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Lower bound on the paired domination number of a tree

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Abstract

We prove that the paired domination number $\gamma_p(T)$ of a tree T on $n > 1$ vertices and with n_1 end-vertices satisfies the inequality $\gamma_p(T) \geq (n+2-n_1)/2$ and we characterize the trees for which $\gamma_p(T) = (n+2-n_1)/2$.

1 Introduction

In this paper, all graphs considered will be finite and without multiple loops or edges. A set $D \subseteq V(G)$ is a *dominating set* of a graph G if every vertex in $V(G) - D$ is adjacent to least one vertex in D. A set $D \subseteq V(G)$ is a *paired dominating set* of G if it is dominating and the induced subgraph $\langle D \rangle$ has a perfect matching. The *paired domination number* $\gamma_p(G)$ is the cardinality of a smallest paired dominating set D in G. This type of domination was introduced by Haynes and Slater in [4, 5] and is studied, for example, in [1, 7, 8, 9].

Let $n(G)$ be the cardinality of the vertex set $V(G)$. The *open neighbourhood* of a vertex $x \in V(G)$, denoted by $N_G(x)$, is the set $\{v \in V(G) : d_G(v,x) = 1\}$, where $d_G(v,x)$ is the distance between v and x in G. The set $N_G[x] = N_G(x) \cup \{x\}$ is called the *closed neighbourhood* of x in G. For a set $X \subseteq V(G)$, the *closed neighbourhood* $N_G[X]$ is defined to be $\bigcup_{x \in X} N_G[x]$. The *private neighbourhood of a vertex x with respect to a set $D \subseteq V(G)$* is the set $PN_G[x,D] = N_G[x] - N_G[D - \{x\}]$. Let $\Omega(G)$ be the set of all end-vertices of G, that is the set of vertices degree 1, and let $n_1(G)$ be the cardinality of $\Omega(G)$. A vertex v is called a *support* if v is a neighbour of an end-vertex. The *diameter* $\operatorname{diam}(G)$ of a connected graph G is the number $\max_{u,v \in V(G)} d_G(u,v)$. A *double star* $S(p,r)$, where p and r are positive integers, is the tree obtained from stars $K_{1,p}$ and $K_{1,r}$ by adding the edge joining one central vertex of $K_{1,p}$ with one central vertex of $K_{1,r}$.

For unexplained terms and symbols see [2, 3].

Lemańska [6] has given a lower bound on the domination number of a tree T in terms of $n(T)$ and $n_1(T)$. In this paper we present a similar lower bound on the paired domination number of a tree. We have two aims in this paper: to prove that the paired domination number $\gamma_p(T)$ of a tree T on $n(T) > 1$ vertices satisfies inequality $\gamma_p(T) \geq (n(T) + 2 - n_1(T))/2$ and to give a constructive characterization of the trees for which $\gamma_p(T) = (n(T) + 2 - n_1(T))/2$.

2 Results

We begin with a basic property of a paired dominating set.

Observation 1 *If v is a support in G, then v is in every paired dominating set of G.* ■

Let D be a minimum paired dominating set of a tree T. By $\Omega_l(T)$ we denote the set of all end-vertices which belong to any longest path in T. We say that D has property $\mathcal{F}$ if the number $|\Omega_l(T) \cap D|$ is as small as possible.

Lemma 1 *If T is a tree with $\gamma_p(T) > 2$, then there exists an edge $e \in E(T)$ such that $\gamma_p(T) = \gamma_p(T_1) + \gamma_p(T_2)$, where T_1 and T_2 are the components of $T - e$.*

Proof. Let T be a tree with $\gamma_p(T) > 2$ and let D be a minimum paired dominating set with property $\mathcal{F}$ in T. Then $\mathrm{diam}(T) > 3$ and we consider two cases:

Case 1. If $\Omega_l(T) \cap D \neq \emptyset$, then there exists a longest path $S = (s_0, s_1, \ldots, s_l)$ in T such that s_0 and s_1 belong to D. In this case s_2 also belongs to D, as otherwise $D' = D - \{s_0\} \cup \{s_2\}$ would be a minimum paired dominating set of T with $|\Omega_l(T) \cap D'| < |\Omega_l(T) \cap D|$, a contradiction. Now it is easy to observe that if T_1 and T_2 are the components of $T - s_1s_2$ containing s_1 and s_2 respectively, then $\{s_0, s_1\}$ and $D - \{s_0, s_1\}$ are minimum paired dominating sets in T_1 and T_2 respectively, and therefore $\gamma_p(T_1) = 2$, while $\gamma_p(T_2) = \gamma_p(T) - 2$.

Case 2. Assume now that $\Omega_l(T) \cap D = \emptyset$, and let $S = (s_0, s_1, \ldots, s_l)$ be a longest path in T. In this case $s_0 \notin D$, $s_1, s_2 \in D$, and s_1s_2 is an edge of a perfect matching of $\langle D \rangle$. We claim that $d_T(v) = 1$ for each vertex $v \in N_T(s_2) - V(S)$. Suppose on the contrary, that there exists $v \in N_T(s_2) - V(S)$ with $d_T(v) > 1$. Thus, since S is a longest path in T, every vertex belonging to $N_T(v) - \{s_2\}$ has degree 1. Therefore, v is a support and from Observation 1, $v \in D$. Since $v \in D$ and $\Omega_l(T) \cap D = \emptyset$, the edge vs_2 belongs to a perfect matching of $\langle D \rangle$, which is impossible as the edge s_1s_2 already belongs to the same perfect matching. This proves the claim. We consider two subcases: $s_3 \in PN_T[s_2, D]$ and $s_3 \notin PN_T[s_2, D]$.

Subcase 2.1. If $s_3 \in PN_T[s_2, D]$, then it is easy to observe that $d_T(s_3) = 2$. In addition, if T_1 and T_2 are the components of $T - s_3s_4$ containing s_3 and s_4 respectively, then $\gamma_p(T_1) = 2$ and $\gamma_p(T_2) = \gamma_p(T) - 2$.

Subcase 2.2. If $s_3 \notin PN_T[s_2, D]$ and if T_1 and T_2 are the components of $T - s_2s_3$ containing s_2 and s_3 respectively, then $\gamma_p(T_1) = 2$ and $\gamma_p(T_2) = \gamma_p(T) - 2$.

Thus, in any event the statement holds. ■

Theorem 2 *If T is a tree on $n(T) > 1$ vertices, then*

$$n_1(T) \geq n(T) + 2 - 2\gamma_p(T).$$

Proof. We proceed by induction on $\gamma_p(T)$. If T is a tree with $\gamma_p(T) = 2$, then T is a star or a double star, and it is easy to observe that $n_1(T) \geq n(T) - 2 = n(T) + 2 - 2\gamma_p(T)$.

Assume now that the result is true for all trees T' with $2 \leq \gamma_p(T') \leq j$ and let T be a tree with $\gamma_p(T) = j + 2$. Let D be a minimum paired dominating set of T. In this case $\mathrm{diam}(T) > 3$ and by Lemma 1, there exists an edge $e \in E(T)$ such that $\gamma_p(T) = \gamma_p(T_1) + \gamma_p(T_2)$, where T_1 and T_2 are the components of $T - e$. It is immediate that $n(T_1)+n(T_2) = n(T)$ and $n_1(T_1)+n_1(T_2) \leq n_1(T)+2$. By induction hypothesis, $n_1(T_1) \geq n(T_1)+2-2\gamma_p(T_1)$ and $n_1(T_2) \geq n(T_2)+2-2\gamma_p(T_2)$. Therefore

$$\begin{aligned} n_1(T) + 2 \geq n_1(T_1) + n_1(T_2) &\geq (n(T_1) + 2 - 2\gamma_p(T_1)) + (n(T_2) + 2 - 2\gamma_p(T_2)) \\ &= (n(T_1) + n(T_2)) + 2 - 2(\gamma_p(T_1) + \gamma_p(T_2)) + 2 \\ &= n(T) + 2 - 2\gamma_p(T) + 2 \end{aligned}$$

and consequently,

$$n_1(T) \geq n(T) + 2 - 2\gamma_p(T).$$

■

We are now in a position to provide a constructive characterization of the trees T for which $n_1(T) = n(T)+2-2\gamma_p(T)$. For this purpose, we introduce the following operation: if T_1 and T_2 are vertex disjoint trees, then by $T_1 \oplus T_2$ we denote a tree obtained from T_1 and T_2 by adding an edge joining an end-vertex of T_1 with an end-vertex of T_2.

Let $\mathcal{R}_p$ denote the family of trees such that:
(i) Every double star $S(p, r)$ belongs to $\mathcal{R}_p$;
(ii) $T_1 \oplus T_2$ belongs to $\mathcal{R}_p$ if only T_1 and T_2 belong to $\mathcal{R}_p$.

Observation 2 *If T is a tree belonging to the family $\mathcal{R}_p$, then either T is a double star or there are double stars $S_1, \ldots, S_j$ $(j \geq 2)$ such that $T = (\ldots(S_1 \oplus S_2) \oplus \cdots \oplus S_{j-1}) \oplus S_j$.*

Lemma 3 *If T is a tree belonging to the family $\mathcal{R}_p$, then*

$$n_1(T) = n(T) + 2 - 2\gamma_p(T).$$

Proof. If T is a double star, then $\gamma_p(T) = 2$, $n_1(T) = n(T) - 2$ and certainly $n_1(T) = n(T) + 2 - 2\gamma_p(T)$. Otherwise, if T is a tree obtained from j double stars $S_1, \ldots, S_j$ $(j \geq 2)$, then it is easily seen that $\gamma_p(T) = 2j$. Moreover,

$$n(T) = \sum_{i=1}^{j} n(S_i) = \sum_{i=1}^{j} (n_1(S_i) + 2),$$

and

$$n_1(T) = \sum_{i=1}^{j} n_1(S_i) - 2(j-1).$$

It is easy to check that the equality $n_1(T) = n(T) + 2 - 2\gamma_p(T)$ holds. ■

Lemma 4 *If T is a tree with $n(T) > 1$ and $n_1(T) = n(T) + 2 - 2\gamma_p(T)$, then T belongs to the family $\mathcal{R}_p$.*

Proof. We proceed by induction on $\gamma_p(T)$. If $\gamma_p(T) = 2$ then $\operatorname{diam}(T) \leq 3$ and $n_1(T) = n(T) + 2 - 2\gamma_p(T) = n(T) - 2$. Hence $n(T) \geq 4$ and there are exactly two supports in T. Therefore T is a double star.

Assume now that the result is true for all trees T' with $2 \leq \gamma_p(T') \leq j$, and let T be a tree with $\gamma_p(T) = j + 2$ and such that $n_1(T) = n(T) + 2 - 2\gamma_p(T)$.

Lemma 1 implies that there exists an edge $e \in E(T)$ such that $\gamma_p(T) = \gamma_p(T_1) + \gamma_p(T_2)$, where T_1 and T_2 are the components of $T - e$. It is immediate that $n(T_1) + n(T_2) = n(T)$. Moreover, $n_1(T_1) + n_1(T_2) \leq n_1(T) + 2$. By Theorem 2, $n_1(T_1) \geq n(T_1) + 2 - 2\gamma_p(T_1)$ and $n_1(T_2) \geq n(T_2) + 2 - 2\gamma_p(T_2)$. Therefore,

$$n_1(T) \geq n_1(T_1) + n_1(T_2) - 2 \geq n(T) + 2 - 2\gamma_p(T).$$

As $n_1(T) = n(T) + 2 - 2\gamma_p(T)$ we conclude that

$$n_1(T) = n_1(T_1) + n_1(T_2) - 2 = n(T) + 2 - 2\gamma_p(T),$$

which implies that

$$\begin{aligned} n_1(T_1) + n_1(T_2) &= n_1(T) + 2 \\ n_1(T_1) &= n(T_1) + 2 - 2\gamma_p(T_1) \\ n_1(T_2) &= n(T_2) + 2 - 2\gamma_p(T_2). \end{aligned}$$

Thus, by induction T_1 and T_2 belong to the family $\mathcal{R}_p$ and, if $e = uv$ was the edge we removed from T to obtain T_1 and T_2, then $d_{T_1}(u) = d_{T_2}(v) = 1$, that is u and v are end-vertices in T_1 and T_2 respectively. Therefore, $T = T_1 \oplus T_2$ and we conclude that $T \in \mathcal{R}_p$. ■

The following result is obvious from Lemmas 3 and 4.

Theorem 5 *If T is a tree on $n(T) > 1$ vertices, then*

$$n_1(T) = n(T) + 2 - 2\gamma_p(T)$$

if and only if T belongs to the family $\mathcal{R}_p$. ■

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