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TREES WITH EQUAL RESTRAINED DOMINATION AND TOTAL RESTRAINED DOMINATION NUMBERS

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Abstract

For a graph $G = (V, E)$, a set $D \subseteq V(G)$ is a *total restrained dominating set* if it is a dominating set and both $\langle D \rangle$ and $\langle V(G) - D \rangle$ do not have isolated vertices. The cardinality of a minimum total restrained dominating set in G is the *total restrained domination number*. A set $D \subseteq V(G)$ is a *restrained dominating set* if it is a dominating set and $\langle V(G) - D \rangle$ does not contain an isolated vertex. The cardinality of a minimum restrained dominating set in G is the *restrained domination number*. We characterize all trees for which total restrained and restrained domination numbers are equal.

Keywords: total restrained domination number, restrained domination number, trees.

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1. Introduction

Let $G = (V(G), E(G))$ be a simple graph with $|V(G)| = n(G)$. The *neighbourhood* $N_G(u)$ of a vertex u is the set of all vertices adjacent to u in G and the *closed neighbourhood* of u is $N_G[u] = N_G(u) \cup \{u\}$. For a set $D \subseteq V(G)$ the *closed neighbourhood* of D is defined to be $\bigcup_{u \in D} N_G[u]$. The *private neighbourhood of a vertex u with respect to a set $D \subseteq V(G)$*, where $u \in D$, is the set $PN_G[u, D] = N_G[u] - N_G[D - \{u\}]$. If $v \in PN_G[u, D]$, then we say that v is a private neighbour of u with respect to the set D.

The *degree* $d_G(u)$ of a vertex u is the number of edges incident to u in G, that is $d_G(u) = |N_G(u)|$. Let $\Omega(G)$ be the set of all leaves of G, that is the set of vertices degree 1. A vertex which is a neighbour of a leaf is called a *support vertex*. Let $S(G)$ be the set of all support vertices in G. The *diameter* $\mathrm{diam}(G)$ of a connected graph G is the maximum distance between two vertices of G, that is $\mathrm{diam}(G) = \max_{u,v \in V(G)} d_G(u,v)$. We say that a set $D \subseteq V(G)$ is *independent*, if the induced subgraph $\langle D \rangle$ has no edge.

A set $D \subseteq V(G)$ is a *dominating set of* G if for every vertex $v \in V(G) - D$ there exists a vertex $u \in D$ such that v and u are adjacent. The minimum cardinality of a dominating set in G is the *domination number* denoted $\gamma(G)$. A minimum dominating set of a graph G is called a $\gamma(G)$-set.

A set $D \subseteq V(G)$ is a *restrained dominating set of* G (RDS) if D is a dominating set and the induced subgraph $\langle V(G) - D \rangle$ does not contain an isolated vertex. The cardinality of a minimum restrained dominating set in G is the *restrained domination number* and is denoted by $\gamma_r(G)$. A minimum RDS of a graph G is called a $\gamma_r(G)$-set. The concept of restrained domination was introduced by Telle and Proskurowski [6], albeit indirectly, as a vertex partitioning problem. Restrained domination was studied further for example by Domke *et al.* [1, 2].

The total restrained domination number of a graph was defined by Ma, Chen and Sun [5]. A set $D \subseteq V(G)$ is a *total restrained dominating set of* G (TRDS) if it is a dominating set and the induced subgraphs $\langle D \rangle$ and $\langle V(G) - D \rangle$ do not contain isolated vertices. The cardinality of a minimum total restrained dominating set in G is the *total restrained domination number* and is denoted by $\gamma_r^t(G)$. A minimum TRDS of a graph G is called a $\gamma_r^t(G)$-set. We note that every graph G without an isolated vertex has a (total) restrained dominating set, since $D = V(G)$ is such a set.

For any graph theoretical parameters λ and μ, we define G to be (λ, μ)-graph if $\lambda(G) = \mu(G)$. Henning has wrtitten an extensive series of papers which give constructive characterizations of trees for which certain domination parameters are equal (see, for example [4]). In this paper we provide a constructive characterization of (γ_r, γ_r^t)-trees. For any unexplained terms and symbols see [3].

2. A Characterization of (γ_r, γ_r^t)-Trees

As a consequence of the definitions of the restrained and total restrained domination numbers we have the following observations.

Observation 1. Let G be a graph without isolated vertices. Then

(i) every leaf is in every $\gamma_r^t(G)$-set;

(ii) every support vertex is in every $\gamma_r^t(G)$-set;

(iii) every leaf is in every $\gamma_r(G)$-set;

(iv) $\gamma(G) \le \gamma_r(G) \le \gamma_r^t(G)$.

Observation 2. Let T be a (γ_r, γ_r^t)-tree. Then each $\gamma_r^t(T)$-set is a $\gamma_r(T)$-set.

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the following two operations defined on a tree T.

- **Operation** $\mathcal{T}_1$. Assume $x \in V(T)$ is a support vertex. Then add a vertex y and the edge xy.
- **Operation** $\mathcal{T}_2$. Assume $x \in V(T)$ is a support vertex. Then add a path $P_4 = (y_1, y_2, y_3, y_4)$ and the edge xy_1.

Let $\mathcal{T}$ be the family of trees such that $\mathcal{T} = \{T : T$ is obtained from P_3 by a finite sequence of Operations $\mathcal{T}_1$ or $\mathcal{T}_2\} \cup \{P_2, P_6\}$. We show first that each tree in the family $\mathcal{T}$ has equal restrained domination number and total restrained domination number.

Lemma 3. *If T belongs to the family $\mathcal{T}$, then T is a (γ_r, γ_r^t)-tree.*

Proof. We proceed by induction on the number $s(T)$ of operations required to construct the tree T. If $s(T) = 0$, then $T \in \{P_2, P_3, P_6\}$ and clearly T is a (γ_r, γ_r^t)-tree. Assume now that T is a tree with $s(T) = k$ for some positive integer k and each tree $T' \in \mathcal{T}$ with $s(T') < k$ is a (γ_r, γ_r^t)-tree. Then T can be obtained from a tree T' belonging to $\mathcal{T}$ by operation $\mathcal{T}_1$ or $\mathcal{T}_2$. We now consider two possibilities depending on whether T is obtained from T' by Operation $\mathcal{T}_1$ or $\mathcal{T}_2$.

Case 1. T is obtained from T' by Operation $\mathcal{T}_1$. Suppose T is obtained from T' by adding a vertex y and the edge xy, where $x \in V(T')$ is a support vertex. Thus y belongs to every $\gamma_r(T)$-set and every $\gamma_r^t(T)$-set. Hence $\gamma_r(T) = \gamma_r(T') + 1$ and $\gamma_r^t(T) = \gamma_r^t(T') + 1$. Since $\gamma_r(T') = \gamma_r^t(T')$ and $\gamma_r(T) \le \gamma_r^t(T)$, we conclude that $\gamma_r(T) = \gamma_r^t(T)$.

Case 2. T is obtained from T' by Operation $\mathcal{T}_2$. Suppose T is obtained from T' by adding a path (y_1, y_2, y_3, y_4) and the edge xy_1, where $x \in V(T')$ is a support vertex. Then x and y_3 are support vertices in T and y_4 is a leaf. Hence x, y_3 and y_4 belong to every $\gamma_r^t(T)$-set and for this reason

$\gamma_r^t(T) \geq \gamma_r^t(T') + 2$. On the other hand, any $\gamma_r^t(T')$-set may be extended to a TRDS of T by adding to it y_3 and y_4. Thus $\gamma_r^t(T) = \gamma_r^t(T') + 2$.

Now let D be a $\gamma_r(T)$-set. Then $y_4 \in D$ and $N_T[y_2] \cap D \neq \emptyset$. For this reason $\gamma_r(T) \geq \gamma_r(T') + 2$. On the other hand, $\gamma_r(T) \leq \gamma_r^t(T) = \gamma_r^t(T') + 2 = \gamma_r(T') + 2$. We conclude that $\gamma_r(T) = \gamma_r(T') + 2$ and consequently, $\gamma_r(T) = \gamma_r^t(T)$. ∎

We now show that every (γ_r, γ_r^t)-tree belongs to the family $\mathcal{T}$. It is clear that P_2 is a (γ_r, γ_r^t)-tree and P_2 belongs to the family $\mathcal{T}$. Therefore from now on we consider only trees T with $n(T) \geq 3$.

Lemma 4. *Let T be a (γ_r, γ_r^t)-tree with $n(T) \geq 3$ and let D_r^t be a minimum total restrained dominating set of T. If $u, v \in D_r^t$ and $uv \in E(T)$, then either u or v is a leaf.*

Proof. It is possible to see that the statement is true for all trees T with diameter 2 and 3. For this reason we consider only trees with diameter at least 4. Suppose T is a (γ_r, γ_r^t)-tree, $u, v \in D_r^t$, $uv \in E(T)$ and neither u nor v is a leaf. We consider three cases.

Case 1. u is an isolated vertex in $\langle (V(T) - D_r^t) \cup \{u\} \rangle$ and v is an isolated vertex in $\langle (V(T) - D_r^t) \cup \{v\} \rangle$. Since neither u nor v is a leaf, we conclude that $D_r^t - \{u, v\}$ is a RDS of T of cardinality smaller than $\gamma_r(T)$, a contradiction.

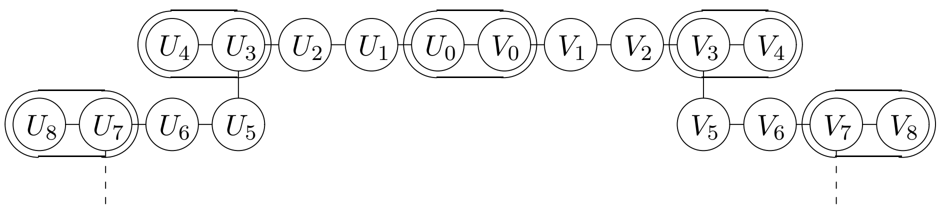

Figure 1. Illustration for *Case* 2 of the proof of Lemma 4.

Case 2. Both $\langle (V(T) - D_r^t) \cup \{u\} \rangle$ and $\langle (V(T) - D_r^t) \cup \{v\} \rangle$ are without isolated vertices. Then since T is a (γ_r, γ_r^t)-tree, we conclude that $D_r^t - \{u\}$ and $D_r^t - \{v\}$ are not dominating sets of T. Therefore, both u and v have a private neighbour with respect to D_r^t. Let $U_0 = \{u\}$ and $V_0 = \{v\}$ and denote by U_1 and V_1 the sets of private neighbours of u and v with respect to D_r^t, respectively. Of course, $U_1 \cap V_1 = \emptyset$ and $U_1 \cup V_1$ is an independent set of vertices, because T is a tree. Since D_r^t is a TRDS, each vertex of $U_1 \cup V_1$ has a neighbour in $V(T) - D_r^t$. Denote by U_2 and V_2 the sets of all vertices of $V(T) - D_r^t$ which are neighbours of vertices of U_1 and V_1, respectively. Observe that $U_2 \cap V_2 = \emptyset$, $U_1 \cap U_2 = \emptyset$, $V_1 \cap V_2 = \emptyset$ and $U_2 \cup V_2$ is an independent set of vertices. Since T is a tree, no two vertices of $U_1 \cup V_1$ have common neighbour in $U_2 \cup V_2$, so $|U_1| \leq |U_2|$ and $|V_1| \leq |V_2|$. Moreover, since D_r^t is a dominating set of T, each vertex of $U_2 \cup V_2$ has a neighbour in D_r^t. Denote by U_3 and V_3 the sets of all vertices belonging to D_r^t which are neighbours of vertices of U_2 and V_2, respectively. Since T is a tree, $(U_3 \cup V_3) \cap \{u, v\} = \emptyset$, $U_3 \cap V_3 = \emptyset$, $U_3 \cup V_3$ is an independent set of vertices, $|U_2| \leq |U_3|$ and $|V_2| \leq |V_3|$. Finally, since D_r^t is a TRDS of T, each vertex of $U_3 \cup V_3$ has a neighbour in D_r^t. Denote by U_4 and V_4 the sets of all vertices belonging to D_r^t which are neighbours of vertices of U_3 and V_3, respectively. Since T is a tree, $(U_4 \cup V_4) \cap \{u, v\} = \emptyset$, $(U_4 \cup V_4) \cap (U_3 \cup V_3) = \emptyset$, $U_4 \cap V_4 = \emptyset$, $U_4 \cup V_4$ is an independent set of vertices, $|U_3| \leq |U_4|$ and $|V_3| \leq |V_4|$. Define U_5 to be the set of vertices of $V(T) - U_2$ which are private neighbours with respect to D_r^t of vertices belonging to U_3 and define V_5 to be the set of vertices of $V(T) - V_2$ which are private neighbours with respect to D_r^t of vertices belonging to V_3. Denote by U_6 and V_6 the sets of all vertices of $V(T) - D_r^t$ which are neighbours of vertices of U_5 and V_5, respectively, and so on.

Generally, let k be a non-negative integer. Define U_{4k+5} to be the set of vertices of $V(T) - U_{4k+2}$ which are private neighbours with respect to D_r^t of vertices belonging to U_{4k+3} and define V_{4k+5} to be the set of vertices of $V(T) - V_{4k+2}$ which are private neighbours with respect to D_r^t of vertices belonging to V_{4k+3}. Since D_r^t is a TRDS, each vertex of $U_{4k+1} \cup V_{4k+1}$, where $k \geq 0$, has a neighbour in $V(T) - D_r^t$. Let U_{4k+2} be the set of all vertices of $V(T) - D_r^t$ which are neighbours of vertices of U_{4k+1} and let V_{4k+2} be the set of all vertices of $V(T) - D_r^t$ which are neighbours of vertices of V_{4k+1}. Since D_r^t is a dominating set, each vertex of $U_{4k+2} \cup V_{4k+2}$ has a neighbour in D_r^t. Denote by U_{4k+3} the set of all vertices belonging to D_r^t which are neighbours of vertices of U_{4k+2} and denote by V_{4k+3} the set of all vertices belonging to D_r^t which are neighbours of vertices of V_{4k+2}. Finally, since D_r^t is a TRDS of T, each vertex of $U_{4k+3} \cup V_{4k+3}$ has a neighbour in D_r^t. Denote by U_{4k+4} and V_{4k+4} the sets of all vertices belonging to D_r^t which are neighbours of vertices of U_{4k+3} and V_{4k+3}, respectively. Since T is a finite tree, there exist the smallest integer i such that $U_{4i+5} = \emptyset$ and the smallest integer j such that $V_{4j+5} = \emptyset$.

Since T is a tree, we conclude that no two vertices of $U_{4k+1} \cup V_{4k+1}$ have common neighbour in $U_{4k+2} \cup V_{4k+2}$. This implies that $|U_{4k+1}| \leq |U_{4k+2}|$ and $|V_{4k+1}| \leq |V_{4k+2}|$. Similarly, $|U_{4k+2}| \leq |U_{4k+3}|$ and $|V_{4k+2}| \leq |V_{4k+3}|$. Further, $|U_{4k+3}| \leq |U_{4k+4}|$ and $|V_{4k+3}| \leq |V_{4k+4}|$. Moreover, every two of defined sets are disjoint.

Now consider the set $D = D_r^t - (U_3 \cup U_7 \cup \cdots \cup U_{4i+3} \cup V_3 \cup V_7 \cup \cdots \cup V_{4j+3} \cup \{u, v\}) \cup U_1 \cup U_5 \cup \cdots \cup U_{4i+1} \cup V_1 \cup V_5 \cup \cdots \cup V_{4j+1}$. It is possible to observe that D is a dominating set of T and $\langle V(T) - D \rangle$ does not contain an isolated vertex. Hence D is a RDS of T. Moreover $|D| < |D_r^t|$, which implies that T is not a (γ_r, γ_r^t)-tree, a contradiction.

Case 3. Either $\langle (V(T) - D_r^t) \cup \{u\} \rangle$ or $\langle (V(T) - D_r^t) \cup \{v\} \rangle$ contains an isolated vertex, say u is an isolated vertex in $\langle (V(T) - D_r^t) \cup \{u\} \rangle$. Then since T is a (γ_r, γ_r^t)-tree, we conclude that $D_r^t - \{v\}$ is not a dominating set of T. Let j and $V_0, V_1, \ldots, V_{4j+5}$ have the same meaning and properties as in previous case. Consider the set $D = D_r^t - (V_3 \cup V_7 \cup \cdots \cup V_{4j+3} \cup \{u, v\}) \cup V_1 \cup V_5 \cup \cdots \cup V_{4j+1}$. It is easy to observe that D is a dominating set of T and $\langle V(T) - D \rangle$ does not contain an isolated vertex. Hence D is a RDS of T. Moreover $|D| < |D_r^t|$, which implies that T is not a (γ_r, γ_r^t)-tree, a contradiction.

This proves the statement. ■

The above Lemma together with Lemma 1 imply what follows.

Corollary 5. *If T is a (γ_r, γ_r^t)-tree with $n(T) \geq 3$, then $\Omega(T) \cup S(T)$ is the unique $\gamma_r^t(T)$-set and $\gamma_r(T) = \gamma_r^t(T) = |\Omega(T) \cup S(T)|$.*

Corollary 6. *If T is a (γ_r, γ_r^t)-tree with $n(T) \geq 3$, then $S(T)$ is a $\gamma(T)$-set and $\gamma(T) = |S(T)|$.*

Corollary 7. *If T is a (γ_r, γ_r^t)-tree with $n(T) \geq 3$, then $\gamma_r^t(T) = \gamma(T) + |\Omega(T)|$.*

Lemma 8. *Let T be a (γ_r, γ_r^t)-tree with $n(T) \geq 3$. If $u, v \in S(T)$, then $d_T(u, v) \geq 3$.*

Proof. It is possible to verify that the statement is true for all trees with diameter between 2 and 5. For this reason we consider only trees with diameter at least 6.

Let T be a (γ_r, γ_r^t)-tree with $n(T) \geq 3$ and let D_r^t be a $\gamma_r^t(T)$-set. By Corollary 5, $u, v \in D_r^t$ and by Lemma 4, u and v are not adjacent. Suppose that $d_T(u, v) = 2$ and let x be the neighbour of u and v in T. Lemma 4 implies that x is not a support vertex and as x is not a leaf, $x \notin D_r^t$. Since both $\langle (V(T) - D_r^t) \cup \{u\} \rangle$ and $\langle (V(T) - D_r^t) \cup \{v\} \rangle$ are without isolated vertices and T is a (γ_r, γ_r^t)-tree, we deduce that $D_r^t - \{u\}$ and $D_r^t - \{v\}$ are not dominating sets of T. Therefore, both u and v have a private neighbour with respect to D_r^t. Let j and $V_0, V_1, \ldots, V_{4j+5}$ have the same meaning and properties as in the proof of Lemma 4. Consider the set $D = D_r^t - (V_3 \cup V_7 \cup \cdots \cup V_{4j+3} \cup \{v\}) \cup V_1 \cup V_5 \cup \cdots \cup V_{4j+1}$. It is possible to observe that D is a dominating set of T and $\langle V(T) - D \rangle$ does not contain an isolated vertex. Hence D is a RDS of T. Moreover $|D| < |D_r^t|$, which implies that T is not a (γ_r, γ_r^t)-tree, a contradiction. ■

Corollary 9. *If T is a (γ_r, γ_r^t)-tree with $n(T) \geq 3$, then each vertex of $V(T) - S(T)$ has exactly one neighbour in $S(T)$.*

Corollary 10. *If T is a (γ_r, γ_r^t)-tree with $n(T) \geq 3$, then $S(T)$ is the unique $\gamma(T)$-set.*

Lemma 11. *If T is a (γ_r, γ_r^t)-tree with $n(T) \geq 3$, then T belongs to the family $\mathcal{T}$.*

Proof. It is easily seen that the statement is true for all trees with with diameter between 2 and 5. For this reason we consider only trees with diameter at least 6.

Let T be a (γ_r, γ_r^t)-tree and assume that the result holds for all trees on $n(T) - 1$ and fewer vertices. We proceed by induction on the number of vertices of a (γ_r, γ_r^t)-tree. Let $P = (s_0, s_1, \ldots s_l)$, $l \geq 6$, be a longest path in T and let D_r^t be a $\gamma_r^t(T)$-set. We consider two cases.

Case 1. $d_T(s_1) > 2$. In this case s_1 is a neighbour of at least two leaves of T. Denote $T' = T - s_0$. Of course $D_r^t - \{s_0\}$ is a TRDS of T', so $\gamma_r^t(T') \leq \gamma_r^t(T) - 1$. Moreover, any $\gamma_r^t(T')$-set may be extended to a $\gamma_r^t(T)$-set by adding to it s_0, so $\gamma_r^t(T') = \gamma_r^t(T) - 1$. By similar arguments it may be concluded that $\gamma_r(T') = \gamma_r(T) - 1$. Hence, $\gamma_r(T') = \gamma_r^t(T')$. Consequently, T' is a (γ_r, γ_r^t)-tree and by induction hypothesis, $T' \in \mathcal{T}$. As s_1 is a support vertex in T', we deduce that T may be obtained from T' by Operation $\mathcal{T}_1$.

Case 2. $d_T(s_1) = 2$. Then Corollary 5 and Lemma 8 imply that $d_T(s_2) = 2$ and s_3 is not a support vertex. Moreover, s_3 is a neighbour of exactly one support vertex, say x.

Suppose that $x \neq s_4$. Then s_4 is not a support vertex, but s_4 is a neighbour of exactly one support vertex, say y. Denote $A = N_T(s_3) - \{x\} - V(P)$ and observe that since x is a support vertex, Lemma 8 implies that $A \cap S(T) = \emptyset$. Corollary 9 says that each vertex of A has exactly one neighbour in $S(T)$. Let A' be the set of neighbours of vertices of A which belong to $S(T)$. Hence $s_0, s_1, x, y \in D_r^t$ and $s_2, s_3, s_4 \notin D_r^t$. Consider the set $D = D_r^t - \{s_1, y\} - A' \cup \{s_3\}$. It is easy to observe that D is a dominating set in T and $\langle V(T) - D\rangle$ does not contain an isolated vertex. Hence D is a RDS of T. Moreover $|D| < |D_r^t|$ even when $A = \emptyset$, which implies that T is not a (γ_r, γ_r^t)-tree, a contradiction. Therefore s_4 is the unique support vertex in $N_T(s_3)$.

Now suppose that $d_T(s_3) > 2$. Denote $A = N_T(s_3) - V(P)$ and observe that since $d_T(s_3) > 2$, $A \neq \emptyset$. Moreover, since s_4 is a support vertex, $A \cap S(T) = \emptyset$. Let A' be the set of neighbours of vertices of A which belong to $S(T)$. Then $s_0, s_1, s_4 \in D_r^t$ and $s_2, s_3 \notin D_r^t$. Consider the set $D = ((D_r^t - \{s_1\}) - A') \cup \{s_3\}$. It is easy to observe that D is a dominating set of T and $\langle V(T) - D\rangle$ does not contain an isolated vertex. Hence D is a RDS of T. Moreover $|D| < |D_r^t|$, which implies that T is not a (γ_r, γ_r^t)-tree, a contradiction. Therefore $d_T(s_3) = 2$ and s_4 is the unique neighbour of s_3 belonging to $S(T)$.

Denote $T' = T - \{s_0, s_1, s_2, s_3\}$. Of course s_0 and s_1 belong to every $\gamma_r^t(T)$-set. For this reason, $\gamma_r^t(T') \leq \gamma_r^t(T) - 2$. Since s_4 is a support vertex in T', any $\gamma_r^t(T')$-set may be extended to a TRDS of T by adding to it s_0 and s_1, so $\gamma_r^t(T') = \gamma_r^t(T) - 2$. Further, $\gamma_r(T') \leq \gamma_r^t(T') = \gamma_r^t(T) - 2 = \gamma_r(T) - 2$ and any $\gamma_r(T')$-set may be extended to a RDS of T by adding to it s_0 and s_3. Hence $\gamma_r(T') = \gamma_r(T) - 2$ and so $\gamma_r(T') = \gamma_r^t(T')$. Consequently, T' is a (γ_r, γ_r^t)-tree and by induction hypothesis, $T' \in \mathcal{T}$. As s_4 is a support vertex in T', we conclude that T may be obtained from T' by Operation $\mathcal{T}_2$. ∎

As an immediate consequence of Lemmas 4 and 11 we have the following characterization of (γ_r, γ_r^t)-trees.

Theorem 12. *A tree T is a (γ_r, γ_r^t)-tree if and only if T belongs to the family $\mathcal{T}$.*

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