ .* ■



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