

SURVEY OF MODERN TRENDS IN ANALYSIS OF CONTINUUM DAMAGE MECHANICS

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Abstract: A brief review of the damage mechanics literature is given. As this area of scientific research is very modern, the authors have restricted themselves to about 100 most important books and papers. Basic equations to introduce the isotropic model in the framework of thermodynamics are given in a form easily applicable in numerical simulations.

Keywords: continuum damage mechanics, ductile damage, fatigue damage, viscoplasticity

1. Introduction

Damage mechanics is one of the most important and interesting branches of solid mechanics: although it is still developing, it has already been applied to many engineering problems. The pioneer of the damage parameter proposal was Murzewski, who put forward a probabilistic interpretation of the decohesion parameter (see Murzewski [1] and [2]). Following in his footsteps, Kachanov ([3, 4]) and then Rabotnov ([5, 6]) formulated the famous equation of damage growth under creep conditions for the uniaxial state of stress. The theory was formulated in the 1950's and one of the first papers in the Western scientific literature was published by Odqvist and Hult [7] in the early sixties. Since that time, a number of theoretical and experimental investigations have been performed; see *e.g.* Chrzanowski [8], Chaboche [9], Krajcinovic [10] and Lemaitre [11].

The present paper consists in two main parts. In the first part, a literature review is presented. The literature concerning continuum damage mechanics is very extensive. The authors have paid particular attention to selected publications concerning the widely defined area of continuum damage mechanics. In the second part, the most popular damage evolution equations are collected, based on the damage strain energy function introduced by Lemaitre [12].

2. Literature survey

2.1. Theoretical papers

In this section papers are collected presenting the theory and a general description of the damage phenomenon and the current developments in the damage theory.

A review of the main features of continuum damage mechanics, its present possibilities and possible future developments was given by Chaboche [13]. The author considered damage definitions and measures, damage growth equations and anisotropy effects. Particular attention was given to the possible links between different approaches to damage, especially between the damage continuum mechanics definition and information obtained from the material science.

A presentation of notes concerning the two-scalar damage effect tensors, a formulation of conditions of thermodynamic admissibility of such tensors and verification of damage effect tensors known from the literature from the point of view of the derived conditions was given by Ganczarski and Barwacz [14].

Borst and Abellan [15] derived a framework for proper and consistent description of discontinuity as a result of the damaging process in a continuous medium. The process was described using a gradient-enhanced damage theory.

Mika [16] modelled deterioration of material properties such as rupture toughness, strength, and rigidity, as well as lifetime reductions by a symmetric second-order damage tensor introduced into the constitutive description, employing the theory of tensor function representations. The growth of damage and state failure were described by the one-parameter model of damage evolution and a three-parameter failure criterion.

Rosa *et al.* [17] analysed the response of some continuum damage mechanics models which incorporated a correlation between the triaxiality factor and logarithmic plastic strain. These correlation laws, also affecting the true stress-true strain material curve, were derived from experimental data processed using the Bridgman method.

Micilio *et al.* showed in [18], with the aid of a numerical simulation, that any plastic strains induced important geometrical effects in the evaluation of the elastic modulus of metals, which had a significant influence on the evaluation of the scalar damage parameter.

Souchet [19] offered consistent theories describing damage processes within the framework of effective stress and internal parameters. Damage was related to irreversible changes of small vicinities surrounding material points in the body.

Ibijola [20] presented the concept of plastic relaxation, an alternative formulation of the damage strain energy function, the force associated with the damage variable and a model for ductile plastic evolution.

Steinmann and Carol [21] derived a framework for geometrically nonlinear continuum damage mechanics, which allowed formulation of damage by second-order tensors capable of describing anisotropic damage states. Numerical examples demonstrated the applicability of the proposed framework.

The mechanical behaviour of two packaging paper materials subjected to tensile loading until complete breakage was investigated by Isaksson *et al.* [22]. A model for isotropic strain hardening coupled with anisotropic damage was discussed.

Constitutive relations, including a gradient-enhanced damage model, were developed within a thermo-dynamical framework.

Alfredsson and Stigh [23] derived a framework for developing constitutive laws for engineering materials. Their framework was based on physically motivated assumptions concerning the mechanical and thermal behaviour of plastically deformed and damaged materials.

2.2. Books and monographs

Formulation of damage models for a variety of materials and their identification procedures were discussed by Allix and Hild [24]. This book contains a series of lectures given by O. Allix, D. Krajcinovic, A. C. F. Cocks, F. Hild, D. Baptiste, A. Dragon, P. Ladevèze, J. Lemaitre, J. Mazars, O. Allix, R. de Borst, J. H. A. Schipperen and Z. P. Bažant.

The state-of-art in damage development in plastic body has been given in [25], which includes over 140 papers from AEPA '96, or the 3rd Asia-Pacific Symposium on Advances in Engineering Plasticity and its Applications, including case studies showing applications of plasticity in interdisciplinary or unconventional areas. Among others, modelling of elastic-plastic materials based on damage mechanics and studies on low-cycle fatigue strength and fatigue crack propagation rate for structural materials.

Basics and practical aspects of the mechanical behaviour and damage of materials, as well as significant links between various complementary approaches, were given by Miannay *et al.* [26]. The book contains a selection of papers presented at the EUROMAT 2000 Conference.

A comprehensive study of damage mechanics was presented by Krajcinovic [27]. In this book, all concepts are carefully identified and defined in the micro- and macroscopic scales.

An interesting study of damage mechanics was given by Voyiadjis and Kattan [28], including a detailed discussion of isotropic damage mechanics in the scalar formulation and anisotropic damage mechanics in the tensor formulation.

The basic principles of damage mechanics of composite materials, damage characterization by internal variables, and damage evolution in laminates were described by Talreja [29].

A large number of papers devoted to impact, blast and shock loading, indicating a strong research interest in high loading rates were included in [30], which contains papers of the Seventh International Symposium on Structural Failure and Plasticity.

A survey of one-dimensional and three-dimensional damage models for elastic and inelastic solids were presented by Skrzypek and Ganczarski [31], who offered examples of practical applications, numerical procedures, and computer codes.

Kattan and Voyiadjis [32] presented research on damage mechanics with applications to FEM. These authors paid particular attention to programming FEM to incorporate applications of damage mechanics and to research on the separation of voids and cracks in continuum damage mechanics.

An interesting introduction to anisotropic behaviour of damaged materials was given by Skrzypek *et al.* [33]. The book deals with the anisotropic damage mechanics involving micromechanical aspects and thermo-mechanical coupling.

The background of continuum damage mechanics, numerical analysis of damage, ductile failures, low cycle fatigue, creep, creep-fatigue and dynamic failures, high cycle fatigue, failure of brittle and quasi-brittle materials were presented by Lemaitre and Desmorat [34].

Last but not least, the Trends in Mechanics of Materials series should be mentioned, published by the Institute of Fundamental Technological Research of the Polish Academy of Sciences (see *e.g.* [35] and [36]), covering various aspects of damage and fracture in brittle and ductile materials caused by quasi-static and dynamic loadings.

2.3. Different types of the damage phenomenon

2.3.1. Fatigue damage

A fatigue damage accumulation model related to ductility exhaustion was developed by Chen and Plumtree [37] using continuum damage mechanics. The authors applied this model to study fatigue damage evolution in a steel pressure vessel. Two-level cyclic tests were presented to verify the damage accumulation model.

A non-linear continuum damage mechanics model was formulated by Dattoma *et al.* [38] to take into account the material damage evolution for various load levels. Fatigue test data for hardened and tempered steel were used to verify the proposed model and the results showed good agreement with the predicted fatigue life under complex load sequences.

A multiaxial fatigue damage model based on the critical plane approach was proposed by Zolochovsky and Obataya [39]. Two different physical mechanisms of fatigue damage development in each potential failure plane were considered. The proposed model simultaneously reproduces fatigue damage-induced anisotropy, the influence of positive and negative mean stresses, unilateral fatigue damage, as well as the micro crack closure effect and fatigue behaviour under variable amplitude loading.

Bhattacharya and Ellingwood [40] presented a continuum damage mechanics-based approach that estimated cumulative fatigue damage and predicted crack initiation from fundamental principles of thermodynamics and mechanics.

Zolochovski *et al.* [41] presented a continuum damage mechanics model associated with the equivalent strain concept for low cycle fatigue failure of initially isotropic materials under biaxial loading conditions and presented experimental data for steel. An interesting review of low cycle fatigue damage models for polycrystalline materials was given by Fatemi and Yang [42].

A nonlinear continuum damage mechanics model was employed by Jing *et al.* to predict low-cycle fatigue damage [43] and creep-fatigue [44] life of a steam turbine's high pressure rotor. The effects of mean stress were taken into account and the damage was accumulated nonlinearly.

A continuum damage mechanics model which improves Lemaitre's theory [45] in interpreting high cycle fatigue damage problems was proposed by Xiao *et al.* [46]. This model was derived from a brittle damage mechanism, so it cannot be used to cope with strain control problems such as low cycle fatigue and creep, which have to be described using the ductile damage model.

2.3.2. Cracking

Dhar *et al.* [47] proposed a damage mechanics model to study void growth and initiation of cracks. A failure curve was presented for the AISI-1090 steel material and a large deformation finite element analysis was carried out.

Crack tip stress and strain fields were generalized for various plane strain and plane stress fracture specimens by Fashang and Kuang [48]. Continuum damage mechanics was used by these authors to quantify the constraints of the fracture specimens and the effects of specimen configuration on the onset of ductile fracture growth.

Chung *et al.* [49] simulated damage and crack propagation on the basis of continuum damage mechanics. Additionally, they utilized the parallel computing technique for precise analysis with the large-scale structural model.

Moyer *et al.* [50] examined an application of uncoupled continuum damage mechanics to predict stable crack growth in the 2219-T87 aluminium alloy. They presented a formulation of predictive damage methodology and its implementation into a finite element code.

An analysis of cracks' growth in piezoelectric ceramics with the double cantilever beam model was performed by Mizuno and Honda [51]. Damage within piezoelectric ceramics was represented by a damage variable based on continuum damage mechanics and its effect on material properties was taken into account in a constitutive equation of piezoelectric ceramics.

2.3.3. Fracture

A model based on continuum damage mechanics and the Hertzian contact theory was developed by Tavares and King [52] to describe fracture by repeated impacts. The model was validated with experimental data from repeated-impact tests.

In [53], Leski presented a numerical analysis of a rotor blade damaged in combat. A finite element model of the blade was built and tested and, based on experimental results, four instances of damage were chosen for a numerical analysis of stress distribution in a damaged rotor blade.

2.3.4. Creep analysis including damage

The effect of damage evolution on creep and creep fracture of brittle materials was treated theoretically by Salganik and Gotlib [54]. The authors presented a comparative analysis of the continuum and the discontinuum damage mechanics approaches.

A boundary element formulation of creep damage problems for three-dimensional problems using isoparametric quadratic elements was presented by Gun [55]. The Euler method with an automatic time-step control scheme was implemented for time integrations. Creep rupture life was predicted using a continuum damage mechanics approach. The proposed formulation was applied to finite element benchmark problems (the uniaxial, perforated plate and the multi-material cross-weld problems).

Finite element creep continuum damage results were given by Becker *et al.* [56]. The authors used the FE-DAMAGE program and the ABAQUS-UMAT damage code in their numerical simulations. The results of a series of tests used to represent uniaxial, biaxial, triaxial and multi-material creep and damage behaviour were presented.

A concept of coupling between the heat transfer equation and the damage evolution in solids subject to creep at elevated temperatures was presented by Skrzypek and Ganczarski [57]. The second-order damage tensor which appears in the constitutive equations of creep and damage and in the combined heat flow-radiation rule plays serves as the feedback-introducing variable. Axisymmetric cylinders and disks subject to creep damage under combined mechanical and thermal loadings were considered as examples.

A formulation of coupled thermo-creep-damage in 3D rotationally-symmetric structures in the case of combined reverse-cyclic mechanical and thermal loads was presented by Ganczarski and Forys [58]. The thermo-damage coupling was described by the modified Fourier heat flux equation, where the second-rank tensor of thermal conductivity was defined with the damage tensor as an argument. The crack closure/opening effects were incorporated by new, effective stress definitions for tension or compression in constitutive equations.

Perrin and Hayhurst [59] discussed the use of the creep continuum damage mechanics method analyze the deformation and type IV creep failure of ferritic steel cross-weld specimens. The authors applied the finite element creep continuum damage mechanics method in numerical modelling, with physically-based constitutive equations.

2.4. The damage phenomenon with respect to material

2.4.1. Ductile materials

Wang [60] proposed a complete set of constitutive equations including damage for elastic-plastic material and has used it to predict forming limits for a sheet, presenting a comparison of the predicted and experimental data.

Lee *et al.* [61] suggested the instrumented indentation technique for estimating fracture toughness of ductile materials. This technique, based on the indentation energy to the characteristic fracture initiation point, may be closely related to a material's resistance to fracture and the characteristic fracture initiation point can be determined with the basic concepts of continuum damage mechanics.

Saanouni *et al.* [62] analyzed numerically the prediction of central burst defects in axisymmetric cold extrusions using two-dimensional finite element analyses accounting for the ductile damage effect. The coupling between ductile damage and the thermo-elasto-plastic constitutive equations was formulated in the framework of thermodynamics of irreversible processes and the continuum damage mechanics theory.

2.4.2. Brittle materials

The brittle damage constitutive equation developed by Chow and Yang [63] was discussed and coupled to finite element method calculations for non-linear elastic deformation behaviour of graphite by Kaji *et al.* [64]. This model is achieved by introducing a damage surface similar to the yield function in the conventional theory of plasticity.

A local continuum damage theory and its non-local variant were applied by de Vree [65] to model the failure behaviour of a construction made of macroscopically

brittle material. A numerical implementation of the approaches was performed in a finite element code.

Skrzypek and Kuna-Ciskał [66] developed methods of damage and fracture modelling in elastic-brittle materials and structures, based on the local approach to fracture and the finite element method. Examples of analysis of 2D structures subjected to damage growth and fracture were presented.

2.4.3. Laminates and composites

A continuum damage mechanics model was applied by Varna *et al.* [67] to predict stiffness reduction in $[S, 90_n]_s$ GF/EP laminates due to transverse cracking in the 90 layers. The continuum damage mechanics constants were calculated using the stiffness versus the crack density data for a cross-ply laminate of the same material. The crack closure technique and Monte Carlo simulations were used to model the evolution of damage.

The concept of homogenization was utilized by Allen [68] to develop simplified methodologies for predicting the response of solids that develop multi-scale damage. Layered composite analyses were presented to exemplify the concept of homogenization.

A composite ply failure model based on continuum damage mechanics was proposed by Edlund and Volgers [69]. The model was derived using continuum damage mechanics and a thermodynamic formulation with internal variables accounting for plastic and damage effects. A set of material parameters was identified and comparisons with experimental results were made.

Nguyen and Khaleel [70] developed a micro-macro mechanistic approach to matrix cracking in randomly-oriented short-fibre composites. The macroscopic response was obtained using a continuum damage mechanics description and a finite element formulation which incorporated the effect of damage on mechanical behaviour through a damage variable linked to crack density.

A continuum damage mechanics model was developed by Rangavan and Ghosh [71] for fibre-reinforced composites. The model was constructed on the basis of rigorous micromechanical analysis of a representative volume element using the Voronoi cell FEM.

A mathematical model was developed by Williams *et al.* [72] to predict damage growth and its effects on the response of laminated fibre-reinforced plastic composites. The formulation was based on the sub-laminate response, in recognition of the fact that the laminate response was driven not only by the lamina's properties but also by the ply interactions through the stacking sequence and damage growth.

Zou *et al.* [73] developed a model for delamination, a crucial mode of interfacial damage, in composite laminate in the context of the continuum damage mechanics. In this model, delaminations were assumed to be confined to interfaces located between predefined sublaminates.

Kumar and Talreja [74] presented a constitutive model for linear viscoelastic orthotropic solids containing a fixed level of distributed cracks. The model was formulated in the continuum damage mechanics framework using internal variables taken as second-order tensors. The model was applied to a specific case of cross-ply laminates with a transverse matrix of cracks.

2.4.4. Concrete and rock

A unified formulation for damage analysis of steel and reinforced concrete frame members was described by Flórez and López [75], based on the methods of continuum damage mechanics and the notion of inelastic hinges.

A continuum damage mechanics model was proposed by Peng and Meyer [76] to describe the inelastic behaviour of concrete reinforced with randomly distributed short fibres. The proposed model can take into account the effect of fibre content on damage development and the effect of volumetric stress on the deviatoric inelastic response. The validity of the proposed model was demonstrated by comparing theoretical and experimental results for plain and fibre-reinforced concrete specimens subjected to triaxial stress histories.

Faming and Zongjin [77] developed an analytical model for the tensile behaviour of fibre-reinforced concrete (FRC) exhibiting a strain hardening response, based on the principles of continuum damage mechanics. Assuming a parallel bar model in which the fibres and the concrete were joined by parallel-series components, an equilibrium equation was established for FRC. Good agreement was obtained between the stress-strain curves predicted in the model and the experimental ones.

A continuum damage mechanics model with irreversible thermodynamics and internal state variables was developed by [78] Cheng and Dusseault to simulate the behaviour of geo-materials. Comparisons of model predictions with experimental data for rock and concrete under uniaxial and triaxial compression were presented. The authors obtained excellent agreement of their numerical simulations with experimental data.

A continuum damage approach was used by Shao and Khazraei [79] to describe the damage behaviour of brittle rock. Two crack growth models were identified by the authors: time-independent mechanical growth due to stresses variation and time-dependent subcritical growth mainly due to stress corrosion.

2.4.5. Ceramics

Maire and Chaboche [80] modelled the damage elastic properties of ceramic matrix composites in the framework of continuum damage mechanics. Weigel *et al.* [81] developed micromechanically-based continuum damage mechanics material laws for fibre-reinforced ceramics. The micromechanical model was derived for the damage and failure mechanisms observed in 2D C/C-SiC composite samples subjected to compression, tension and shear.

2.4.6. Graphite

A failure model for nuclear graphite was described by Zou [82] in the context of continuum damage mechanics. Damage initiation was governed by the stress criterion and the full formation of a crack by the fracture-mechanics criterion. Numerical predictions for tensions in L-shaped and channel-section graphite specimens were presented for various corner radii.

A continuum damage mechanics-based elasto-plastic damage theory was presented by Bielski *et al.* [83]. Weak elastic-plastic dissipation coupling was assumed by the use of plastic and damage dissipation potentials, wherein only isotropic plasticity and damage hardening were included, whereas kinematic hardening was neglected. An

incremental representation of the elastic-damage constitutive equations was derived. Numerical examples were presented for the FCD400 spheroidized graphite cast iron, illustrating the model's capability to describe elastic-plastic damage evolution under monotonic loading.

2.4.7. Elastomers

Chagnon *et al.* [84] demonstrated the ability and limitations of continuum damage mechanics in describing the Mullins effect in elastomers. The considered material was natural rubber, assumed to be non-linear elastic, isotropic and incompressible, with viscous effects not taken into account. The general framework of hyperelasticity with damage was derived.

2.4.8. Human bone

In [85], Taylor *et al.* developed a technique to simulate the tensile fatigue behaviour of human cortical bone. A combined continuum damage mechanics and finite element analysis approach was used to predict the number of cycles till failure, modulus degradation and accumulation of permanent strain of human cortical bone specimens.

2.5. The damage phenomenon with respect to structure

2.5.1. Beams and trusses

Larissa *et al.* [86] presented in detail theoretical and numerical aspects of the behaviour of spatial trusses undergoing large displacements, heavy strains and damage.

Based on convolution-type constitutive equations for linear viscoelastic material with damage and the hypotheses of Timoshenko beams, Cheng *et al.* [87] derived the equations governing the quasi-static and dynamical behaviour of Timoshenko beams with damage.

Gerstmayr *et al.* [88] presented a numerical algorithm describing the development of plastic and damage zones in vibrating structural elements of the beam type.

2.5.2. Plates and shells

Ganczarski and Skrzypek [89] and Ganczarski [90] presented analyses of damage mechanisms and their influence on heat flow in a medium-thick Reissner plate. Adapting the concept of thermodamage coupling to continuum damage mechanics, two 2D coupled problems were formulated: (i) heat transfer through non-homogeneous, partly damaged material and (ii) a Reissner membrane-plate made of aluminium alloy, subjected to brittle damage resulting from mechanical and thermal loadings.

Daudonet [91] proposed several models describing the swelling and dynamic rupture of plates subjected to explosions.

2.5.3. 3D cases

Baaser and Gross [92] simulated damage evolution in cracked cylindrical shells subjected to uniform pressure using finite 3D-continuum elements. They applied the Gurson damage model to describe the behaviour of steel.

2.6. Methods and procedures

Kořakowski [93] described the application of a structural reanalysis technique known as the Virtual Distortion Method, coupled with an optimisation technique called the Gradient Projection Method, to the damage identification problem in static structural analysis.

A special finite element procedure for dynamic and static analysis of civil engineering structures including elasto-viscoplastic models with damage was presented by Ambroziak [94]. The UVSCPL user-defined subroutine was proposed for introducing the elasto-viscoplastic model into the MSC.Marc system.

Addressi *et al.* [95] proposed a plastic non-local damage model for studying the mechanical response of structural elements made of concrete. The authors described a numerical procedure based on an implicit technique for integration of plastic and damage evolution equations.

2.7. Large deformations approach

The continuum damage mechanics theory, together with large deformation elastic-plastic finite element analysis, was used by Dar *et al.* [96] to predict crack growth initiation in ductile materials. A local crack growth initiation criterion was proposed and experiments were conducted on standard specimens and also simulated numerically in order to test its validity.

In [97], Menzel and Steinmann proposed a framework for and an outline of the corresponding numerical treatment of overall modelling of anisotropic materials under large strains, incorporating second-order internal variables in a thermodynamically consistent way.

Shin *et al.* [98] presented a finite element analysis model for material and geometrical non-linearities due to large plastic deformations of ductile materials.

2.8. Stochastic

A stochastic treatment of the damage process in the form of multiple cracking in discontinuous random fibre-reinforced brittle-matrix composites was presented by Wu and Li [99]. Characteristic features of damage accumulation under arbitrary stochastic conditions were studied by Silberschmidt [100] in terms of continuum damage mechanics. In this paper he proposed a modification for the kinetic equation of damage evolution for stochastic conditions.

Bhattacharya and Ellingwood [101] started from the first principles of thermodynamics and recognized intrinsic energy fluctuations in matter, obtaining a stochastic differential equation of ductile damage growth in a deformable body.

Shen *et al.* [102] described a probabilistic distribution model of stochastic fatigue damage. The authors calculated cumulative fatigue damage distribution after a period of stochastic loading.

3. Constitutive equations for isotropic damage evolution

Damage of a material means its progressive internal deterioration o under loading, including a loss of effective cross-sectional area:

$$D = \frac{A - \tilde{A}}{A}, \quad (1)$$

where A is the overall cross-section area, while \tilde{A} is the effective cross-section area (see *e.g.* [12]). According to the hypothesis of effective stress, the effective stress, $\tilde{\sigma}$, is specified using the scalar damage variable D as:

$$\tilde{\sigma} = \frac{\sigma}{1-D}. \quad (2)$$

Using free energy, ψ , and the hypothesis that the elasticity and plasticity behaviours are uncoupled ($\psi = \psi^E + \psi^I$), it is possible to derive the constitutive law of damaged materials. In the case of the isotropic damage and isothermal conditions, the elastic part of the free energy function can be expressed as follows [45]:

$$\psi^E = \frac{(1-D)}{2\rho} (C_{ijnm} \varepsilon_{ij}^E \varepsilon_{nm}^E), \quad (3)$$

where ρ is density, C_{ijnm} – components of the elasticity tensor of the fourth order, ε_{nm}^E – components of the elastic strain tensor.

In paper [11], Lemaitre proposed a strain equivalence hypothesis according to which the strain behaviour of a damaged material can be replaced with that of the virgin material with the stress component calculated as:

$$\rho_{ij} = \rho \frac{\partial \psi^E}{\partial \varepsilon_{ij}^E} = (1-D) C_{ijnm} \varepsilon_{nm}^E. \quad (4)$$

The elastic strain component, ε_{ij}^E , can be derived from Equation (4) as:

$$\varepsilon_{ij}^E = \left(\frac{1}{1-D} \right) \left(\frac{1+v}{E} \sigma_{ij} - \frac{v}{E} \sigma_{kk} \delta_{ij} \right). \quad (5)$$

For the one-dimensional case, the formula assumes a simple form:

$$\varepsilon^E = \frac{\sigma}{(1-D)E}, \quad (6)$$

where E and v are Young's modulus and Poisson's ratio, while δ_{ij} is the Kronecker delta. For small strain problems, additive decomposition of the total strain can be assumed:

$$\varepsilon_{ij} = \varepsilon_{ij}^E + \varepsilon_{ij}^I. \quad (7)$$

It should be noted that the damage parameter can be measured directly as the variations of the Young modulus of damaged material,

$$(1-D)E = \tilde{E} \Rightarrow D = 1 - \frac{\tilde{E}}{E}. \quad (8)$$

The damage strain energy is defined as follows [12, 92]:

$$Y = \rho \frac{\partial \psi^E}{\partial D} = -\frac{1}{2} C_{ijnm} \varepsilon_{ij}^E \varepsilon_{nm}^E. \quad (9)$$

Considering that the density of elastic strain energy can be expressed as:

$$W_e = \frac{1}{2} (1-D) C_{ijnm} \varepsilon_{ij}^E \varepsilon_{nm}^E, \quad (10)$$

we can obtain

$$-Y = \frac{W_e}{1-D}. \quad (11)$$

Through the generalized normality rule, the constitutive equations may be derived from the dissipation function, φ :

$$\dot{\varepsilon}_{ij}^I = \lambda \frac{\partial \varphi}{\partial \sigma_{ij}}, \quad (12)$$

where λ is the viscosity multiplier. Then the accumulated strain rate can be calculated as:

$$\dot{p} = \sqrt{\frac{2}{3} \varepsilon_{ij}^I \varepsilon_{ij}^I}. \quad (13)$$

Therefore, the damage evolution equation can be generally expressed as [12]:

$$\dot{D} = -\lambda_D \frac{\partial \varphi}{\partial Y}, \quad (14)$$

where λ_D is the damage multiplier.

4. Models for isotropic damage evolution

Several approaches to damage evolution can be encountered in the literature. We present the most popular ones below, which utilize the damage strain energy function, Y , directly.

In [45], Lemaitre proposed the potential of dissipation for ductile plastic damage written as a power function of Y and linear with respect to \dot{p} :

$$\varphi = \frac{S}{s+1} \cdot \left(\frac{-Y}{S} \right)^{s+1} \cdot \dot{p}, \quad (15)$$

where S and s are the damage material parameters. Therefore, the evolution of damage is derived from Equation (14), where $\lambda_D = 1$:

$$\dot{D} = \left(-\frac{Y}{S} \right) \cdot \dot{p}. \quad (16)$$

The Y function is given by [45]:

$$-Y = \frac{1}{2(1-D)^2 E} \left(\frac{2}{3}(1+v)\sigma_{eq}^2 + 3(1-2v)\sigma_H^2 \right), \quad (17)$$

where σ_{eq} is the Huber-Mises equivalent stress and σ_H is the hydrostatic stress, expressed as:

$$\sigma_{eq} = \sqrt{\frac{3}{2} \sigma_{ij}^D \sigma_{ij}^D}, \quad \sigma_{ij}^D = \sigma_{ij} \sigma_H \delta_{ij}, \quad \sigma_H = \frac{1}{3} \sigma_{kk}. \quad (18)$$

The identification procedure of the damaged material parameters for the above type of the damage evolution equation can be found in [103]. The authors investigated the INCO718 alloy in detail.

According to [104], Wang assumed the following form of the potential of dissipation:

$$\varphi = g + \frac{Y^2}{2 \cdot c \cdot (1-D)} \cdot \frac{(p-p_0)^{k-1}}{p^{2 \cdot n}}, \quad (19)$$

where

$$g = \frac{\sigma_{eq} - R}{1-D} - \sigma_Y > 0. \quad (20)$$

The initial yield stress, σ_Y , and the isotropic hardening scalar variable, R , are associated with the equivalent plastic strain, p . Wang assumed that [60]:

$$\lambda_D = (1 - D)\dot{p}. \quad (21)$$

Then, the damage evolution can be expressed as:

$$\dot{D} = -\frac{Y}{c} \cdot \frac{(p - p_0)^{k-1}}{p^{2+n}} \cdot \dot{p}, \quad (22)$$

where c and k are material constants and n is the hardening exponent.

Bonora ([105, 106]) proposed the following expression for the damage dissipation potential:

$$\varphi = \left(\frac{1}{2} \cdot \left(-\frac{Y}{S} \right) \cdot \frac{S}{1 - D} \right) \frac{(D_{cr} - D)^{\frac{\alpha-1}{\alpha}}}{p^{\frac{2+n}{n}}}, \quad (23)$$

where D_{cr} is the critical value of the damage variable for which ductile failure occurs, α is the damage exponent that characterizes the shape of the damage evolution curve, S is a material constant and n is the material hardening exponent. Bonora obtained the damage evolution equation in the following form:

$$\dot{D} = -\frac{Y}{S} \cdot \frac{(D_{cr} - D)^{\frac{\alpha-1}{\alpha}}}{p^{\frac{2+n}{n}}} \cdot \dot{p}. \quad (24)$$

Lemaitre's [45] continuum damage mechanics model was used by Dhar *et al.* [47] together with the experimental results of Le Roy *et al.* [107] to derive the following damage growth law:

$$\dot{D} = s_0 \cdot \dot{p} + (s_1 + s_2 \cdot D) \cdot (-Y \cdot \dot{p}), \quad (25)$$

where s_0 , s_1 and s_2 are damage constants.

Xiao used the following form of damage evolution in [46]:

$$\dot{D} = B \cdot Y^{q-1} \cdot \dot{Y}, \quad (26)$$

where $B = \lambda \cdot q \cdot k$, q is the material parameter and k is a function of the material's state variables.

Tai [108] applied the following form of damage evolution:

$$\dot{D} = -\frac{Y}{S} \cdot D \cdot \dot{p}, \quad (27)$$

where S is the damage parameter.

In [109] and [110], Chandrakanth proposed another form of damage evolution:

$$\dot{D} = -\frac{Y}{S} \cdot \left[\frac{1}{(1 - D)^{\frac{1}{n}}} - (1 - D) \right] \cdot \dot{p}, \quad (28)$$

$$\dot{D} = -\frac{Y}{S} \cdot \frac{\dot{p}}{D^{\frac{\alpha}{n}} \cdot p^{\frac{2}{n}}}, \quad (29)$$

where α , n and S are damage parameters.

5. Concluding remarks

Conferences and symposia organized all over the world have emphasized the importance of damage mechanics and for many journals (*e.g.* International Journal of Damage Mechanics) damage mechanics is the main field of interest.

Several variants of the damage equation evolution have been presented in the present paper. They are not universal as they can be used under certain conditions and for certain materials. The authors are aware that also many other damage evolution equations exist, but their engineering applications are limited due to difficulties in determining a large number of material parameters.

It is sometimes necessary to find a quick solution of an engineering problem, but we have to remember that the model parameters applied to numerical simulations usually cannot be taken from the literature, as the conditions under which they were obtained are unknown. Laboratory test results should be registered electronically, including date identification, and carefully verified.

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