General construction of noiseless networks detecting entanglement with the help of linear maps

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We present the general scheme for construction of noiseless networks detecting entanglement with the help of linear, hermiticity-preserving maps. We show how to apply the method to detect entanglement of an unknown state without its prior reconstruction. In particular, we prove there always exists noiseless network detecting entanglement with the help of positive, but not completely positive maps. Then the generalization of the method to the case of entanglement detection with arbitrary, not necessarily hermiticity-preserving, linear contractions on product states is presented.

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I. INTRODUCTION

It has been known that entanglement can be detected with the help of a special class of maps called positive maps [1–3]. In particular there is an important criterion [1] saying that ϱ acting on a given product Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ is separable if and only if for all positive (but not completely positive) maps $\Lambda:\mathcal{B}(\mathcal{H}_B)\to\mathcal{B}(\mathcal{H}_A)$ [4] the following operator:

$$X_\Lambda(\varrho) = [I \otimes \Lambda](\varrho), \tag{1}$$

has all non-negative eigenvalues which usually is written as

$$[I \otimes \Lambda](\varrho) \geq 0. \tag{2}$$

Here by I we denote the identity map acting on $\mathcal{B}(\mathcal{H}_A)$. Since any positivity-preserving map is also Hermiticity preserving, it makes sense to speak about eigenvalues of $X_\Lambda(\varrho)$. However, it should be emphasized that there are many Λ's (and equivalently the corresponding criteria) and to characterize them is a hard and still unsolved problem (see, e.g., Ref. [5] and references therein).

For a long time the above criterion has been treated as purely mathematical. One used to take matrix ϱ (obtained in some *prior* state estimation procedure) and then put it into the formula (2). Then its spectrum was calculated and the conclusion was drawn. However, it can be seen that for, say states acting on $\mathcal{H}_A \otimes \mathcal{H}_B \sim \mathbb{C}^d \otimes \mathbb{C}^d$ and maps $\Lambda:\mathcal{B}(\mathbb{C}^d)\to\mathcal{B}(\mathbb{C}^d)$, the spectrum of the operator $X_\Lambda(\varrho)$ consists of $n_{\mathrm{spec}}=d^2$ elements, while full *prior* estimation of such states corresponds to $n_{\mathrm{est}}=d^4-1$ parameters.

The question was raised [6] as to whether one can perform the test (2) physically without necessity of *prior* tomography of the state ϱ despite the fact that the map $I\otimes\Lambda$ is not physically realizable. The corresponding answer was [6] that one can use the notion of structural physical approximation (SPA) $\widetilde{I\otimes\Lambda}$ of unphysical map $I\otimes\Lambda$ which is physically realizable already, but at the same time the spectrum of the state

$$\widetilde{X}_\Lambda(\varrho) = [\widetilde{I\otimes\Lambda}](\varrho) \tag{3}$$

is just an affine transformation of that of the (unphysical) operator $X_\Lambda(\varrho)$. The spectrum of $\widetilde{X}_\Lambda(\varrho)$ can be measured with the help of the spectrum estimator [7], which requires estimation of only d^2 parameters which (because of affinity) are in one to one correspondence with the needed spectrum of Eq. (2). Note that for $2\otimes 2$ systems (the composite system of two qubits), similar approaches lead to the method of detection of entanglement measures (concurrence [8] and entanglement of formation [9]) without the state reconstruction [10].

The disadvantage of the above method is [11] that realization of SPA requires the addition of the noise to the system (we have to put some controlled ancillas, couple the system, and then trace them out). In Ref. [11] the question was raised about the existence of noiseless quantum networks, i.e., those of which the only input data are (i) unknown quantum information represented by $\varrho^{\otimes m}$, (ii) the controlled measured qubit which reproduces for us the spectrum moments (see Ref. [7]). It was shown that for at least one positive map (transposition) T the noiseless network exists [11]. Such networks for two-qubit concurrence and three-qubit tangle have also been designed [12].

In the present paper we ask a general question: do noiseless networks work only for special maps (functions) or do they exist for any positive map test? In the case of a positive answer to the latter: is it possible to design a general method for constructing them? Can it be adopted to any criteria other than the one defined in Eq. (2)?

For this purpose we first show how to measure a spectrum of the matrix $\Theta(\varrho)$, where $\Theta:\mathcal{B}(\mathbb{C}^m)\to\mathcal{B}(\mathbb{C}^m)$ is an arbitrary linear, Hermiticity-preserving map and ϱ is a given density operator acting on $\mathbb{C}^m$, with the help of only m parameters estimated instead of m^2-1. For bipartite ϱ where $m=d^2$ this gives d^2 instead of d^4-1. This approach is consistent with previous results [13–15] where arbitrary polynomials of elements of a given state ϱ have been considered. In these works it was shown that at most kth degree polynomial of a density matrix ϱ can be measured with the help of two collective observables on k copies of ϱ. In fact one can treat the moments of $\Theta(\varrho)$ which we analyze below as polynomials belonging to such a class. We derive the explicit form of

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observables for the sake of possible future application. Moreover, the approach presented in the present paper allows for quite natural identification of an observable that detects an arbitrary polynomial of the state ϱ subjected to some transformation Θ. Then we provide an immediate application in entanglement detection showing that for suitable Θ the scheme constitutes just the right method for detecting entanglement without prior state reconstruction with the help of either positive map criteria (2) or the linear contraction methods discussed later.

II. GENERAL SCHEME FOR CONSTRUCTION OF NOISELESS NETWORK DETECTING SPECTRUM OF $\Theta(\varrho)$

A. Construction of an observable

Since $m\times m$ matrix $\Theta(\varrho)$ is Hermitian its spectrum may be calculated using only m numbers,

$$\alpha_k \equiv \mathrm{Tr}[\Theta(\varrho)]^k = \sum_{i=1}^{m} \lambda_i^k \quad (k=1,\ldots,m), \tag{4}$$

where λ_i are eigenvalues of $\Theta(\varrho)$. We shall show that all these spectrum moments can be represented by mean values of special observables. To this aim let us consider the permutation operator $V^{(k)}$ defined by the formula

$$V^{(k)}|e_1\rangle|e_2\rangle \otimes \cdots \otimes |e_k\rangle = |e_k\rangle|e_1\rangle \otimes \cdots \otimes |e_{k-1}\rangle, \tag{5}$$

where $(k=1,\ldots,m)$ and $|e_i\rangle$ are vectors from $\mathbb{C}^m$. One can see that $V^{(1)}$ is just an identity operator $\mathbb{1}_m$ acting on $\mathbb{C}^m$. Combining Eqs. (4) and (5) we infer that α_k may be expressed by the relation

$$\alpha_k = \mathrm{Tr}\{V^{(k)}[\Theta(\varrho)]^{\otimes k}\} \tag{6}$$

which is a generalization of the formula from Refs. [6,7] where Θ was (unlike here) required to be a physical operation. At this stage the careful analysis of the right-hand side of Eq. (6) shows that α_k is a polynomial of at most the kth degree in matrix elements of ϱ. This, together with the observation of Refs. [13–15] allows us already to construct a single collective observable that detects α_k. However, for the sake of possible future applications we derive the observable explicitly below. To this aim we first notice that α_k may be obtained using Hermitian conjugation of $V^{(k)}$ which again is a permutation operator but permutes states $|e_i\rangle$ in the reversed order. Therefore all the numbers α_k may be expressed as

$$\alpha_k = \frac{1}{2}\mathrm{Tr}[(V^{(k)} + V^{(k)\dagger})\Theta(\varrho)^{\otimes k}]. \tag{7}$$

Let us focus for a while on the map Θ. Due to its Hermiticity-preserving property it may be expressed as

$$\Theta(\cdot) = \sum_{j=0}^{m^2-1} \eta_j K_j(\cdot)K_j^\dagger \tag{8}$$

with $\eta_j \in \mathbb{R}$ and K_j being linearly independent $m\times m$ matrices. By the virtue of this fact and some well-known properties of the trace, after rather straightforward algebra we may rewrite Eq. (7) as

$$\alpha_k = \frac{1}{2}\mathrm{Tr}[(\Theta^\dagger)^{\otimes k}(V^{(k)} + V^{(k)\dagger})\varrho^{\otimes k}], \tag{9}$$

where $\Theta^\dagger$ is a dual map to Θ and is given by $\Theta^\dagger(\cdot) = \sum_i \eta_i K_i^\dagger(\cdot)K_i$. Here we have applied a map $(\Theta^\dagger)^{\otimes k}$ on the operator $V^{(k)}+V^{(k)\dagger}$ instead of applying $\Theta^{\otimes k}$ to $\varrho^{\otimes k}$. This apparently purely mathematical trick with the aid of the fact that the square brackets in the above contain a Hermitian operator allows us to express the numbers α_k as a mean value of some observables in the state $\varrho^{\otimes k}$. Indeed, introducing

$$\mathcal{O}_\Theta^{(k)} = \frac{1}{2}[(\Theta^\dagger)^{\otimes k}(V^{(k)} + V^{(k)\dagger})] \tag{10}$$

we arrive at

$$\alpha_k = \mathrm{Tr}[\mathcal{O}_\Theta^{(k)}\varrho^{\otimes k}]. \tag{11}$$

In general, a naive measurement of all mean values would require the estimation of much more parameters that m. But there is a possibility of building a unitary network that requires the estimation of exactly m parameters using the idea that we recall and refine below.

Finally, let us notice that the above approach generalizes measurements of polynomials of elements of ϱ in the sense that it shows explicitly how to measure the polynomials of elements of $\Theta(\varrho)$. Of course, this is only of rather conceptual importance since both issues are mathematically equivalent and have the origin in Refs. [13–15].

B. Detecting mean of an observable by measurement on a single qubit revised

Let $\mathcal{A}$ be an arbitrary observable (it may be even infinite dimensional) which spectrum lies between finite numbers $a_{\mathcal{A}}^{\min}$ and $a_{\mathcal{A}}^{\max}$ and σ be a state acting on $\mathcal{H}$. In Ref. [16] it has been pointed out that the mean value $\langle\mathcal{A}\rangle_\sigma = \mathrm{Tr}\,\mathcal{A}\sigma$ may be estimated in process involving the measurement of only one qubit. This fact is in good agreement with further proof that single qubits may serve as interfaces connecting quantum devices [17]. Below we recall the mathematical details of the measurement proposed in Ref. [16]. At the beginning one defines the following numbers:

$$a_{\mathcal{A}}^{(-)} \equiv \max\{0, -a_{\mathcal{A}}^{\min}\}, \quad a_{\mathcal{A}}^{(+)} \equiv a_{\mathcal{A}}^{(-)} + a_{\mathcal{A}}^{\max}, \tag{12}$$

and observe that the Hermitian operators

$$V_0 = \sqrt{(a_{\mathcal{A}}^{(-)}\mathbb{1}_{\mathcal{H}} + \mathcal{A})/a_{\mathcal{A}}^{(+)}} \tag{13}$$

and

$$V_1 = \sqrt{\mathbb{1}_{\mathcal{H}} - V_0^\dagger V_0} \tag{14}$$

satisfy $\sum_{i=0}^{1} V_i^\dagger V_i = \mathbb{1}_{\mathcal{H}}$ [18] and as such define a generalized quantum measurement which can easily be extended to a unitary evolution (see Appendix A of Ref. [20] for a detailed description). Consider a partial isometry on the Hilbert space $\mathbb{C}^2 \otimes \mathcal{H}$ defined by the formula

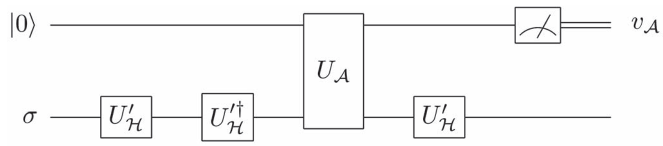

FIG. 1. General scheme of a network for estimating mean value of an observable $\mathcal{A}$, with a bounded spectrum, in a given state σ. Both $U'_{\mathcal{H}}$ and its conjugate $U'^{\dagger}_{\mathcal{H}}$ standing before $U_{\mathcal{A}}$ can obviously be removed as they give rise to identity, last unitary on the bottom wire can be removed as it does not impact measurement statistics on the top qubit. However, they have been put to simplify subsequent network structure.

$$\tilde{U}_{\mathcal{A}} = \sum_{i=0}^{1} |i\rangle\langle 0| \otimes V_i = \begin{pmatrix} V_0 & 0 \\ V_1 & 0 \end{pmatrix}. \tag{15}$$

The first Hilbert space $\mathbb{C}^2$ represents the qubit which shall be measured in order to estimate the mean value $\langle\mathcal{A}\rangle_\sigma$. The partial isometry can always be extended to unitary $U_{\mathcal{A}}$ such that if it acts on $|0\rangle\langle 0|\otimes\sigma$ then the final measurement of observable σ_z [19] on the first (qubit) system gives probabilities "spin-up" (of finding it in the state $|0\rangle$) and "spin-down" (of finding in state $|1\rangle$), respectively, of the form

$$p_0 = \mathrm{Tr}(V_0^\dagger V_0 \varrho), \quad p_1 = \mathrm{Tr}(V_1^\dagger V_1 \varrho) = 1 - p_0. \tag{16}$$

One of the possible extensions of $\tilde{U}_{\mathcal{A}}$ to the unitary on $\mathbb{C}^2 \otimes \mathcal{H}$ is the following:

$$U_{\mathcal{A}} = \begin{pmatrix} V_0 & -V_1 \\ V_1 & V_0 \end{pmatrix} = \mathbb{1}_2 \otimes V_0 - i\sigma_y \otimes V_1. \tag{17}$$

The unitarity of $U_{\mathcal{A}}$ follows from the fact that operators V_0 and V_1 commute. Due to the practical reasons instead of unitary operation representing positive operator valued measure (POVM) $\{V_0, V_1\}$ we shall consider

$$U^{\det}(\mathcal{A}, U'_{\mathcal{H}}) = (\mathbb{1}_2 \otimes U'_{\mathcal{H}}) U_{\mathcal{A}} (\mathbb{1}_2 \otimes U'_{\mathcal{H}})^\dagger, \tag{18}$$

where $\mathbb{1}_2$ is an identity operator on the one-qubit Hilbert space $\mathbb{C}^2$ and $U'_{\mathcal{H}}$ is an arbitrary unitary operation that acts on $\mathcal{H}$ and simplifies the decomposition of $U_{\mathcal{A}}$ into elementary gates. Now if we define a mean value of measurement of σ_z on the first qubit after action of the network (which sometimes may be called visibility):

$$v_{\mathcal{A}} = \mathrm{Tr}[(\sigma_z \otimes \mathbb{1}_{\mathcal{H}})(\mathbb{1}_2 \otimes U'_{\mathcal{H}}) U_{\mathcal{A}} \mathcal{P}_0 \otimes \sigma U_{\mathcal{A}}^\dagger (\mathbb{1}_2 \otimes U'_{\mathcal{H}})^\dagger], \tag{19}$$

where $\mathcal{P}_0$ is a projector onto state $|0\rangle$, i.e., $\mathcal{P}_0 = |0\rangle\langle 0|$, then we have an easy formula for the mean value of the initial observable $\mathcal{A}$:

$$\langle\mathcal{A}\rangle_\sigma = a_{\mathcal{A}}^{(+)} p_0 - a_{\mathcal{A}}^{(-)} = a_{\mathcal{A}}^{(+)} \frac{v_{\mathcal{A}} + 1}{2} - a_{\mathcal{A}}^{(-)}. \tag{20}$$

A general scheme of a network estimating the mean value (20) is provided in Fig. 1. We put an additional unitary operation on the bottom wire after unitary $U_{\mathcal{A}}$ (which does not change the statistics of the measurement on control qubit) and divided identity operator into two unitaries acting on that wire which explicitly shows how simplification introduced in Eq. (18) works in practice.

Now one may ask if the mean value $\langle\mathcal{A}\rangle_\sigma$ belongs to some fixed interval, i.e.,

$$c_1 \leqslant \langle\mathcal{A}\rangle_\sigma \leqslant c_2, \tag{21}$$

where c_1 and c_2 are real numbers belonging to the spectrum of $\mathcal{A}$, i.e., $[a_{\mathcal{A}}^{\min}, a_{\mathcal{A}}^{\max}]$ (e.g., if $\mathcal{A}$ is an entanglement witness and we want to check the entanglement of a state σ then we can put $c_1=0$ and $c_2=a_{\mathcal{A}}^{\max}$, and condition (21) reduces to $\langle\mathcal{A}\rangle_\sigma \geqslant 0$). Then one easily infers that the condition (21) rewritten for visibility is

$$2\frac{c_1 + a_{\mathcal{A}}^{(-)}}{a_{\mathcal{A}}^{(+)}} - 1 \leqslant v_{\mathcal{A}} \leqslant 2\frac{c_2 + a_{\mathcal{A}}^{(-)}}{a_{\mathcal{A}}^{(+)}} - 1. \tag{22}$$

Having the general network estimating $v_{\mathcal{A}}$, one needs to decompose an isometry $U_{\mathcal{A}}$ onto elementary gates. One of the possible ways to achieve this goal is, as we shall see below, to diagonalize the operator V_0. Hence we may choose $U'_{\mathcal{H}}$ [see Eq. (18)] to be

$$U'_{\mathcal{H}} = \sum_{\mathbf{k}} |\mathbf{k}\rangle\langle\phi_{\mathbf{k}}| \tag{23}$$

with $|\phi_k\rangle$ being normalized eigenvectors of V_0 indexed by a binary number with length 2^k. Since V_0 and V_1 commutes, this operation diagonalizes V_1 as well. By virtue of these facts, Eq. (18) reduces to

$$U^{\det}(\mathcal{A}, U'_{\mathcal{H}}) = \sum_{\mathbf{k}} U_{\mathbf{k}} \otimes |\mathbf{k}\rangle\langle\mathbf{k}|, \tag{24}$$

with unitaries (as previously indexed by a binary number)

$$U_{\mathbf{k}} = \sqrt{\lambda_{\mathbf{k}}}\mathbb{1}_2 - i\sqrt{1-\lambda_{\mathbf{k}}}\sigma_y, \tag{25}$$

where $\lambda_{\mathbf{k}}$ are eigenvalues of V_0. So in fact we have a combination of operations on the first qubit controlled by 2^k wires. All this combined gives us the network shown in Fig. 2.

Now we are in the position to combine all the elements presented so far and show how, if put together, they provide the general scheme for constructing a noiseless network for the spectrum of $\Theta(\varrho)$ for a given quantum state ϱ. For the sake of clarity below we itemize all steps necessary to obtain the spectrum of $\Theta(\varrho)$:

(i) Take all observables $\mathcal{O}^{(k)}(k=1,\ldots,m)$ defined by Eq. (10).

(ii) Construct unitary operations $U_{\mathcal{O}^{(k)}}$ according to the given prescription. Consider the unitary operation ($U'_{\mathcal{H}}$ arbitrary). Find decomposition of the operation into elementary quantum gates and minimize the number of gates in the de-

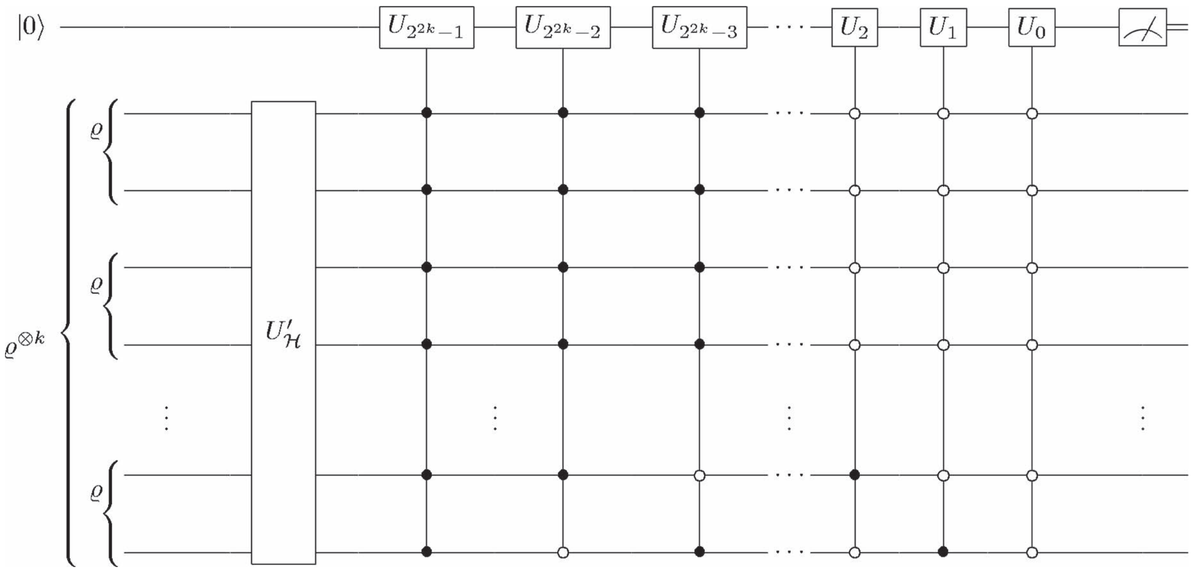

FIG. 2. Noiseless network for estimating moments of $\Theta(\varrho)$ with ϱ being a bipartite mixed state, i.e., density matrix acting on $\mathbb{C}^2\otimes\mathbb{C}^2$.

composition with respect to $U'_{\mathcal{H}}$. Build the (optimal) network found in this way.

(iii) Act with the network on initial state $\mathcal{P}_0\otimes\varrho^{\otimes k}$.

(iv) Measure the "visibilities" $v_{\mathcal{O}_{\Theta}^{(k)}}$ $(k=1,\ldots,m)$ according to Eq. (19)

(v) Using Eq. (20) calculate the values of α_k $(k=1,\ldots,m)$ representing the moments of $\Theta(\varrho)$.

C. Detecting entanglement with networks: Example

The first obvious application of the presented scheme is entanglement detection *via* positive but not completely positive maps. In fact for any bipartite state $\varrho\in\mathcal{B}(\mathcal{H}_A\otimes\mathcal{H}_B)$ we only need to substitute Θ with $I_A\otimes\Lambda_B$ with Λ_B being some positive map. Then application of the above scheme immediately reproduces all the results of the schemes from Ref. [6] but without additional noise added (the presence of which required more precision in the measurement of visibility).

As an illustrative example consider $\Lambda_B=T$, i.e., Θ is the partial transposition on the second subsystem (usually denoted by T_B or by Γ), in $2\otimes 2$ systems. Due to the fact that partial transposition is trace-preserving we need only three numbers α_k $(k=2,3,4)$, measurable via observables

$$\mathcal{O}_T^{(2)}=V_1^{(2)}\otimes V_2^{(2)} \tag{26}$$

and

$$\mathcal{O}_T^{(3,4)}=\frac{1}{2}(V_1^{(3,4)}\otimes V_2^{(3,4)\dagger}+V_1^{(3,4)\dagger}\otimes V_2^{(3,4)}), \tag{27}$$

where subscripts mean that we exchange first and second subsystems, respectively. The Hermitian conjugation in the above may be replaced by transposition since the permutation operators have real entries. For simplicity we show only the network measuring the second moment of ϱ^{T_B}. The general scheme from Fig. 2 reduces then to the scheme from Fig. 3.

Note that the network can also be regarded as one measuring the purity of a state as $\mathrm{Tr}(\varrho^{T_B})^2=\mathrm{Tr}\varrho^2$. Note that this network is not optimal since an alternative network [7] measuring $\mathrm{Tr}\varrho^2$ requires two controlled swaps.

III. EXTENSION TO LINEAR CONTRACTIONS CRITERIA

The above approach may be generalized to the so-called *linear contractions criteria*. To see this let us recall that the powerful criterion called computable cross norm (CCN) or matrix realignment criterion has recently been introduced [21,22]. This criterion is easy to apply (involves simple permutation of matrix elements) and has been shown [21,22] to be independent on a postive partial trans-

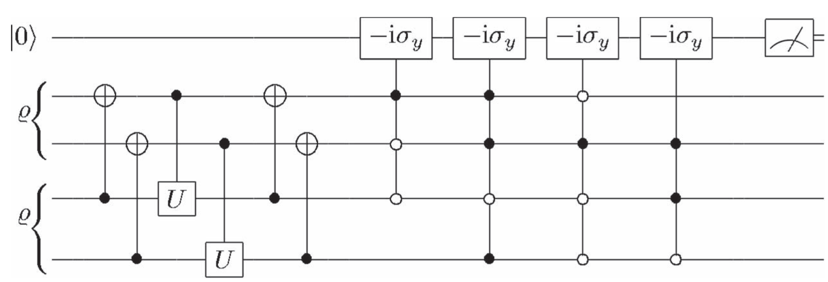

FIG. 3. Network estimating the second moment of partially transposed two-qubit density matrix ϱ. $U'_{\mathcal{H}}$ is decomposed to single qubit gates; here $U=(1/\sqrt{2})(\mathbb{1}_2+i\sigma_y)$.

position (PPT) test [2]. It has been further generalized to the *linear contractions criterion* [23] which we shall recall below. If by $\varrho_{A_i}(i=1,\ldots,n)$ we denote density matrices acting on Hilbert spaces $\mathcal{H}_{A_i}$ and by $\widetilde{\mathcal{H}}$ certain Hilbert space, then for some linear map $\mathcal{R}:\mathcal{B}(\mathcal{H}_{A_1}\otimes\ldots\otimes\mathcal{H}_{A_n})\to\mathcal{B}(\widetilde{\mathcal{H}})$ we have the following:

Theorem [23]. If some $\mathcal{R}$ satisfies

$$\|\mathcal{R}(\varrho_{A_1}\otimes\varrho_{A_2}\otimes\cdots\otimes\varrho_{A_n})\|_{\mathrm{Tr}}\leqslant 1, \tag{28}$$

then for any separable state $\varrho_{A_1A_2\cdots A_n}\in\mathcal{B}(\mathcal{H}_{A_1}\otimes\cdots\otimes\mathcal{H}_{A_n})$ one has

$$\|\mathcal{R}(\varrho_{A_1A_2\cdots A_n})\|_{\mathrm{Tr}}\leqslant 1. \tag{29}$$

The maps $\mathcal{R}$ satisfying Eq. (28) are linear contractions on product states and hereafter they shall be called, in brief, linear contractions. In particular, the separability condition Eq. (29) comprises the generalization of the realignment test to permutation criteria [23,24] (see also Ref. [25]).

The noisy network for entanglement detection with the help of the latter have been proposed in Ref. [26]. Here we improve this result in two ways, namely, by taking into account all maps $\mathcal{R}$ of type (28) (not only permutation maps) and introducing the corresponding noiseless networks instead of noisy ones. For these purposes we need to generalize the lemma from Ref. [26] formulated previously only for real maps $\mathcal{S}:\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{H})$. We represent action of $\mathcal{S}$ on any $\varrho\in\mathcal{B}(\mathcal{H})$ as

$$\mathcal{S}(\varrho)=\sum_{ij,kl}\mathcal{S}_{ij,kl}\mathrm{Tr}(\varrho P_{ij})P_{kl}, \tag{30}$$

where in Dirac notation $P_{xy}=|x\rangle\langle y|$. Let us define the complex conjugate of the map $\mathcal{S}$ *via* the complex conjugation of its elements, i.e.,

$$\mathcal{S}^*(\varrho)=\sum_{ij,kl}\mathcal{S}^*_{ij,kl}\mathrm{Tr}(\varrho P_{ij})P_{kl}, \tag{31}$$

where the asterisk stands for the complex conjugation. The we have the following lemma which is easy to prove by inspection:

Lemma. Let $\mathcal{S}$ be an arbitrary linear map on $\mathcal{B}(\mathcal{H})$. Then the map $\mathcal{S}'\equiv[T\circ\mathcal{S}^*\circ T]$ satisfies $\mathcal{S}'(\varrho)=[\mathcal{S}(\varrho)]^\dagger$.

Now let us come to the initial problem of this section. Suppose then we have $\mathcal{R}$ satisfying Eq. (28) and a given physical source producing copies of a system in state ϱ for which we would like to check Eq. (29). Let us observe that

$$\|\mathcal{R}(\varrho)\|_{\mathrm{Tr}}=\sum_i\sqrt{\gamma_i}, \tag{32}$$

where $\{\gamma_i\}$ are eigenvalues of the operator $X_{\mathcal{R}}(\varrho)=\mathcal{R}(\varrho)\mathcal{R}(\varrho)^\dagger$. Below we show how to find the spectrum $\{\gamma_i\}$. We need to apply our previous scheme from Sec. II to the special case. Let us define the map $L_{\mathcal{R}}=\mathcal{R}\otimes\mathcal{R}'$, where $\mathcal{R}$ is our linear contraction and $\mathcal{R}'$ is defined according to the prescription given in the lemma above, i.e., $\mathcal{R}'=[T\circ\mathcal{R}^*\circ T]$. Let us also put $\varrho'=\varrho^{\otimes 2}$ and apply the scheme presented above to detect the spectrum of $L_{\mathcal{R}}(\varrho')$. It is easy to see that the moments detected in that way are

$$\mathrm{Tr}[L_{\mathcal{R}}(\varrho')]^k=\mathrm{Tr}[\mathcal{R}(\varrho)\mathcal{R}(\varrho)^\dagger]^k=\sum_i\gamma_i^k. \tag{33}$$

From the moments one easily reconstructs $\{\gamma_i\}$ and may check the violation of Eq. (29).

IV. SUMMARY

We have shown how to detect the spectrum of the operator $\Theta(\varrho)$ for the arbitrary linear Hermiticity-preserving map Θ given the source producing copies of the system in state ϱ. The network involved in the measurement is noiseless in the sense of Ref. [11] and the measurement is required only on the controlled qubit. Further we have shown how to apply the method to provide general a noiseless network scheme of detection detecting entanglement with the help of criteria belonging to one of two classes, namely, those involving positive maps and applying linear contractions on product states.

The structure of the proposed networks is not optimal and needs further investigation. Here, however, we have been interested in quite a fundamental question which is interesting by itself: Is it possible to get noiseless network schemes for any criterion from one of the above classes? Up to now their existence was known *only* for the special case of positive partial transpose (cf. Ref. [12]). Here we have provided a positive answer to the question.

Finally let us note that the above approach can be viewed as an application of collective observables [see Eq. (11)]. The general paradigm initiated in Refs. [10,27] has been recently fruitfully applied in the context of general concurrence estimates [28,29] which has been even preliminarily experimentally illustrated. Moreover, recently the universal collective observable detecting any two-quibt entanglement has been constructed [30]. It seems that the present approach needs further analysis from the point of view of collective observables including especially collective entanglement witness (see [27,29]).

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