LENGTH DISTRIBUTION OF FUZZY-END SEGMENTS

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Abstract: We have calculated and discussed the probability density distributions of lengths of fuzzy-end segments, *i.e.* segments the ends of which assume random positions. We performed our calculations for several simple cases in 1, 2 and 3 dimensions: one end fixed, the other assuming a random position, and both ends at random positions. The obtained statistical data may serve as reference data for calculations of stochastic-geometrical properties of complex systems, such as conformations of complicated bolted constructions with clearances (in structure mechanics) or energy transfer processes between molecules in diluted systems (in physics).

Keywords: fuzzy-end segments, stochastic geometry

1. Introduction

Knowing the Cartesian coordinates of two points, one can readily calculate the distance between them. The problem becomes non-trivial if the points' coordinates are known with an incertainity. Depending on the probability density of the points' coordinates one obtains different probability densities, $P(d)$, to find distance d between the random points. It is not difficult to write general expressions for $P(d)$, but analytical calculations turn out to be rather camplicated even for the simplest cases.

In the paper we calculate and discuss distributions of distances between the ends of a fuzzy-end segment, *i.e.* a segment the ends of which assume random positions around their ideal (unperturbed) locations. The probability densities of the ends' shifts have been assumed to be uniform within 1D, 2D and 3D spheres. In particular, we consider in turn the following random displacements of the ends of a segment, say AB, within:

– two segments situated along the segment itself, *viz.* along the x axis (1D case described in Section 2),
– two co-planar circles in the 2D xy plane centred at points A and B, respectively of radii r_{A} and r_{B} (2D case described in Section 3),
– two spheres in the 3D xyz space centred at points A and B, respectively of radii r_{A} and r_{B} (3D case discussed in Section 4).

Whenever possible, the distance distributions are calculated analytically, otherwise they are calculated numerically.

The 1D, 2D and 3D cases listed above are analysed in Sections 2-4. In Section 5 potential applications of the results are proposed and concluding remarks made.

2. The 1D case

Let us begin with a trivial 1D case and consider two situations: a segment with one end fixed and the second at a random position (Subsection 2.1), and a segment with both ends at random positions (Subsection 2.2).

2.1. Random position of one end

Let us assume that one end, A, of a segment remains fixed at $x=0$, while the other, $\mathrm{B_{rand}}$, assumes random positions with a probability density $\rho_{\mathrm{B}}(x)$ in the interval $[L-r,L+r]$, $r<L$ (L being the lenght of an "unperturbed" segment). Obviously, the probability density of finding distance $\mathrm{AB_{rand}}$ to be equal to d is:

$$P(d)=\rho_{\mathrm{B}}(d)\chi_{[L-r,L+r]}(d), \tag{1}$$

where χ_I is the indicator function of set I. Let us note that the $\rho^*_{\mathrm{B}}(x)=L\rho_{\mathrm{B}}(Lx)$ function is also the probability density in the "normalized" interval, $[1-r/L,1+r/L]$, and thus we can proceed using one parameter only, r/L, *viz.* the relative perturbation amplitude. The following scaling rules for central momenta can be easily obtained. Let us indicate with $\overline{d}$ the average distance with respect to density $P(d)$, and with $\overline{d}^*$ the average distance with respect to the corresponding density in the "normalized" interval. We then have:

$$\overline{d}=\int\limits_{L-r}^{L+r} x\rho_{\mathrm{B}}(x)dx=L\int\limits_{1-r/L}^{1+r/L} y\rho^*_{\mathrm{B}}(y)dy=L\overline{d}^*. \tag{2}$$

Similarly,

$$\sigma^2=\int\limits_{L-r}^{L+r}(x-\overline{d})^2\,\rho_{\mathrm{B}}(x)\,dx=L^2\int\limits_{1-r/L}^{1+r/L}(y-\overline{d}^*)^2\rho^*_{\mathrm{B}}(y)\,dy=L^2\sigma^{2*}, \tag{3}$$

and then, as $(x-\overline{d})/\sigma=(y-\overline{d}^*)/\sigma^*$ when $x=Ly$,

$$\beta=\int\limits_{L-r}^{L+r}\left(\frac{x-\overline{d}}{\sigma}\right)^3\rho_{\mathrm{B}}(x)\,dx=\int\limits_{1-r/L}^{1+r/L}\left(\frac{y-\overline{d}^*}{\sigma^*}\right)^3\rho^*_{\mathrm{B}}(y)\,dy=\beta^*, \tag{4}$$

$$K=\int\limits_{L-r}^{L+r}\left(\frac{x-\overline{d}}{\sigma}\right)^4\rho_{\mathrm{B}}(x)\,dx=\int\limits_{1-r/L}^{1+r/L}\left(\frac{y-\overline{d}^*}{\sigma^*}\right)^4\rho^*_{\mathrm{B}}(y)\,dy=K^*. \tag{5}$$

Let us consider a uniform probability density $\rho_{\mathrm{B}}(x)=1/2r$ of displacements of $\mathrm{B_{rand}}$ in interval $[L-r,L+r]$ as a simple example. We obtain

$$\overline{d}=L,\quad \sigma=r/\sqrt{3},\quad \beta=0,\quad K=9/5 \tag{6}$$

or, using the relative perturbation amplitude, *viz.* the r/L ratio,

$$\overline{d}^*=1,\quad \sigma^*=\frac{1}{\sqrt{3}}\frac{r}{L},\quad \beta^*=0,\quad K^*=9/5. \tag{7}$$

2.2. Random positions of both ends

Let us now consider a more complicated situation, where both ends of segment AB can assume random positions along the segment itself, $x_{\mathrm{A}}\in[-r_{\mathrm{A}},r_{\mathrm{A}}]$, $x_{\mathrm{B}}\in[L-r_{\mathrm{B}},L+r_{\mathrm{B}}]$, $r_{\mathrm{A}}+r_{\mathrm{B}}<L$, with respective probability densities $\rho_{\mathrm{A}}(x)$ and $\rho_{\mathrm{B}}(x)$ (assuming that $r_{\mathrm{A}}\le r_{\mathrm{B}}$).

Let us fix x_{A}. There is only one point in $[L-r_{\mathrm{B}},L+r_{\mathrm{B}}]$, $x_{\mathrm{B}}=x_{\mathrm{A}}+d$ distanced from x_{A} by d. The probability density of finding distance d between x_{A} and x_{B} is:

$$P(d)=\begin{cases} 0 & \text{for } d<L-r_{\mathrm{A}}-r_{\mathrm{B}},\\ \displaystyle\int\limits_{L-r_{\mathrm{B}}-d}^{r_{\mathrm{A}}}\rho_{\mathrm{A}}(x)\,\rho_{\mathrm{B}}(x+d)\,dx & \text{for } L-r_{\mathrm{A}}-r_{\mathrm{B}}\le d<L+r_{\mathrm{A}}-r_{\mathrm{B}},\\ \displaystyle\int\limits_{-r_{\mathrm{A}}}^{r_{\mathrm{A}}}\rho_{\mathrm{A}}(x)\,\rho_{\mathrm{B}}(x+d)\,dx & \text{for } L+r_{\mathrm{A}}-r_{\mathrm{B}}\le d<L+r_{\mathrm{B}}-r_{\mathrm{A}},\\ \displaystyle\int\limits_{-r_{\mathrm{A}}}^{L+r_{\mathrm{B}}-d}\rho_{\mathrm{A}}(x)\,\rho_{\mathrm{B}}(x+d)\,dx & \text{for } L+r_{\mathrm{B}}-r_{\mathrm{A}}\le d\le L+r_{\mathrm{A}}+r_{\mathrm{B}},\\ 0 & \text{for } d>L+r_{\mathrm{A}}+r_{\mathrm{B}}. \end{cases} \tag{8}$$

$\rho^*_{\mathrm{A}}(x)=L\rho_{\mathrm{A}}(Lx)$ is the probability density in the "normalized" interval $[-r_{\mathrm{A}}/L, r_{\mathrm{A}}/L]$ and $\rho^*_{\mathrm{B}}(x)=L\rho_{\mathrm{B}}(Lx)$ is the probability density in the "normalized" interval $[1-r_{\mathrm{B}}/L,1+r_{\mathrm{B}}/L]$. If we indicate with $f^*(d)$ the corresponding probability density of finding distance d between a point located in the normalized interval centred on the origin and a point in the normalized interval centred on 1, it is easy to demonstrate that $Lf(Ld)=f^*(d)$, and then, like in the previous paragraph, we have

$$\overline{d}=L\overline{d}^*,\quad \sigma^2=L^2\sigma^{2*},\quad \beta=\beta^*,\quad K=K^*. \tag{9}$$

In a particular case of uniform probability densities of random positions within the two intervals centred on 0 and 1, *viz.* $\rho_{\mathrm{A}}(x)=\frac{1}{2r_{\mathrm{A}}}\chi_{[-r_{\mathrm{A}},r_{\mathrm{A}}]}(x)$ and $\rho_{\mathrm{B}}(x)=\frac{1}{2r_{\mathrm{B}}}\chi_{[L-r_{\mathrm{B}},L+r_{\mathrm{B}}]}(x)$, one has:

$$P(d) = P(d) = \begin{cases} 0 & \text{for } d < L - r_{\mathrm{A}} - r_{\mathrm{B}}, \\ \dfrac{r_{\mathrm{A}} + r_{\mathrm{B}} + d - L}{4 r_{\mathrm{A}} r_{\mathrm{B}}} & \text{for } L - r_{\mathrm{A}} - r_{\mathrm{B}} \le d < L + r_{\mathrm{A}} - r_{\mathrm{B}}, \\ \dfrac{1}{2 r_{\mathrm{B}}} & \text{for } L + r_{\mathrm{A}} - r_{\mathrm{B}} \le d < L + r_{\mathrm{B}} - r_{\mathrm{A}}, \\ \dfrac{r_{\mathrm{A}} + r_{\mathrm{B}} + L - d}{4 r_{\mathrm{A}} r_{\mathrm{B}}} & \text{for } L + r_{\mathrm{B}} - r_{\mathrm{A}} \le d \le L + r_{\mathrm{A}} + r_{\mathrm{B}}, \\ 0 & \text{for } d > L + r_{\mathrm{A}} + r_{\mathrm{B}}, \end{cases} \tag{10}$$

and

$$\overline{d} = L, \quad \sigma^2 = \frac{r_{\mathrm{A}}^2 + r_{\mathrm{B}}^2}{3}, \quad \beta = 0, \quad K = \frac{9}{5} + \frac{12}{5}\left(\frac{r_{\mathrm{A}} r_{\mathrm{B}}}{r_{\mathrm{A}}^2 + r_{\mathrm{B}}^2}\right)^2. \tag{11}$$

The graph of (10) is a trapeze and then the probability density is not differentiable in d at points $L - r_{\mathrm{A}} - r_{\mathrm{B}}$, $L + r_{\mathrm{A}} - r_{\mathrm{B}}$, $L - r_{\mathrm{A}} + r_{\mathrm{B}}$, $L + r_{\mathrm{A}} + r_{\mathrm{B}}$. The derivability of the probability density at points $L - r_{\mathrm{A}} - r_{\mathrm{B}}$ and $L + r_{\mathrm{A}} + r_{\mathrm{B}}$ is generally obtained if ρ_{A} or $\rho_{\mathrm{B}}(x)$ is 0 at these points, and the derivability at points $L + r_{\mathrm{A}} - r_{\mathrm{B}}$ and $L - r_{\mathrm{A}} + r_{\mathrm{B}}$ if $\rho_{\mathrm{B}}(x)$ is derivable at any point of the $[L - r_{\mathrm{B}}, L + r_{\mathrm{B}}]$ interval.

For $r_{\mathrm{A}} = r_{\mathrm{B}} = r$, the middle formula of (8) disappears and the graph of (10) is triangular and symmetric with respect to $d = L$. Relations (11) are then simplified to:

$$\overline{d} = L, \quad \sigma = \sqrt{\frac{2}{3}} r, \quad \beta = 0, \quad K = \frac{12}{5} \tag{12}$$

or, using normalized parameter r/L, to:

$$\overline{d}^* = 1, \quad \sigma^* = \sqrt{\frac{2}{3}}\left(\frac{r}{L}\right), \quad \beta^* = 0, \quad K^* = \frac{12}{5}. \tag{13}$$

3. The 2D case

3.1. Random position of one end

Let us now consider a circle, C, of radius r and centre at $(L,0)$, and a point A at the origin $(0,0)$ on a 2D plane (see Figure 1). Let us indicate with $\rho(x,y)$ the density of probability of finding a point, $\mathrm{P}_{\mathrm{rand}}$, within circle C. The probability density of finding point $\mathrm{P}_{\mathrm{rand}}$ in C at distance d from point A is:

$$P(d|\mathrm{A}) = \chi_{[L-r, L+r]}(d) \cdot \int_{-y_S}^{y_S} \frac{d}{\sqrt{d^2 - y^2}} \rho\left(\sqrt{d^2 - y^2}, y\right) dy\,, \tag{14}$$

where y_S is the ordinate of point S, given by

$$y_S = \sqrt{d^2 - \frac{(L^2 - r^2 + d^2)^2}{4L^2}}, \tag{15}$$

and $-y_S$ is the ordinate of point s (see Figure 1).

Function $\rho^*(x,y) = L^2 \rho(Lx, Ly)$ is the probability density in the normalized circle C^* of radius r/L centred at $(1,0)$. If we indicate with $P^*(d|\mathrm{A})$ the corresponding probability density of finding a point $\mathrm{P}_{\mathrm{rand}}$ within C^* at distance d from A, then

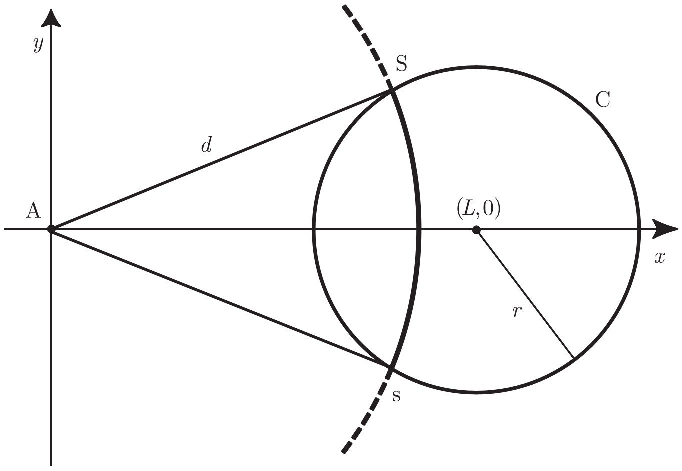

Figure 1. Point, A, at the origin of the axis and circle, C, of radius r, centred at point $(L,0)$. The solid line connecting points s and S represents the points of C at distance d from A

$P^*(d|\mathrm{A}) = LP(Ld|\mathrm{A})$ and we obtain (9) again. In the case of a uniform distribution on C, let

$$\xi_r(s,t) = 2s \arcsin\left(\frac{\sqrt{(s^2-(t-r)^2)((t+r)^2-s^2)}}{2st}\right) \tag{16}$$

be the length of the arc being the intersection between a circle of radius r and the circumference of a circle of radius s with a centre distant by t from the centre of C. Then, the distances' probability density is:

$$P(d) = P(d|\mathrm{A}) = \chi_{[L-r,L+r]}(d)\,\frac{2d}{\pi r^2}\arcsin\left(\frac{\sqrt{((L+r)^2-d^2)(d^2-(L-r)^2)}}{2Ld}\right). \tag{17}$$

Figure 2 shows the relative graphs for various values of r.

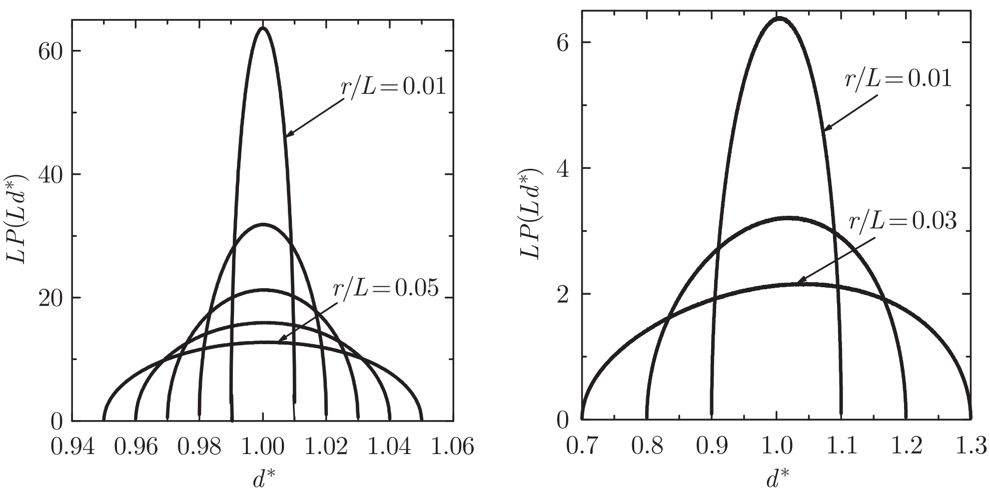

Figure 2. Probability densities $P(d)$ (Equation (17)) for several values of r/L

The central momenta, calculated numerically, obey the following empirical relations:

$$\overline{d}^* = 1 + 0.125\left(\frac{r}{L}\right)^2, \qquad \sigma^* = \frac{1}{2}\frac{r}{L}, \qquad \beta^* = -\frac{1}{4}\frac{r}{L}, \qquad K^* = 2 + 0.165\left(\frac{r}{L}\right)^2 \tag{18}$$

or (see relations (9)):

$$\overline{d} = L + \frac{0.125}{L}r^2, \qquad \sigma = \frac{1}{2}r, \qquad \beta = -\frac{1}{4}\frac{r}{L}, \qquad K = 2 + 0.165\left(\frac{r}{L}\right)^2. \tag{19}$$

3.2. Random positions of both ends

Let us consider two circles, C_A and C_B, of radii r_A and r_B ($r_A \le r_B$), respectively centred at points $A = (0,0)$ and $B = (L,0)$, as shown in Figures 3 and 4. Let $\rho_A(x,y)$ and $\rho_B(x,y)$ be the densities of probability of finding the segment's ends in C_A and C_B, respectively.

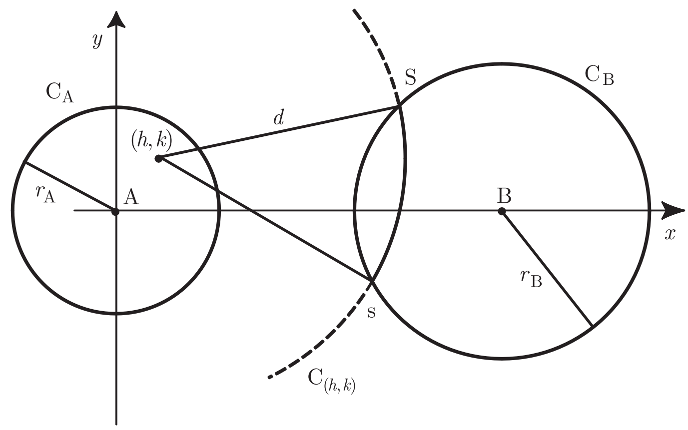

Figure 3. Two circles of different radii. The solid line connecting points s and S represents the points of C_B at distance d from the point of circle C_A of coordinates (h,k)

The density of probability of finding a P_{rand} in C_B, distanced from the point positioned at (h,k) in C_A is expressed as follows:

$$\begin{aligned} P(d|(h,k)) = {} & \chi_{\left[\sqrt{(L-h)^2+k^2}-r_B,\,\sqrt{(L-h)^2+k^2}+r_B\right]}(d)\cdot \\ & \int_{y_s}^{y_S} \rho_B\left(h+\sqrt{d^2-(y-k)^2},y\right)\frac{d}{\sqrt{d^2-(y-k)^2}}\,dy, \end{aligned} \tag{20}$$

where y_s and y_S are the ordinates of s and S, which represent intersections between the circumference of C_B and of the circle centered in (h,k) and radius d ($C_{(h,k)}$ in Figure 3). It is easy to obtain the expression for the density of probability to find two P_{rand}, one on C_A and the other on C_B, at distance d:

$$P(d) = \iint_{D_d} P(d|(h,k))\rho_A(h,k)\,dh\,dk, \tag{21}$$

where D_d:

– for $L-r_\mathrm{A}-r_\mathrm{B}\leq d<L+r_\mathrm{A}-r_\mathrm{B}$ is the intersection between $\mathrm{C_A}$ and the circle of radius d, centered at $(L-r_\mathrm{B},0)$,
– for $L+r_\mathrm{A}-r_\mathrm{B}\leq d<L-r_\mathrm{A}+r_\mathrm{B}$ coincides with $\mathrm{C_A}$,
– for $L-r_\mathrm{A}+r_\mathrm{B}\leq d\leq L+r_\mathrm{A}+r_\mathrm{B}$ is the intersection between $\mathrm{C_A}$ and the complementary of the circle of radius d, centered at $(L+r_\mathrm{B},0)$.

As in the previous section, $\rho^*_\mathrm{B}(x,y)=L^2\rho_\mathrm{B}(Lx,Ly)$ is the probability density in the normalized circle C^*_B of radius r_B/L centred at (1,0) and $\rho^*_\mathrm{A}(x,y)=L^2\rho_\mathrm{A}(Lx,Ly)$ is the probability density in the normalized circle C^*_A of radius r_A/L centred at (0,0). Then $P^*(d|(h,k)=LP(Ld|(Lh,Lk))$ and $P^*(d)=LP(Ld)$, so that relations (9) are valid again.

If we assume that ρ_A and ρ_B are uniform, we can calculate $P(d)$ in the following way. Let us first fix point D$=(h,0)$ on $\mathrm{C_A}$; all points of $\mathrm{C_B}$ remaining at distance d from D coincide with the intersection of circle $\mathrm{C_B}$ with the circumference of the circle of radius d centred in D ($\mathrm{C_D}$ in Figure 4).

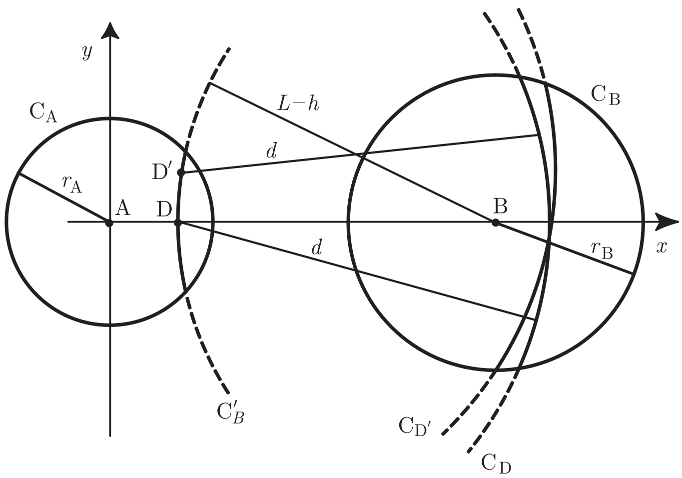

Figure 4. Two circles of different radii. The arc, intersection between circumference $\mathrm{C_D}$ of radius d and centre D and circle $\mathrm{C_B}$ has the same length as the arc corresponding to the circumference of radius d and centers D′, *viz.* $\mathrm{C_{D'}}$. The same is true for all points represented by the solid line of circumference $\mathrm{C'_B}$, of radius $L-h$ centred at B

Then (see Equation (16)):

$$P(d|(h,0))=\chi_{_H}(h,d)\frac{\xi_{r_\mathrm{B}}(d,L-h)}{\pi r_\mathrm{B}^2} \tag{22}$$

with

$$H\equiv\left\{(h,d):\begin{cases}L-r_\mathrm{B}-d\leq h\leq r_\mathrm{A} & L-r_\mathrm{A}-r_\mathrm{B}\leq d<L+r_\mathrm{A}-r_\mathrm{B}\\ -r_\mathrm{A}\leq h\leq r_\mathrm{A} & L+r_\mathrm{A}-r_\mathrm{B}\leq d<L+r_\mathrm{B}-r_\mathrm{A}\\ -r_\mathrm{A}\leq h\leq L+r_\mathrm{B}-d & L+r_\mathrm{B}-r_\mathrm{A}\leq d\leq L+r_\mathrm{A}+r_\mathrm{B}\\ \emptyset & \text{otherwise}\end{cases}\right\}. \tag{23}$$

For every point D′ of the arc intersection between C_A and the circumference of C'_B, there is a corresponding arc, of the same lengh as that corresponding to D, of points on C_B at distance d (see Figure 4). We then have:

$$P(d)=\frac{1}{\pi^2 r_A^2 r_B^2}\int_{-r_A}^{r_A}\chi_H(h,d)\xi_{r_B}(d,L-h)\xi_{r_A}(L-h,L)\,dh. \tag{24}$$

The integral in Equation (24) most problablly cannot be calculated analytically or approximated in any convenient way. The results of numerical integration (using the Mathematica program [1]) for several values of $r_A=r_B$ and $L=1$, are shown in Figure 5.

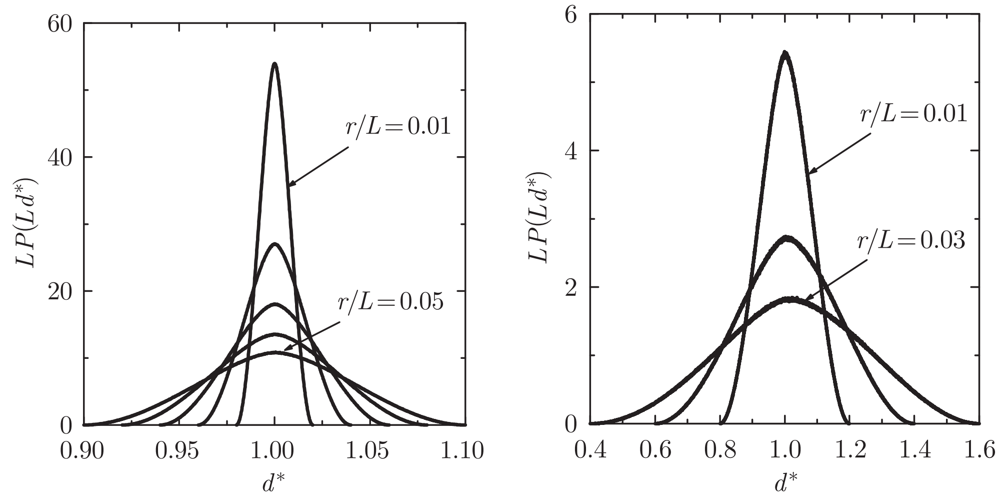

Figure 5. Probability densities (24) for several values of the r/L ratio

The following empirical relations concerning the central momenta are obeyed with very good approximation:

$$\overline{d}^*=1+\frac{1}{4}\left(\frac{r}{L}\right)^2,\quad \sigma^*=\frac{1}{\sqrt{2}}\frac{r}{L},\quad \beta^*=-0.16\frac{r}{L},\quad K^*=2.5+\frac{1}{5}\left(\frac{r}{L}\right)^2. \tag{25}$$

4. The 3D case

4.1. Random position of one end

Let S be a sphere of radius r centred at $(L,0,0)$ and let point A remain at the origin $(0,0,0)$ of a 3D space. Let us indicate with $\rho(x,y,s)$ the density of probability of finding a point, P_{rand}, within S.

The density of probability of finding in S a point P_{rand} at distance d from point A is

$$P(d|A)=\chi_{[L-r,L+r]}(d)\cdot\iint_C\frac{d}{\sqrt{d^2-s^2-y^2}}\rho\left(\sqrt{d^2-s^2-y^2},y,s\right)dy\,ds, \tag{26}$$

where C is the circle centred $(0,0)$ in the y,s coordinates and radius

$$r'=\sqrt{d^2-\frac{(d^2+L^2-r^2)^2}{4L^2}}. \tag{27}$$

Function $\rho^*(x,y,s) = L^3\rho(Lx,Ly,Ls)$ is the probability density in the normalized sphere S^* of radius r/L and centred at (1,0,0). If we indicate with $P^*(d|A)$ the corresponding probability density of finding a point, P_{rand}, within S^* at distance d from A, then $P^*(d|A) = LP(Ld|A)$ and relations (9) hold also in the present case.

The surface of the spherical bowl being the intersection between a sphere of radius r and the surface of a sphere of radius w, distanced by d from the centres, is:

$$\psi_r(w,d) = \frac{\pi w}{d}\,(r^2-(d-w)^2). \tag{28}$$

Thus, in the particular case of a uniform distribution on S, the probability density for the distances is:

$$P(d) = P(d|A) = \chi_{[L-r,L+r]}(d)\frac{3}{4\pi r^3}\psi_r(d,L) = \chi_{[L-r,L+r]}(d)\frac{3d}{4Lr^3}\left(r^2-(L-d)^2\right). \tag{29}$$

We have shown the relative graphs for various values of r/L in Figure 6.

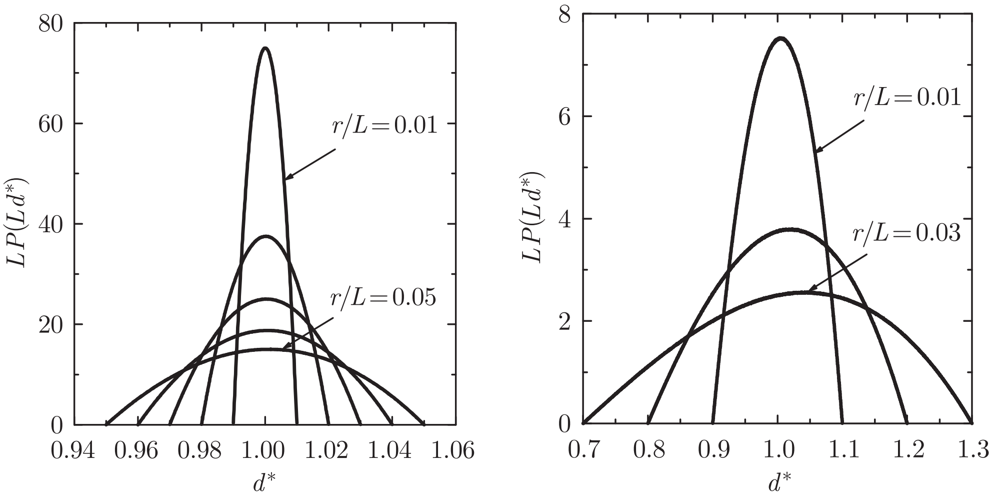

Figure 6. Probability densities according to Equation (29) for several values of the r/L ratio

The maximum of $P(d|A)$ distribution is achieved at

$$d_{max} = \frac{2+\sqrt{1+3\left(\frac{r}{L}\right)^2}}{3}. \tag{30}$$

The following exact formulae for central momenta can be obtained analytically:

$$\overline{d}^* = 1+\frac{1}{5}\left(\frac{r}{L}\right)^2, \tag{31}$$

$$\sigma^* = \sqrt{\frac{1}{5}\left(\frac{r}{L}\right)^2-\left(\frac{1}{5}\left(\frac{r}{L}\right)^2\right)^2}, \tag{32}$$

$$\beta^* = \frac{2r}{7L}\frac{\left(-15+7\left(\frac{r}{L}\right)^2\right)}{\left(5-\left(\frac{r}{L}\right)^2\right)^{3/2}}, \tag{33}$$

$$K^* = \frac{375 - 90\left(\frac{r}{L}\right)^2 - 21\left(\frac{r}{L}\right)^4}{7\left(-5 + \left(\frac{r}{L}\right)^2\right)^2}. \tag{34}$$

4.2. Random positions of both ends

We now consider two spheres, S_A and S_B, of radii r_A and r_B ($r_A \le r_B$) centred at points $A = (0,0,0)$ and $B = (L,0,0)$, and let $\rho_A(x,y,s)$ and $\rho_B(x,y,s)$ be the probabilities of finding a point P_{rand} in S_A and S_B, respectively.

The density of probability of finding a point P_{rand} in S_B at distance d from point (h,k,t) can be expressed as follows:

$$P(d|(h,k,t)) = \chi_{\left[\sqrt{(L-h)^2+k^2+t^2}-r_B,\sqrt{(L-h)^2+k^2+t^2}+r_B\right]}(d) \cdot \iint_W \rho_B(h+\sqrt{d^2-(y-k)^2-(s-t)^2},y,s)\frac{d}{\sqrt{d^2-(y-k)^2-(s-t)^2}}\,dy\,ds, \tag{35}$$

where W is the circle the circumference of which represents the extreme solutions of intersections between S_B and the sphere centred at (h,k,t) of radius d. Like in the 2D case, it is easy to obtain the expression for the density of probability of finding two P_{rand}, one in S_A and the other in S_B, at distance d:

$$P(d) = \iiint_{D_z} P(d|(h,k,s))\rho_A(h,k,s)\,dh\,dk\,ds, \tag{36}$$

where D_Z:

- for $L - r_A - r_B \le d < L + r_A - r_B$ is the intersection between S_A and the sphere centred at $(L - r_B, 0, 0)$ of radius d,
- for $L + r_A - r_B \le d < L - r_A + r_B$ coincides with S_A,
- for $L - r_A + r_B \le d \le L + r_A + r_B$ is the intersection between S_A and the complementary of the sphere centred at $(L + r_B, 0, 0)$ of radius d.

Like in the previous sections, $\rho_B^*(x,y,s) = L^3\rho_B(Lx,Ly,Ls)$ is the probability density in the normalized sphere S_B^* of radius r_B/L and centred at $(1,0,0)$, while $\rho_A^*(x,y,s) = L^3\rho_A(Lx,Ly,Ls)$ is the probability density in the normalized sphere S_A^* of radius r_A/L and centred at $(0,0,0)$. Then $P^*(d|(h,k,t) = LP(Ld|(Lh,Lk,Lt))$ and $P^*(d) = LP(Ld)$, so that the scaling relations (9) remain valid.

If we assume, like in Section 3.2, that ρ_A and ρ_B are uniform, we can repeat the same arguments where the arcs become spherical bowls. Then

$$P((d)|(h,0,0)) = \chi_H(h,d)\frac{3}{4(L-h)r_B^3}\,d\,(r_B^2 - ((L-h)-d)^2) \tag{37}$$

(see Equation (23)), and

$$P(d) = \frac{3}{4\pi r_A^3}\frac{3}{4\pi r_B^3}\int_{-r_A}^{r_A} \chi_H(h,d)\psi_{r_B}(d,L-h)\psi_{r_A}(L-h,L)\,dh. \tag{38}$$

We can calculate $P(d)$ analitically and obtain:

$$P(d)=\begin{cases}\dfrac{3d(s-t)^3\left(20r_\mathrm{A}r_\mathrm{B}-4s^2+3st+t^2\right)}{160Lr_\mathrm{A}^3r_\mathrm{B}^3} & \text{for } L-r_\mathrm{A}-r_\mathrm{B}\le d<L+r_\mathrm{A}-r_\mathrm{B},\\ \dfrac{-3d\left(-4r_\mathrm{B}^2-2r_\mathrm{B}s+s^2+5t^2\right)}{20Lr_\mathrm{B}^3} & \text{for } L+r_\mathrm{A}-r_\mathrm{B}\le d<L-r_\mathrm{A}+r_\mathrm{B},\\ \dfrac{3d(s+t)^3\left(20r_\mathrm{A}r_\mathrm{B}-4s^2-3st+t^2\right)}{160Lr_\mathrm{A}^3r_\mathrm{B}^3} & \text{for } L-r_\mathrm{A}+r_\mathrm{B}\le d<L+r_\mathrm{A}+r_\mathrm{B},\end{cases} \quad (39)$$

where $s=r_\mathrm{A}+r_\mathrm{B}$ and $t=L-d$. The probability distributions given by Equations (39) for several values of the r/L ratio are shown in Figure 7.

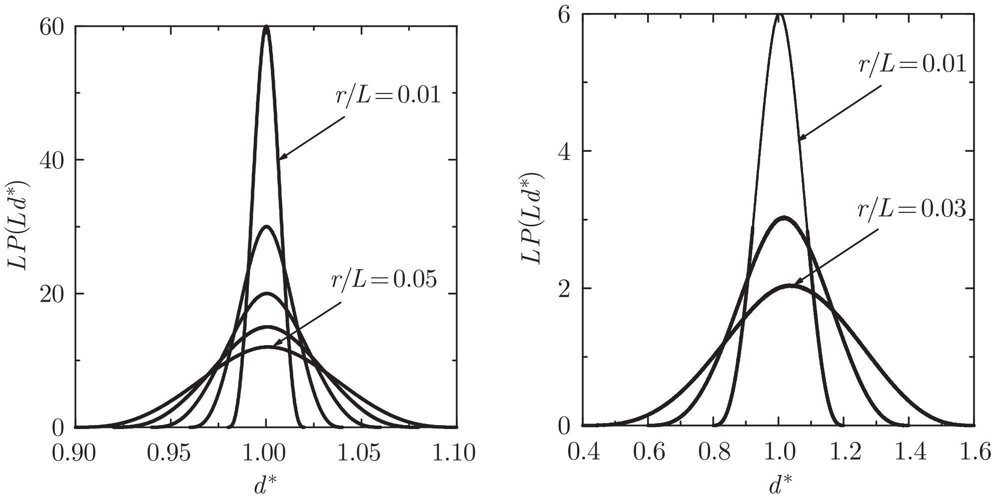

Figure 7. Probability densities, $P(d)$, calculated with formulae (39) for several values of the r/L ratio

The following exact results can be obtained analytically:

$$\overline{d}^*=1+\frac{1}{5}s, \quad (40)$$

$$\sigma^*=\frac{1}{5}\sqrt{(5-s)s}, \quad (41)$$

$$\beta^*=\frac{14s^3-30s^2+60p}{7(s(5-s))^{3/2}}, \quad (42)$$

$$K^*=\frac{3}{7}\frac{-7s^4-30s^3+125s^2-80sp+100p}{(s(5-s))^2}, \quad (43)$$

where $s=\left(\frac{r_\mathrm{A}}{L}\right)^2+\left(\frac{r_\mathrm{B}}{L}\right)^2$ and $p=\left(\frac{r_\mathrm{A}}{L}\right)^2\left(\frac{r_\mathrm{B}}{L}\right)^2$.

5. Possible applications and concluding remarks

In order to verify the correctness of the analytical and numerical results, all the distributions of distances considered above have been calculated using the Monte Carlo method. The Monte Carlo extracted distributions of distances obtained from 10^7 trials are smooth: the fluctuations amplitude does not exceed the line thickness in the plots of analytical results. The identity of our analytically calculated resultsand those obtained with the Monte Carlo method suggests that our calculations are correct.

Basic statistical parameters, as defined in elementary descriptive statistics, have been calculated for all the obtained distance distributions. In particular, the arithmetic mean, $\overline{d}$, standard deviation, σ, the Fischer asymmetry coefficient, β, and the Pearson coefficient of curtosis, K, have been considered. The analytical results for central momenta, simplified for $r_A = r_B$, are given in Table 1. The β parameter assumes null value for symmetric distributions. The higher the absolute value of β, the greater the degree of asymmetry. Positive and negative values of β respectively correspond to distributions with right and left asymmetry. The K parameter measures the deviation of a given distribution from the normal distribution. For exactly normal distributions one has $K = 3$. For $0 < K < 3$ the analyzed distribution is platicurtic, *i.e.* flatter than the normal distribution, while it is leptocurtic for $3 < K < \infty$, *i.e.* more peaked than the normal distribution.

Table 1. Summary of the obtained analytical results: $\rho = \frac{r}{L} = \frac{r_A}{L} = \frac{r_B}{L}$, f&r means one end fixed and one at random position, r&r means both ends at random positions

	1D		2D		3D	
	f&r	**r&r**	**f&r**	**r&r**	**f&r**	**r&r**
$\overline{d}^*$	1	1	$1+\frac{1}{8}\rho^2$	$1+\frac{1}{4}\rho^2$	$1+\frac{1}{5}\rho^2$	$1+\frac{2}{5}\rho^2$
σ^*	$\sqrt{\frac{1}{3}}\rho$	$\sqrt{\frac{2}{3}}\rho$	$\frac{1}{2}\rho$	$\sqrt{\frac{1}{2}}\rho$	$\frac{1}{5}\rho\sqrt{5-\rho^2}$	$\frac{\sqrt{2}}{5}\rho\sqrt{5-2\rho^2}$
β^*	0	0	$-\frac{1}{4}\rho$	$-\frac{4}{25}\rho$	$\frac{2}{7}\rho\frac{-15+7\rho^2}{(5-\rho^2)^{3/2}}$	$\frac{\sqrt{2}}{7}\rho\frac{-15+28\rho^2}{(5-2\rho^2)^{3/2}}$
K^*	$\frac{9}{5}$	$\frac{12}{5}$	$2+.165\rho^2$	$\frac{5}{2}+\frac{1}{5}\rho^2$	$\frac{3}{7}\frac{125-30\rho^2-7\rho^4}{(5-\rho^2)^2}$	$\frac{6}{7}\frac{75-50\rho^2-14\rho^4}{(5-2\rho^2)^2}$

Fuzzy-end segment length distributions calculated in the present paper may find applications in many various fields of applied and technical sciences, *e.g.* in analyzing times of constant-speed propagation of signals when emitters and/or receivers occupy random positions. End-to-end distributions are also of interest in macromolecular physics, see for instance [2–6]. However, perhaps the most straightforward application is in mechanics, in the analysis of slackened skeletal structures. Mathematical models of building structures make allowance for more and more imperfections occurring in real structures. The presence of gaps (clearances) in structural bolted connections is another structural imperfection recently taken under consideration. The problem is not entirely new, but few works are known to deal with the problem of slackened systems. An important contribution in this research area has been that of Andrzej Gawęcki [7–9], who published several works concerning elasto-plastic structures slackened with intentionally created gaps at joints. The presence of clearances at connections may induce geometric instability of the system. Under loading, relative motion occurs between the structural and connecting elements of the slackened structure. But after important geometry changes, the structure becomes kinematically stable. The description of changes between the "ideal" and "real" structures poses a real problem. They depend on many reasons: the number of connections in a structure, the type of bolted connections (shear or tension connection), dimensional tolerances of structural elements and the loading history. A general description of variation displacements is necessary to describe virtual motion of the connecting elements. The known solutions of the problem concern only lap joints (shear connection) with a possible displacement of bolts in drilled holes in the longitudinal direction. However, in reality elements may move in any direction. Therefore, a probabilistic attitude to the problem, as that discussed in the present work, may be very helpful.

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Appendix

Let us first list the needed integrals:

$$\int \frac{1}{\sqrt{r^2-t^2}}dt = \arcsin(t/r),$$

$$\int \frac{t^2}{\sqrt{r^2-t^2}}dt = \frac{1}{2}\left(-t\sqrt{r^2-t^2}+r^2\arcsin\left(\frac{t}{r}\right)\right),$$

$$\int \frac{t^4}{\sqrt{r^2-t^2}}dt = r^4\int \sin^4 y\,dy = \frac{r^4}{8}\left(-2\cos y(\sin^3 y+\frac{3}{2}\sin y)+3y\right) \quad (t = r\sin y),$$

and thus the corresponding definite integrals will be:

$$S_0 \equiv \int_{-r}^{+r} \frac{1}{\sqrt{r^2-t^2}}dt = \pi, \quad S_2 \equiv \int_{-r}^{+r} \frac{t^2}{\sqrt{r^2-t^2}}dt = \pi\frac{r^2}{2}, \quad S_4 \equiv \int_{-r}^{+r} \frac{t^4}{\sqrt{r^2-t^2}}dt = \frac{3}{8}\pi r^4.$$

Please note that

$$S_{2n-1} \equiv \int_{-r}^{+r} \frac{t^{2n-1}}{\sqrt{r^2-t^2}}dt = 0 \qquad \forall n \in \mathbf{N}.$$

The following integrals can be calculated using the $zt = 1-2z$ substitution:

$$\begin{aligned}\int \frac{1}{(t+2)\sqrt{r^2-t^2}}dt &= -\int \frac{1}{\sqrt{-1+4z+(-4+r^2)\,z^2}}dz\\ &= (4-r^2)^{-\frac{1}{2}}\arcsin\left(\frac{2+\left(-4+r^2\right)z}{r}\right),\end{aligned}$$

$$\begin{aligned}\int \frac{1}{(t+2)^2\sqrt{r^2-t^2}}dt &= -\int \frac{z}{\sqrt{-1+4z+(-4+r^2)\,z^2}}dz =\\ &= \left(4-r^2\right)^{-1}\sqrt{-1+4z+(-4+r^2)\,z^2}+\\ &\quad \left(4-r^2\right)^{-\frac{3}{2}} 2\arcsin\left(\frac{2+\left(-4+r^2\right)z}{r}\right),\end{aligned}$$

$$\begin{aligned}\int \frac{1}{(t+2)^3\sqrt{r^2-t^2}}dt &= -\int \frac{z^2}{\sqrt{-1+4z+(-4+r^2)\,z^2}}dz\\ &= (4-r^2)^{-2}\frac{1}{2}\left(6-\left(-4+r^2\right)z\right)\sqrt{-1+4z+(-4+r^2)\,z^2}+\\ &\quad (4-r^2)^{-\frac{5}{2}}\frac{1}{2}\left(8+r^2\right)\arcsin\left(\frac{2+\left(-4+r^2\right)z}{r}\right),\end{aligned}$$

so that the corresponding definite integrals are as follows:

$$I_1 \equiv \int_{-r}^{+r} \frac{1}{(t+2)\sqrt{r^2-t^2}} dt = -\int_{\frac{1}{2-r}}^{\frac{1}{2+r}} \frac{1}{\sqrt{-1+4z+(-4+r^2)\,z^2}} dz = \frac{\pi}{\sqrt{4-r^2}}$$
$$= \frac{\pi}{2} + r^2\frac{\pi}{16} + r^4\frac{3\pi}{256} + r^6\frac{5\pi}{2048} + O(r^8),$$

$$I_2 \equiv \int_{-r}^{+r} \frac{1}{(t+2)^2\sqrt{r^2-t^2}} dt = -\int_{\frac{1}{2-r}}^{\frac{1}{2+r}} \frac{z}{\sqrt{-1+4z+(-4+r^2)\,z^2}} dz = \frac{2\pi}{(4-r^2)^{\frac{3}{2}}}$$
$$= \frac{\pi}{4} + r^2\frac{3\pi}{32} + r^4\frac{15\pi}{512} + O(r^6),$$

$$I_3 \equiv \int_{-r}^{r} \frac{1}{(t+2)^3\sqrt{r^2-t^2}} dt = -\int_{\frac{1}{2-r}}^{\frac{1}{2+r}} \frac{z^2}{\sqrt{-1+4z+(-4+r^2)\,z^2}} dz = \frac{\pi\left(8+r^2\right)}{2(4-r^2)^{\frac{5}{2}}}$$
$$= \frac{\pi}{8} + r^2\frac{3\pi}{32} + r^4\frac{45\pi}{1024} + O(r^6).$$

Let us now consider the probability density,

$$f_r(z) = \begin{cases} \frac{2}{\pi r^2} z \arcsin\sqrt{1-\frac{(1-r^2+z^2)^2}{4z^2}} & 1-r \le z \le 1+r, \\ 0 & \text{elsewhere,} \end{cases} \tag{A1}$$

and the following momenta:

$$M_1(r) \equiv \mathbf{E}_r(\xi-1) = \int_{-\infty}^{\infty} (z-1) f_r(z) dz = \frac{2}{\pi r^2}\int_{1-r}^{1+r} (z-1) z \arcsin\sqrt{1-\frac{(1-r^2+z^2)^2}{4z^2}} dz,$$

$$M_2(r) \equiv \mathbf{E}_r(\xi-1)^2 = \int_{-\infty}^{\infty} (z-1)^2 f_r(z) dz = \frac{2}{\pi r^2}\int_{1-r}^{1+r} (z-1)^2 z \arcsin\sqrt{1-\frac{(1-r^2+z^2)^2}{4z^2}} dz,$$

$$M_3(r) \equiv \mathbf{E}_r(\xi-1)^3 = \int_{-\infty}^{\infty} (z-1)^3 f_r(z) dz = \frac{2}{\pi r^2}\int_{1-r}^{1+r} (z-1)^3 z \arcsin\sqrt{1-\frac{(1-r^2+z^2)^2}{4z^2}} dz,$$

$$M_4(r) \equiv \mathbf{E}_r(\xi-1)^4 = \int_{-\infty}^{\infty} (z-1)^4 f_r(z) dz = \frac{2}{\pi r^2}\int_{1-r}^{1+r} (z-1)^4 z \arcsin\sqrt{1-\frac{(1-r^2+z^2)^2}{4z^2}} dz.$$

All of the above momenta can be expressed using:

$$g_n(r) \equiv \int_{1-r}^{1+r} (z-1)^n z \arcsin\sqrt{1-\frac{(1-r^2+z^2)^2}{4z^2}} dz \quad n=1,2,3,4.$$

In order to calculate them asymptotically for small r we may consider that $g_n(0)=0$ and write

$$g_n(r)=\int_0^r g_n'(t)dt, g_n'(r)=2r\int_{1-r}^{1+r}\frac{z(z-1)^n}{\sqrt{(1+r-z)(1-r+z)(-1+r+z)(1+r+z)}}dz$$
$$=2r\int_{1-r}^{1+r}\frac{z(z-1)^n}{\sqrt{(r^2-(z-1)^2)((z+1)^2-r^2)}}dz=2r\int_{-r}^{r}\frac{(t+1)t^n}{\sqrt{(r^2-t^2)}}\frac{1}{\sqrt{(t+2)^2-r^2}}dt.$$

Before considering the asymptotic behaviour of $g_n'(r)$ we note that for $-r\le t\le r$,

$$\frac{1}{\sqrt{(t+2)^2-r^2}}=\frac{1}{t+2}+\frac{r^2}{2}\frac{1}{(t+2)^3}+O(r^4)\quad |O(r^4)|<r^4,$$

$$\left|\int_{-r}^{+r}\frac{(t+1)t^n}{\sqrt{r^2-t^2}}O(r^4)dt\right|\le r^4\int_{-r}^{+r}\left|\frac{(t+1)t^n}{\sqrt{r^2-t^2}}\right|dt\le r^{4+n}4\int_0^r\frac{1}{\sqrt{r^2-t^2}}dt=r^{4+n}4\frac{1}{2}S_0$$
$$=O(r^{4+n}).$$

For $g_1'(r)$ we have:

$$g_1'(r)=2r\left(\int_{-r}^{+r}\frac{(t+1)t}{\sqrt{r^2-t^2}(t+2)}dt+\frac{r^2}{2}\int_{-r}^{+r}\frac{(t+1)t}{\sqrt{r^2-t^2}(t+2)^3}dt+\int_{-r}^{+r}\frac{(t+1)t}{\sqrt{r^2-t^2}}O(r^4)dt\right).$$

Since $(t+1)t=(t+2)^2-3(t+2)+2$,

$$g_1'(r)=2r\int_{-r}^{+r}\frac{(t+2)^2-3(t+2)+2}{\sqrt{r^2-t^2}(t+2)}dt+r^3\int_{-r}^{+r}\frac{(t+2)^2-3(t+2)+2}{\sqrt{r^2-t^2}(t+2)^3}dt+O(r^6)$$
$$=2r\left(\int_{-r}^{+r}\frac{t}{\sqrt{r^2-t^2}}dt-\int_{-r}^{+r}\frac{1}{\sqrt{r^2-t^2}}dt+2\int_{-r}^{+r}\frac{1}{\sqrt{r^2-t^2}(t+2)}dt\right)+$$
$$r^3\left(\int_{-r}^{+r}\frac{1}{\sqrt{r^2-t^2}(t+2)}dt-3\int_{-r}^{+r}\frac{1}{\sqrt{r^2-t^2}(t+2)^2}dt+2\int_{-r}^{+r}\frac{1}{\sqrt{r^2-t^2}(t+2)^3}dt\right)+$$
$$O(r^6),$$

and finally

$$g_1'(r)=2r(S_1-S_0+2I_1)+r^3(I_1-3I_2+2I_3)+O(r^6)=$$
$$=2r\left(-\pi+\pi+r^2\frac{\pi}{8}+r^4\pi\frac{3}{128}+O(r^6)\right)+r^3\left(\frac{\pi}{2}+r^2\frac{\pi}{16}+O(r^4)-3\frac{\pi}{4}-r^2\frac{9\pi}{32}+\right.$$
$$\left.O(r^4)+\frac{\pi}{4}+r^2\frac{3\pi}{16}+O(r^4)\right)+O(r^6)$$
$$=r^3\pi\left(\frac{1}{4}+\frac{1}{2}-\frac{3}{4}+\frac{1}{4}\right)+r^5\pi\left(\frac{3}{64}+\frac{1}{16}-\frac{9}{32}+\frac{3}{16}\right)+O(r^6)$$
$$=r^3\frac{\pi}{4}+r^5\frac{\pi}{64}+O(r^6).$$

For $g_2'(r)$ we have:

$$g_2'(r) = 2r\left(\int_{-r}^{+r}\frac{(t+1)t^2}{\sqrt{r^2-t^2}(t+2)}dt + \frac{r^2}{2}\int_{-r}^{+r}\frac{(t+1)t^2}{\sqrt{r^2-t^2}(t+2)^3}dt + \int_{-r}^{+r}\frac{(t+1)t^2}{\sqrt{r^2-t^2}}O(r^4)dt\right).$$

Since $(t+1)t^2 = (t+2)^3 - 5(t+2)^2 + 8(t+2) - 4$

$$\begin{aligned}
g_2'(r) = 2r&\int_{-r}^{+r}\frac{(t+2)^3-5(t+2)^2+8(t+2)-4}{\sqrt{r^2-t^2}(t+2)}dt + \\
&r^3\int_{-r}^{+r}\frac{(t+2)^3-5(t+2)^2+8(t+2)-4}{\sqrt{r^2-t^2}(t+2)^3}dt + O(r^7) \\
&= 2r\left(\int_{-r}^{+r}\frac{t^2}{\sqrt{r^2-t^2}}dt\int_{-r}^{+r}\frac{t}{\sqrt{r^2-t^2}}dt + 2\int_{-r}^{+r}\frac{1}{\sqrt{r^2-t^2}}dt - 4\int_{-r}^{+r}\frac{1}{\sqrt{r^2-t^2}(t+2)}dt\right) + \\
&r^3\left(\int_{-r}^{+r}\frac{1}{\sqrt{r^2-t^2}}dt - 5\int_{-r}^{+r}\frac{1}{\sqrt{r^2-t^2}(t+2)}dt + 8\int_{-r}^{+r}\frac{1}{\sqrt{r^2-t^2}(t+2)^2}dt - \right. \\
&\left. 4\int_{-r}^{+r}\frac{1}{\sqrt{r^2-t^2}(t+2)^3}dt\right) + O(r^7),
\end{aligned}$$

and thus

$$\begin{aligned}
g_2'(r) &= 2r\,(S_2 - S_1 + 2S_0 - 4I_1) + r^3\,(S_0 - 5I_1 + 8I_2 - 4I_3) + O(r^7) \\
&= 2r\left(\pi\frac{r^2}{2} + 2\pi - 2\pi - r^2\frac{\pi}{4} - r^4\pi\frac{3}{64} + O(r^6)\right) + \\
&\quad r^3\left(\pi - \pi\frac{5}{2} - r^2\pi\frac{5}{16} + O(r^4) + 2\pi + r^2\pi\frac{3}{4} + O(r^4) - \frac{\pi}{2} - r^2\pi\frac{3}{8} + O(r^4)\right) + O(r^7) \\
&= r^3\pi\left(1 - \frac{1}{2} + 1 - \frac{5}{2} + 2 - \frac{1}{2}\right) + r^5\pi\left(-\frac{3}{32} - \frac{5}{16} + \frac{3}{4} - \frac{3}{8}\right) + O(r^7) \\
&\quad r^3\frac{\pi}{2} - r^5\frac{\pi}{32} + O(r^7)
\end{aligned}$$

Similarly, for $g_3'(r)$ we have:

$$g_3'(r) = 2r\left(\int_{-r}^{+r}\frac{(t+1)t^3}{\sqrt{r^2-t^2}(t+2)}dt + \frac{r^2}{2}\int_{-r}^{+r}\frac{(t+1)t^3}{\sqrt{r^2-t^2}(t+2)^3}dt + \int_{-r}^{+r}\frac{(t+1)t^3}{\sqrt{r^2-t^2}}O(r^4)dt\right).$$

Since $(t+1)t^3 = (t+2)^4 - 7(t+2)^3 + 18(t+2)^2 - 20(t+2) + 8$,

$$\begin{aligned}
g_3'(r) = 2r&\int_{-r}^{+r}\frac{(t+2)^4-7(t+2)^3+18(t+2)^2-20(t+2)+8}{\sqrt{r^2-t^2}(t+2)}dt + \\
&r^3\int_{-r}^{+r}\frac{(t+2)^4-7(t+2)^3+18(t+2)^2-20(t+2)+8}{\sqrt{r^2-t^2}(t+2)^3}dt + O(r^8)
\end{aligned}$$

$$= 2r\left(\int_{-r}^{+r}\frac{t^3}{\sqrt{r^2-t^2}}dt - \int_{-r}^{+r}\frac{t^2}{\sqrt{r^2-t^2}}dt + 2\int_{-r}^{+r}\frac{t}{\sqrt{r^2-t^2}}dt - 4\int_{-r}^{+r}\frac{1}{\sqrt{r^2-t^2}}dt + 8\int_{-r}^{+r}\frac{1}{\sqrt{r^2-t^2}(t+2)}dt\right) + r^3\left(\int_{-r}^{+r}\frac{t}{\sqrt{r^2-t^2}}dt - 5\int_{-r}^{+r}\frac{1}{\sqrt{r^2-t^2}}dt + 18\int_{-r}^{+r}\frac{1}{\sqrt{r^2-t^2}(t+2)}dt - 20\int_{-r}^{+r}\frac{1}{\sqrt{r^2-t^2}(t+2)^2}dt + 8\int_{-r}^{+r}\frac{1}{\sqrt{r^2-t^2}(t+2)^3}dt\right) + O(r^8)$$

and

$$\begin{aligned} g_3'(r) &= 2r\,(S_3 - S_2 + 2S_1 - 4S_0 + 8I_1) + r^3\,(S_1 - 5S_0 + 18I_1 - 20I_2 + 8I_3) + O(r^8) \\ &= 2r\left(-r^2\frac{\pi}{2} - 4\pi + 4\pi + r^2\frac{\pi}{2} + r^4\frac{3\pi}{32} + O(r^6))\right) + \\ &\quad r^3\left(-5\pi + 9\pi + r^2\pi\frac{9}{8} + O(r^4) - 5\pi - r^2\pi\frac{15}{8} + O(r^4) + \pi + r^2\pi\frac{3}{4} + O(r^4)\right) + O(r^8) \\ &= r^3\pi\,(-1+1-5+9-5+1) + r^5\pi\left(\frac{3}{16} + \frac{9}{8} - \frac{15}{8} + \frac{3}{4}\right) + O(r^8) = r^5\pi\frac{3}{16} + O(r^7). \end{aligned}$$

Finally, for $g_4'(r)$ we have:

$$g_4'(r) = 2r\left(\int_{-r}^{+r}\frac{(t+1)t^4}{\sqrt{r^2-t^2}(t+2)}dt + \frac{r^2}{2}\int_{-r}^{+r}\frac{(t+1)t^4}{\sqrt{r^2-t^2}(t+2)^3}dt + \int_{-r}^{+r}\frac{(t+1)t^4}{\sqrt{r^2-t^2}}O(r^4)dt\right).$$

Here, since $(t+1)t^4 = (t+2)^5 - 9(t+2)^4 + 32(t+2)^3 - 56(t+2)^2 + 48(t+2) - 16$,

$$\begin{aligned} g_4'(r) &= 2r\int_{-r}^{+r}\frac{(t+2)^5 - 9(t+2)^4 + 32(t+2)^3 - 56(t+2)^2 + 48(t+2) - 16}{\sqrt{r^2-t^2}(t+2)}dt + \\ &\quad r^3\int_{-r}^{+r}\frac{(t+2)^5 - 9(t+2)^4 + 32(t+2)^3 - 56(t+2)^2 + 48(t+2) - 16}{\sqrt{r^2-t^2}(t+2)^3}dt + O(r^9) \\ &= 2r\left(\int_{-r}^{+r}\frac{t^4}{\sqrt{r^2-t^2}}dt - \int_{-r}^{+r}\frac{t^3}{\sqrt{r^2-t^2}}dt + 2\int_{-r}^{+r}\frac{t^2}{\sqrt{r^2-t^2}}dt - 4\int_{-r}^{+r}\frac{t}{\sqrt{r^2-t^2}}dt + \right. \\ &\quad \left. 8\int_{-r}^{+r}\frac{1}{\sqrt{r^2-t^2}}dt - 16\int_{-r}^{+r}\frac{1}{\sqrt{r^2-t^2}(t+2)}dt\right) + \\ &\quad r^3\left(\int_{-r}^{+r}\frac{t^2}{\sqrt{r^2-t^2}}dt - 5\int_{-r}^{+r}\frac{t}{\sqrt{r^2-t^2}}dt + 18\int_{-r}^{+r}\frac{1}{\sqrt{r^2-t^2}}dt + \right. \end{aligned}$$

$$56\int_{-r}^{+r}\frac{1}{\sqrt{r^2-t^2}(t+2)}dt+48\int_{-r}^{+r}\frac{1}{\sqrt{r^2-t^2}(t+2)^2}dt-16\int_{-r}^{+r}\frac{1}{\sqrt{r^2-t^2}(t+2)^3}dt\Bigg)+ O(r^9)$$

and

$$\begin{aligned}
g_4'(r) &= 2r\,(S_4-S_3+2S_2-4S_1+8S_0-16I_1)+r^3\,(S_2-5S_1+18S_0-\\
&\quad 56I_1+48I_2-16I_3)+O(r^9)\\
&= 2r\left(r^4\pi\frac{3}{8}+r^2\pi+8\pi-8\pi-r^2\pi-r^4\pi\frac{3}{16}-r^6\pi\frac{5}{128}+O(r^8)\right)+\\
&\quad r^3\Bigg(r^2\frac{\pi}{2}+18\pi-28\pi-r^2\pi\frac{7}{2}-r^4\pi\frac{21}{32}+O(r^6)+\\
&\quad 12\pi+r^2\pi\frac{9}{2}+r^4\pi\frac{45}{32}+O(r^6)-2\pi-r^2\pi\frac{3}{2}-r^4\pi\frac{45}{64}+O(r^6)\Bigg)+O(r^9)\\
&= r^3\pi(2-2+18-28+12-2)+r^5\pi\left(\frac{3}{4}-\frac{3}{8}+\frac{1}{2}-\frac{7}{2}+\frac{9}{2}-\frac{3}{2}\right)+\\
&\quad r^7\pi\left(-\frac{5}{64}-\frac{21}{32}+\frac{45}{32}-\frac{45}{64}\right)+O(r^9)\\
&= r^5\pi\frac{3}{8}-r^7\pi\frac{1}{32}+O(r^9).
\end{aligned}$$

Taking into account the above results, we have:

$$g_1(r)=\int_0^r g_1'(t)\,dt=r^4\frac{\pi}{16}+r^6\frac{\pi}{384}+O(r^7),$$

$$g_2(r)=\int_0^r g_2'(t)\,dt=r^4\frac{\pi}{8}-r^6\frac{\pi}{192}+O(r^8),$$

$$g_3(r)=\int_0^r g_3'(t)\,dt=r^6\frac{\pi}{32}+O(r^8),$$

$$g_4(r)=\int_0^r g_4'(t)\,dt=r^6\frac{\pi}{16}-r^8\pi\frac{1}{256}+O(r^{10}),$$

and

$$\begin{aligned}
M_1(r) &= \frac{2}{\pi r^2}g_1(r)=r^2\frac{1}{8}+r^4\frac{1}{192}+O(r^5),\\
M_2(r) &= \frac{2}{\pi r^2}g_2(r)=r^2\frac{1}{4}-r^4\frac{1}{96}+O(r^6),\\
M_3(r) &= \frac{2}{\pi r^2}g_3(r)=r^4\frac{1}{16}+O(r^6),\\
M_4(r) &= \frac{2}{\pi r^2}g_4(r)=r^4\frac{1}{8}-r^6\frac{1}{128}+O(r^8).
\end{aligned}$$

The mean value of probability density distribution is thus:

$$\mu(r)\equiv\mathbf{E}_r(\xi)=M_1(r)+1=1+r^2\frac{1}{8}+O(r^4),$$

the variance is:

$$\sigma^2(r) \equiv \mathbf{E}_r(\xi - 1 - M_1(r))^2 = M_2(r) - M_1^2(r) = r^2\frac{1}{4} - r^4\frac{5}{192} + O(r^6),$$

so that

$$\sigma = r\frac{1}{2} + O(r^3),$$

the skewness parameter is:

$$\begin{aligned}
\beta(r) &\equiv \mathbf{E}_r\left(\frac{\xi - 1 - M_1(r)}{\sigma}\right)^3 \\
&= \frac{1}{\sigma^3}\big(\mathbf{E}_r(\xi-1)^3 - 3M_1(r)\mathbf{E}_r(\xi-1)^2 + 3M_1^2(r)\mathbf{E}_r(\xi-1) - M_1^3(r)\big) \\
&= \frac{1}{\sigma^3}\big(M_3(r) - 3M_1(r)M_2(r) + 2M_1^3(r)\big) \\
&= \frac{r^4\frac{1}{16} + O(r^6) - r^4\frac{3}{32} + O(r^6) + r^6\frac{1}{256} + O(r^8)}{r^3\frac{1}{8} + O(r^5)} \\
&= \frac{-r^4\frac{1}{32} + O(r^6)}{r^3\frac{1}{8} + O(r^5)} = -r\frac{1}{4} + O(r^3),
\end{aligned}$$

and the Pearson coefficient is:

$$\begin{aligned}
K(r) &\equiv \mathbf{E}_r\left(\frac{\xi - 1 - M_1(r)}{\sigma}\right)^4 \\
&= \frac{1}{\sigma^4}\big(\mathbf{E}_r(\xi-1)^4 - 4\mathbf{E}_r(\xi-1)^3M_1(r) + 6M_1^2(r)\mathbf{E}_r(\xi-1)^2 - \\
&\quad 4M_1^3(r)\mathbf{E}_r(\xi-1) + M_1^4(r)\big) \\
&= \frac{1}{\sigma^4}\big(M_4(r) - 4M_3(r)M_1(r) + 6M_1^2(r)M_2(r) - 3M_1^4(r)\big) \\
&= \frac{r^4\frac{1}{8} - r^6\frac{1}{128} + O(r^8) - r^6\frac{1}{32} + O(r^8) + r^6\frac{3}{128} + O(r^8) - r^8\frac{3}{64^2} + O(r^{10})}{r^4\frac{1}{16} - r^6\frac{5}{384} + O(r^8)} \\
&= \frac{r^4\frac{1}{8} - r^6\frac{1}{64} + O(r^8)}{r^4\frac{1}{16} - r^6\frac{5}{384} + O(r^8)} = \frac{2 - r^2\frac{1}{4} + O(r^4)}{1 - r^2\frac{5}{24} + O(r^4)} \\
&= \frac{2\big(1 - r^2\frac{5}{24} + O(r^4)\big) + r^2\frac{5}{12} - r^2\frac{1}{4} + O(r^4)}{1 - r^2\frac{5}{24} + O(r^4)} = 2 + r^2\frac{1}{6} + O(r^4).
\end{aligned}$$

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