NUMERICAL ANALYSIS OF THE FINITE AMPLITUDE PLANE WAVE PROPAGATION PROBLEM

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The paper presents the results of the theoretical investigations of the finite amplitude plane wave propagation problem. The case of harmonic plane pressure wave of megahertz frequency propagating in water is discussed. Mathematical model and some results of numerical calculations are shown. The nonlinear acoustics equation was considered to build the mathematical model. To solve the problem numerically the finite-difference method was applied. The influence of discrete model parameters on the numerical calculations accuracy was studied. The results of computer calculations for different values of physical parameters were also analyzed.

INTRODUCTION

The nonlinear wave propagation problem is described basing on continuity, motion and state equations. The system of these equations is converted to the nonlinear partial differential equation called the nonlinear acoustics equation [1, 2, 3]. This equation has not exact analytical solution. Moreover it has rather complicated form. Therefore the equations which have easier form are used to solve the finite amplitude wave propagation problem in practice. For example, using the quasi-optical assumption the nonlinear acoustics equation is converted to the KZK equation [2, 3]. The Burgers equation is often used to analyze the plane wave propagation. This equation is obtain from the KZK equation assuming that propagated wave is plane one. In general case the nonlinear acoustics equation is solved numerically. The finite-difference method and finite-element method can be used to solve this equation numerically [1].

Analysis of the pressure wave along propagating axis is possible using the nonlinear acoustics equation in one dimensional case. The main aim of this paper was numerical analysis of the finite amplitude wave propagation problem modeled using this equation. The finite-difference method was used to solve the problem numerically. The convergence and accuracy of obtained discrete model are discussed. Additionally some results of numerical calculations obtained for different values of physical parameters are presented.

1. MATHEMATICAL MODEL

We assume that the finite amplitude plane wave is propagated in x axis direction. Mathematical model of this problem is built on the basis on the partial differential equation:

$$\Delta p' - \frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} + \frac{b}{c_0^2\rho_0}\Delta\left(\frac{\partial p'}{\partial t}\right) = -\frac{\varepsilon}{\rho_0 c_0^4}\frac{\partial^2 p'^2}{\partial t^2} \tag{1}$$

were: $p' = p - p_0$ - acoustic pressure,

c_0 - speed of sound,

ρ_0 - medium density at rest,

b - dissipation coefficient of the medium,

ε - nonlinear coefficient,

Δ - Laplace operator.

Equation (1) is obtained from the nonlinear acoustics equation assuming plane wave propagation in water.

The primary pressure wave is given as follows

$$p'(x=0,t) = -p_0 \sin(2\pi f t)\,. \tag{2}$$

were f denote primary wave frequency. Additionally we assume that function $p'=p'(x,t)$ is periodic function of the coordinate t.

When

$$a_0 = \frac{1}{c_0^2},\; a_1 = \frac{b}{c_0^2\rho_0},\; a_2 = \frac{\varepsilon}{\rho_0 c_0^4},$$

we may rewrite Eq. (1) as follows:

$$\Delta p' - a_0\frac{\partial^2 p'}{\partial t^2} + a_1\Delta\left(\frac{\partial p'}{\partial t}\right) = -a_2\frac{\partial^2 p'^2}{\partial t^2} \tag{3}$$

To solve the problem numerically the derivative $\frac{\partial^2 p'^2}{\partial t^2}$ is written by formula

$$\frac{\partial^2 p'^2}{\partial t^2} = \frac{\partial}{\partial t}\left(2p'\frac{\partial p'}{\partial t}\right) = 2\left(\left(\frac{\partial p'}{\partial t}\right)^2 + p'\frac{\partial^2 p'}{\partial t^2}\right)$$

Finally Eq. (3) with boundary condition (2) is solved for fixed distances and fixed time interval, i.e. inside domain $D = \{(x,t) : x \in [0, X_{\max}], t \in [0, T_{\max}]\}$.

2. NUMERICAL SOLUTION

To solve Eq. (3) numerically function $p'(x,t)$ is discretized in both distance x and time t. Now let n designed the nth step in the x direction and m designed the mth time step. Then the net is defined in following way

$$t_m = m\Delta t, \quad x_n = n\Delta x$$
$$\Delta t = \frac{T_{\max}}{N_t}, \quad \Delta x = \frac{X_{\max}}{N_x}.$$

where $m=0,1,\dots,N_t$ and $n=0,1,\dots,N_x-1$. We assume constant dependence between steps Δx and Δt: $\Delta x = c_0 \Delta t$.

Approximating the derivatives $\frac{\partial p'}{\partial t}$ and $\frac{\partial^2 p'}{\partial t^2}$ by

$$\frac{\partial p_n^m}{\partial t} \approx \frac{p_n^m - p_n^{m-1}}{\Delta t}, \quad \frac{\partial^2 p_n^m}{\partial t^2} \approx \frac{p_n^{m+1} - 2p_n^m + p_n^{m-1}}{\Delta t^2}$$

were $p_n^m = p'(x_n, t_m)$, Eq. (3) may be written in following way:

$$\delta p_n^m - a_0 \frac{p_n^{m+1} - 2p_n^m + p_n^{m-1}}{\Delta t^2} + a_1 \delta\left(\frac{p_n^m - p_n^{m-1}}{\Delta t}\right) = -a_2 A(p_n^{m+1}, p_n^m, p_n^{m-1}) \tag{4}$$

where: δ - difference operator of the Laplace operator,

A - difference operator of $\frac{\partial^2 p'^2}{\partial t^2}$.

Using difference equation (4) and knowing values of pressure for $t<t_m$ we can calculate pressure for $t=t_{m+1}$ for all distances from $x=0$ to $x=X_{max}$. Finally we obtain the pressure changes for fixed distances $x_n \in [0, X_{\max}]$ ($n=0,1,\dots,N_x$) in time interval $[0,T_{max}]$ for $t=t_m$ ($m=0,1,\dots,N_t$).

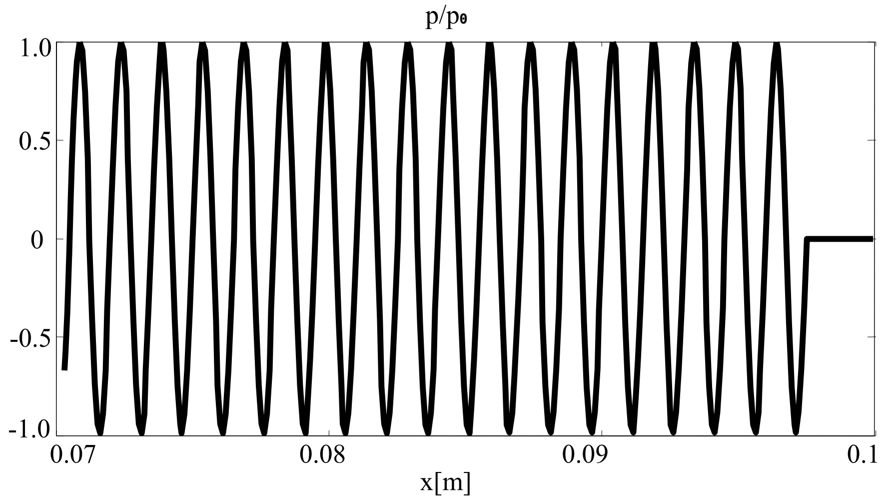
During finite amplitude wave propagation in water we observe its distortion. It means that the harmonic wave shape changes step by step during its propagation. The waveform changes are equivalent with spectrum changes. The harmonic analysis is very often used to investigate wave distortion. The fast Fourier transform is used to calculate spectrum.

3. NUMERICAL INVESTIGATIONS

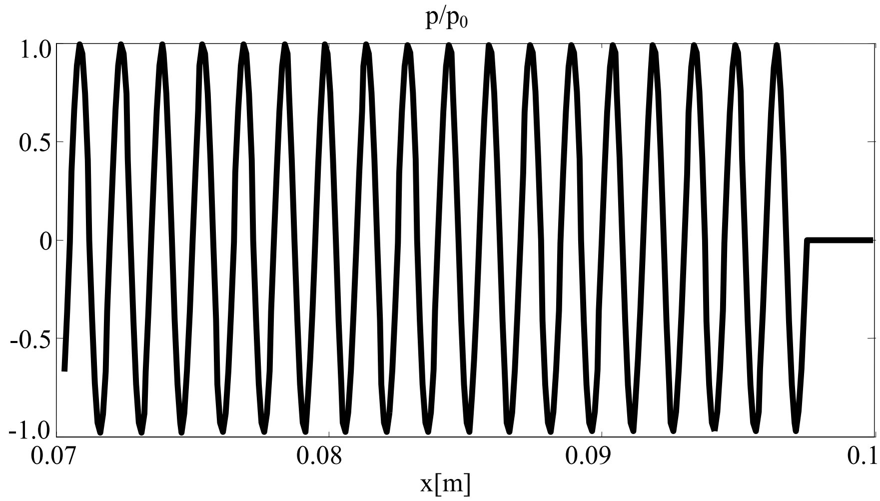
The pressure changes along x axis are the result of computer calculations. Figure 1 presents normalized pressure changes along x axis for time t=65 μs. The numerical calculations were carried out assuming that primary wave which frequency f=1 MHz and amplitude p_0=150 kPa propagates in water where speed c_0=1500 m/s, density ρ_0=1000 kg/m^3, nonlinear coefficient ε=3.5. Calculations were done for dissipation coefficient b=0, b=0.004 and b=0.04 respectively.

For fixed values of physical parameters (static pressure, density, speed of sound, nonlinear coefficient, dissipation coefficient) the correct choice of step sizes is very important during numerical calculations. There are two reasons of it. First of all the accuracy of numerical calculations depends on both step sizes time and space one. Moreover wrong choice of these step sizes can be the reason of wrong results of calculations at all. To analyze this problem, numerical calculations for different step sizes were done. This problem was considered for selected values of dissipation coefficient b, nonlinear coefficient ε and static pressure p_0 and the same values of other physical parameters like earlier.

a)



b)



c)

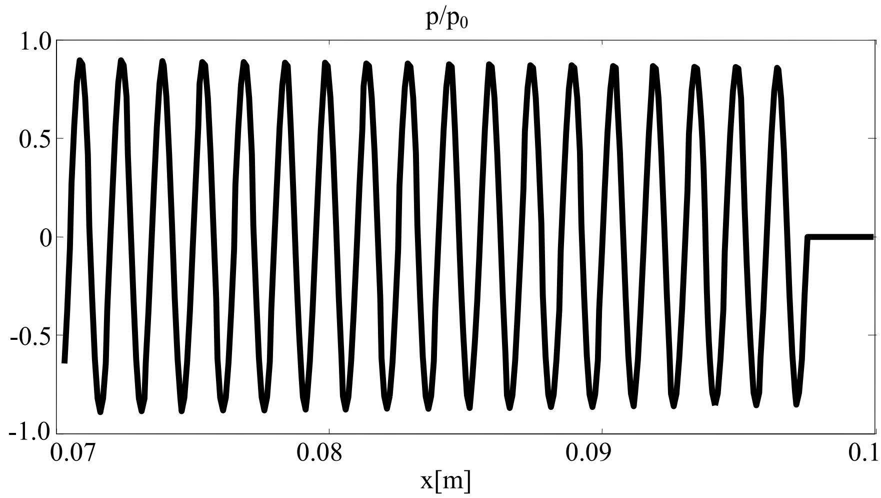


Fig.1 Normalized pressure changes along *x* axis: b=0 (a), b=0.004 (b), b=0.04 (c)

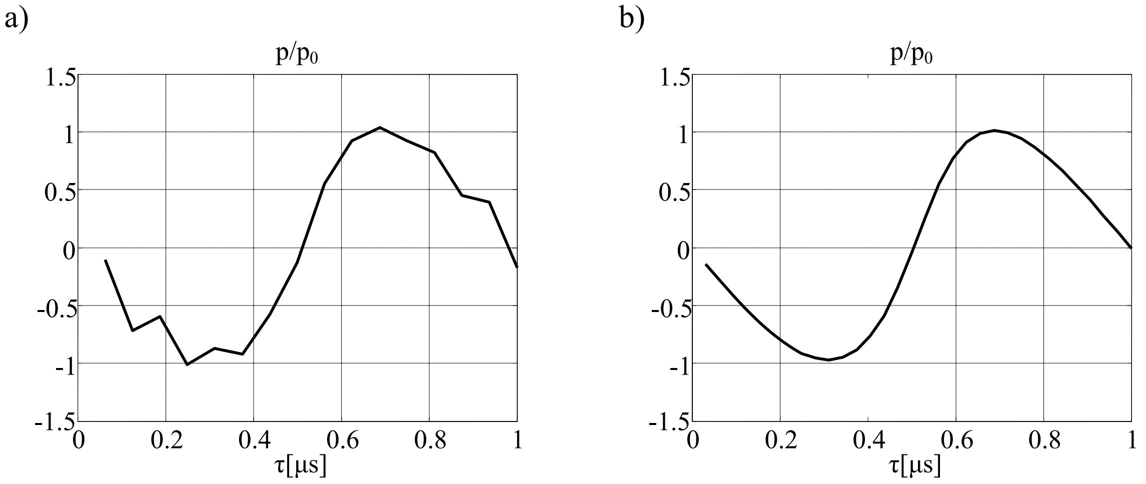


Fig.2 Normalized pressure as a function of time for b=0.004, ε=3.5: Δx=0.1 mm (a), Δx=0.05 mm (b)

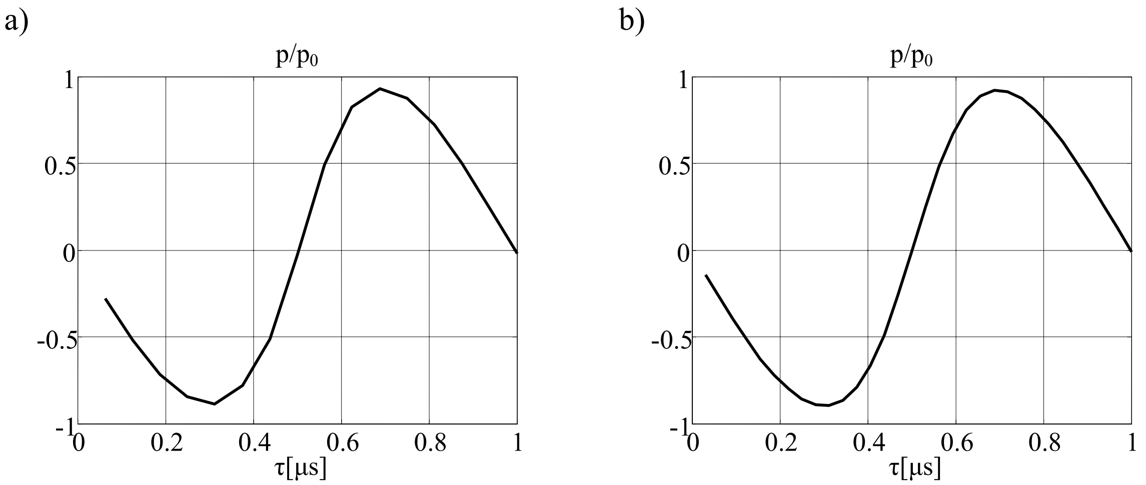


Fig.3 Normalized pressure as a function of time for b=0.04, ε=3.5: Δx=0.1 mm (a), Δx=0.05 mm (b)

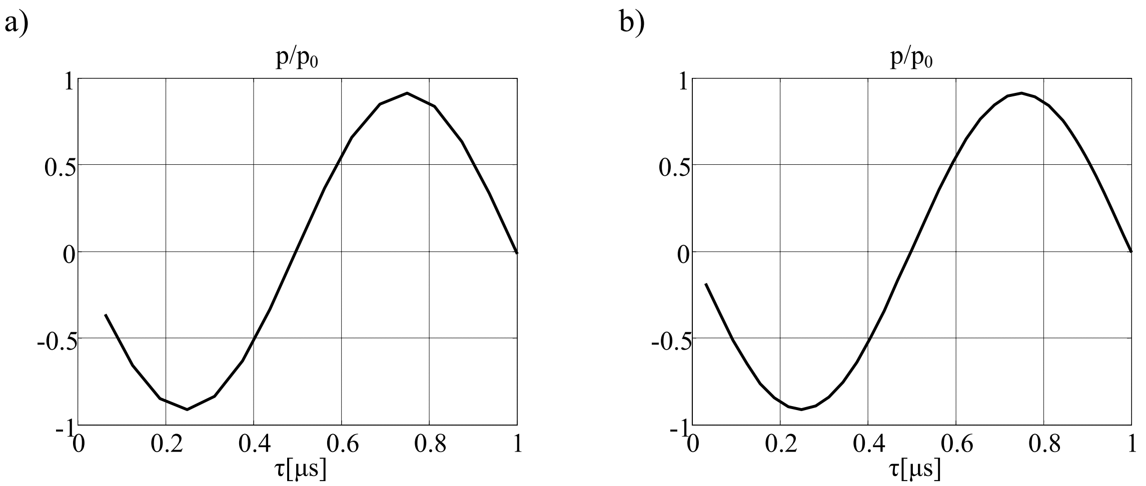


Fig.4 Normalized pressure as a function of time for b=0.04, ε=0: Δx=0.1 mm (a), Δx=0.05 mm (b)

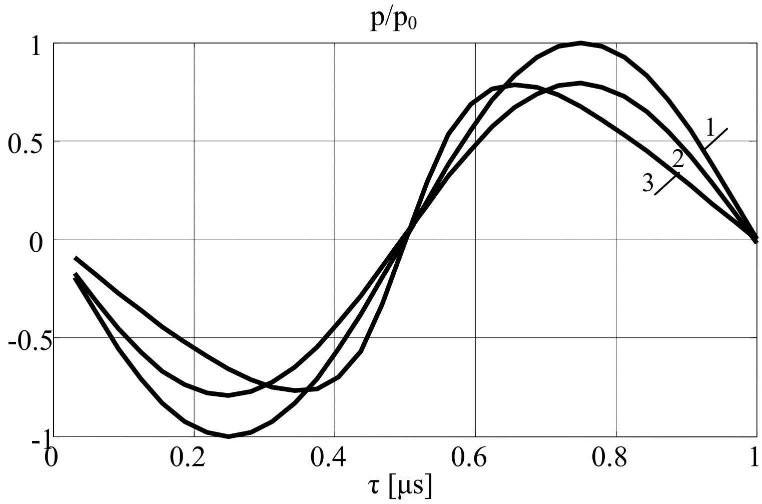


Fig.5 Normalized pressure as a function of time for different values of nonlinear coefficients ε:
1 - primary wave; 2 - ε=0; 3 - ε=3.5

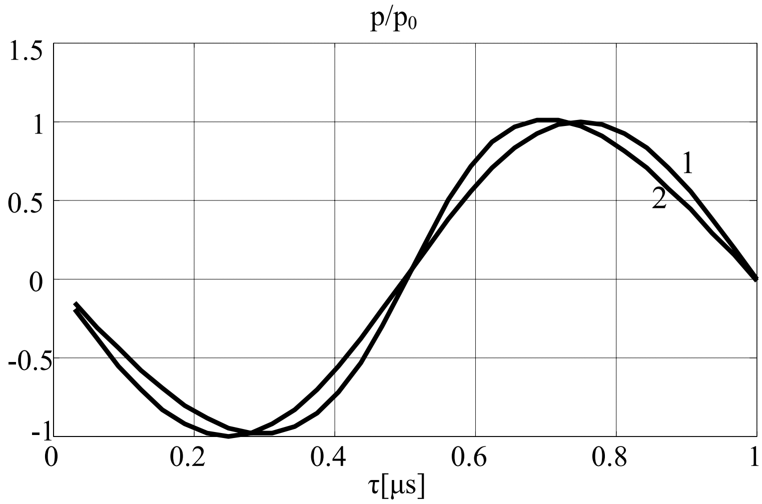


Fig.6 Normalized pressure as a function of time for p_0=150 kPa: 1 - x=0, 2 - x=0.3 m

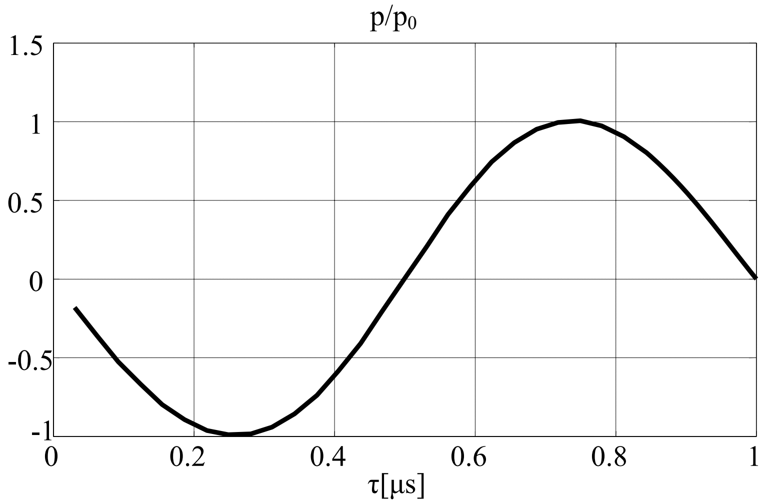


Fig.7 Normalized pressure as a function of time for p_0=10 kPa at x=1 m

Figure 2 presents normalized pressure as function of time at distance x=0.4 m obtained for fixed value of step Δx and twice smaller one ($\tau = x - t/c_0$). Calculations were done for b=0.004 and ε=3.5. Similar results we observe for dissipation coefficient b=0, i.e. when the dissipation effect is not covered at all. The results of numerical calculations obtained for b=0.04 and ε=3.5 are shown in Fig. 3. In this situation we obtain correct calculation results for both values of step Δx. Similar effects of convergence we obtain for b=0.04 and ε=0 which means that it was covered only dissipation effect without nonlinear one (see Fig. 4).

Exact analysis of the numerical results shows that if value of dissipation coefficient increases for fixed value of nonlinear coefficient then the maximum distance which is possible to analyze is longer. It is connected with the fact that proposed numerical method can be used only for continuous solution. In situation when values of dissipation coefficient is small then nonlinear effect dominates and the wave distortion is big. In situation when value of parameter b is big than we observe not only wave distortion but also decrease of its amplitude and then the nonlinear effect are not such big. To illustrate this effect pressure changes for fixed values of physical parameters are presented. Figure 5 shows normalized pressure as a function of time at distance x=1 m for b=0.04 and two different values of nonlinear coefficient. Curve number 1 shows primary wave, curve number 2 shows the result of numerical calculations for ε=0, curve number 3 presents similar result obtained for ε=3.5. Figures 6 and 7 present pressure changes calculated for b=0 and ε=3.5. First of them shows normalized pressure as a function of time calculated assuming that primary wave amplitude is equal p_0=150 kPa. Curve number 1 presents primary wave and curve number 2 the results obtained at distance x=0.3 m. Figure 7 presents pressure as function of time at distance x=1 m calculated when primary waves amplitude p_0=10 kPa.

The pressure distribution for different values of physical and numerical parameters has been presented till now. On the basis of these results it is possible to calculate spectrum changes. Figure 8 presents normalized pressure as a function of time and changes of first four harmonics. Curve number 1 in Fig. 8a presents the pressure changes at distance x=0.1 m and curve number 2 the results obtained for x=0.3 m. The calculations were done for dissipation coefficient b=0.004. Similar results obtained after calculations for b=0.04 shows Fig.9.

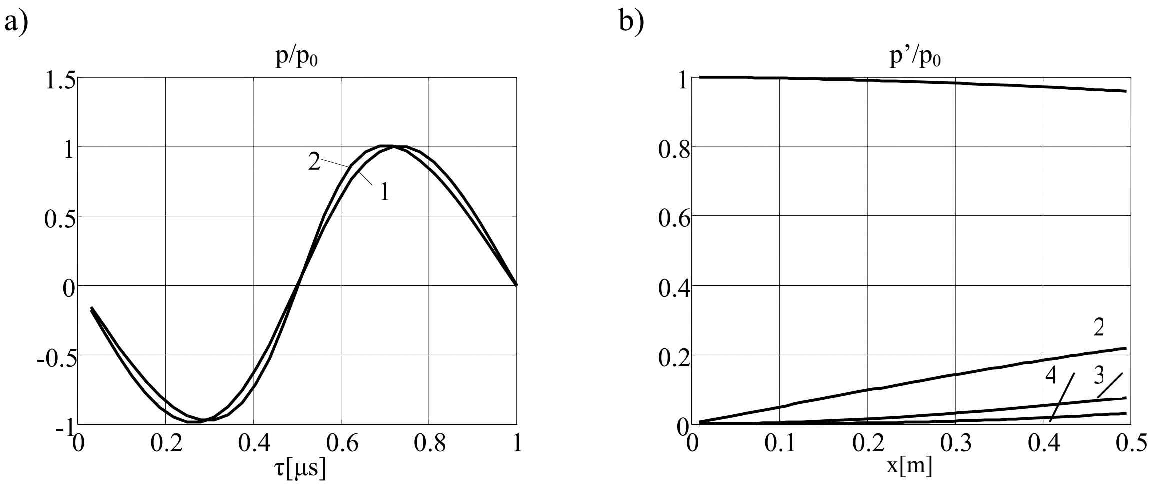


Fig.8 Normalized pressure as a function of time at fixed distances (a) and amplitude of four first harmonics along x axis (b) for b=0.004

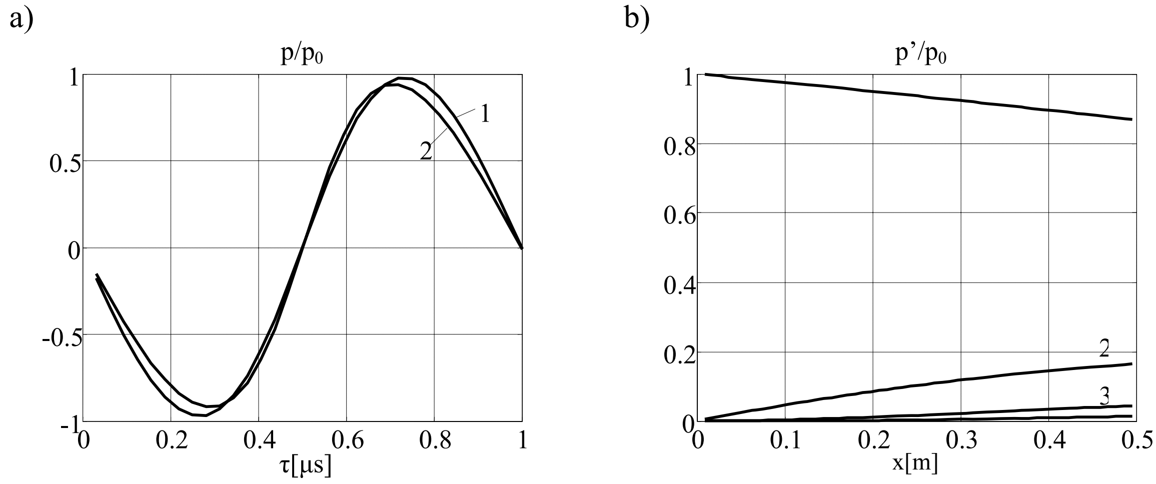


Fig.9 Normalized pressure as a function of time at fixed distances (a) and amplitude of four first harmonic along x axis (b) for b=0.04

4. CONCLUSIONS

The paper presents mathematical model and results of numerical investigations of finite amplitude plane wave propagation problem. The nonlinear acoustics equation was considered to work out mathematical model. The finite-difference method was used to solve the problem numerically. The numerical calculations were done using own computer program which was worked out on the basis on the proposed mathematical and numerical models.

The analysis of the results of numerical calculations shows that proposed numerical method for small distances from the source plane is convergent. However for higher distances from this plane not for all step sizes we obtain correct solution. Generally, for fixed step sizes this method can be used in situation when the wave distortion is enough small. In situation when the nonlinear effect dominates, it is necessary to use smaller values of marching step Δx. Moreover it is important to remember that proposed in this paper difference equation is possible to use only when the solution of the problem is continuous. To study this problem exactly the other approximations of the original differential equation should be analyzed.

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