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Existence of solutions of boundary value problems for differential equations in which deviated arguments depend on the unknown solution

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Abstract

This paper concerns differential equations with boundary conditions. Given are sufficient conditions under which such problems with deviated arguments have a unique solution in a corresponding sector. To obtain existence results we apply a monotone iterative method.

Keywords: Monotone iterative technique; Equations with deviated arguments; Monotone sequences; Convergence; Existence of solutions

1. Introduction

In this paper, we deal with the following problem

$$\begin{cases} x'(t) = f(t, x(\beta(t, x(t)))) \equiv F(x, x)(t), & t \in J, \\ x(0) = \lambda x(T) + k, \end{cases} \tag{1}$$

where

$$F(x, y)(t) = f(t, x(\beta(t, y(t)))) \tag{2}$$

and $J = [0, T]$, $f \in C(J \times \mathbb{R}, \mathbb{R})$, $\beta \in C(J \times \mathbb{R}, \mathbb{R})$, $\lambda, k \in \mathbb{R}$.

If $\lambda = 1$ and $k = 0$, then we have the periodic boundary condition, if $\lambda = -1$ and $k = 0$, then we have the antiperiodic boundary condition, and if $\lambda = 0$, we have an initial condition as special cases of the boundary condition in (1).

To obtain existence results for differential problems someone may use the monotone iterative method, for details see for example [1]. There is a vast literature devoted to the applications of this method to differential equations with initial and boundary conditions. It can be applied to differential problems with deviated arguments, see for example the papers [2–8]. We also apply this technique to problem (1). It is important to indicate that (1) is different from

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corresponding problems investigated in the papers published earlier. Note that in problem (1) a deviated argument β depends on the unknown solution x. It is the first paper when the monotone iterative method is applied to problems of type (1).

The plan of this paper is as follows. Section 2 concerns the case when a parameter $\lambda \geq 0$, while in Section 3 we discuss problem (1) when $\lambda < 0$. In both sections, we formulate sufficient conditions when problem (1) has a unique solution in a corresponding sector. In Section 2, an example is added to illustrate imposed assumptions. A problem more general then (1) is discussed in Section 4.

2. Case $\lambda \geq 0$

Take $y_0, z_0 \in C^1(J, \mathbb{R})$ such that $y_0(t) \leq z_0(t), t \in J$. Let

$$\Omega = \{(t, u) : y_0(t) \leq u \leq z_0(t), t \in J\}.$$

A pair $u, v \in C^1(J, \mathbb{R})$ is called a lower–upper solution of problem (1) for $\lambda \geq 0$ if

$$\begin{cases} u'(t) \leq F(v, v)(t), & t \in J, \quad u(0) \leq \lambda u(T) + k, \\ v'(t) \geq F(u, u)(t), & t \in J, \quad v(0) \geq \lambda v(T) + k. \end{cases}$$

Let us define two sequences $\{y_n, z_n\}$ by relations:

$$\begin{cases} y'_{n+1}(t) = F(z_n, z_n)(t), & t \in J, \quad y_{n+1}(0) = \lambda y_n(T) + k, \\ z'_{n+1}(t) = F(y_n, y_n)(t), & t \in J, \quad z_{n+1}(0) = \lambda z_n(T) + k \end{cases} \tag{3}$$

for $n = 0, 1, \ldots$. Functions y_0, z_0 will be defined later.

A pair $X, Y \in C^1(J, \mathbb{R})$ is called a quasi-solution of (1) if

$$\begin{cases} X'(t) = F(Y, Y)(t), & t \in J, \quad X(0) = \lambda X(T) + k, \\ Y'(t) = F(X, X)(t), & t \in J, \quad Y(0) = \lambda Y(T) + k. \end{cases}$$

A pair $\rho, \gamma \in C^1(J, \mathbb{R})$ is called the minimal and maximal quasi-solution of problem (1) if for any $U, V \in C^1(J, \mathbb{R})$ quasi-solution of (1) we have $\rho(t) \leq U(t), V(t) \leq \gamma(t)$ on J.

Theorem 1. *Assume that*

(H_1) $f \in C(J \times \mathbb{R}, \mathbb{R}), \beta \in C(J \times \mathbb{R}, \mathbb{R})$, *and f is nonincreasing with respect to the last variable,*
(H_2) *a pair $y_0, z_0 \in C^1(J, \mathbb{R})$ is a lower–upper solution of problem (1) for $\lambda \geq 0$, and $y_0(t) \leq z_0(t)$ on J.*
(H_3) $\beta : \Omega \to J, \beta(t, u)$ *is nondecreasing with respect to u for $y_0(t) \leq u \leq z_0(t), t \in J$,*
(H_4) y_0, z_0 *are nondecreasing on J and $f(t, u) \geq 0$ for $t \in J, y_0 \leq u \leq z_0$.*

Then problem (1) has the minimal and maximal quasi-solution in the sector

$$[y_0, z_0]_* = \{u \in C^1(J, \mathbb{R}) : y_0(t) \leq u(t) \leq z_0(t), t \in J\}.$$

Proof. Note that $y_0(t) \leq y_1(t), z_1(t) \leq z_0(t)$ on J. Put $p = y_1 - z_1$. Then $p(0) \leq 0$, and $p'(t) = F(z_0, z_0)(t) - F(y_0, y_0)(t) \leq 0$ because

$$y_0(\beta(t, y_0(t))) \leq y_0(\beta(t, z_0(t))) \leq z_0(\beta(t, z_0(t))).$$

It shows that

$$y_0(t) \leq y_1(t) \leq z_1(t) \leq z_0(t), \quad t \in J.$$

Moreover, in view of assumptions (H_3), (H_4), we have

$$y'_1(t) = F(z_0, z_0)(t) - F(z_1, z_1)(t) + F(z_1, z_1)(t) \leq F(z_1, z_1)(t),$$
$$z'_1(t) = F(y_0, y_0)(t) - F(y_1, y_1)(t) + F(y_1, y_1)(t) \geq F(y_1, y_1)(t)$$

because y_0, z_0 are nondecreasing and

$$z_0(\beta(t, z_0(t))) \geq z_1(\beta(t, z_1(t))), \qquad y_0(\beta(t, y_0(t))) \leq y_1(\beta(t, y_1(t))).$$

By induction, we can show that

$$y_0(t) \le y_1(t) \le \cdots \le y_n(t) \le z_n(t) \le \cdots \le z_1(t) \le z_0(t)$$

for $t \in J$ and $n = 0, 1, \ldots$.

By the Arzeli theorem, $y_n \to y, z_n \to z$, where the pair $y, z \in C^1(J, \mathbb{R})$ is a quasi-solution of problem (1) and $y_0(t) \le y(t) \le z(t) \le z_0(t), t \in J$. Now, we need to show that the pair y, z is the minimal and maximal quasi-solution of (1) in the sector $[y, z_0]_*$. Let $u, v \in [y_0, z_0]_*$ be any quasi-solution of problem (1). Put $p = y_1 - u, q = v - z_1$. Then $p(0) \le 0, q(0) \le 0$, and

$$p'(t) = F(z_0, z_0)(t) - F(u, u)(t) \le 0,$$
$$q'(t) = F(v, v)(t) - F(y_0, y_0)(t) \le 0$$

because

$$z_0(\beta(t, z_0(t))) \ge z_0(\beta(t, u(t))) \ge u(\beta(t, u(t))).$$
$$y_0(\beta(t, y_0(t))) \le y_0(\beta(t, v(t))) \le v(\beta(t, v(t))),$$

Hence $y_1(t) \le u(t), v(t) \le z_1(t), t \in J$. By induction, we can prove that $y_n(t) \le u(t)$ and $v(t) \le z_n(t), t \in J, n = 0, 1, \ldots$. If $n \to \infty$, then we have the assertion of Theorem 1. ■

It is easy to show the following.

Remark 1. Let all assumptions of Theorem 1 hold. If u is any solution of (1) such that $y_0(t) \le u(t) \le z_0(t), t \in J$, then

$$y_n(t) \le u(t) \le z_n(t), \quad t \in J, \ n = 0, 1, \ldots$$

and $y(t) \le u(t) \le z(t), t \in J$, where y, z are from Theorem 1.

Now, we want to formulate sufficient conditions under which problem (1) has a unique solution. First we give the following.

Lemma 1. *Assume that* $\beta \in C(\Omega, J)$, $K, L \in C(J, \mathbb{R}_+)$, $R_+ = [0, \infty)$, $p \in C^1(J, \mathbb{R})$ *and*

$$p'(t) \le K(t)p(t) + L(t)p(\beta(t, w(t))), \quad t \in J, \qquad p(0) = \lambda p(T), \quad \lambda \in [0, 1) \tag{4}$$

for $y_0(t) \le w(t) \le z_0(t), t \in J$. *In addition assume that for* $L^*(t) = K(t) + L(t)$ *we have*

$$\lambda + \int_0^T L^*(t)\mathrm{d}t < 1. \tag{5}$$

Then $p(t) \le 0, t \in J$.

Proof. Suppose that the assertion $p(t) \le 0, t \in J$ is not true. Then, we can find $t_0 \in J$ such that $p(t_0) > 0$. Put

$$p(t_1) = \max_{t \in J} p(t) > 0.$$

Integrating the differential inequality in (4) we obtain

$$p(t) \le p(0) + p(t_1) \int_0^T L^*(s)\mathrm{d}s, \quad t \in J. \tag{6}$$

Then

$$p(0) = \lambda p(T) \le \lambda \left[p(0) + p(t_1) \int_0^T L^*(s)\mathrm{d}s \right].$$

This gives

$$p(0) \le \frac{\lambda}{1 - \lambda} p(t_1) \int_0^T L^*(s)\mathrm{d}s.$$

This and (6) for $t = t_1$ yield

$$p(t_1)\left[1 - \frac{1}{1-\lambda}\int_0^T L^*(s)\mathrm{d}s\right] \le 0.$$

It contradicts the assumption that $p(t_1) > 0$. This shows that $p(t) \le 0$ on J and the proof is complete. ■

Theorem 2. *Let all assumptions of Theorem 1 hold. In addition assume that*

(H_5) *there exists functions $L, M \in C(J, R_+)$, such that*

$$f(t,u) - f(t,\bar{u}) \le L(t)(\bar{u} - u),$$
$$\beta(t,\bar{v}) - \beta(t,v) \le M(t)(\bar{v} - v)$$

if $y_0(t) \le u \le \bar{u} \le z_0(t)$, $y_0(t) \le v \le \bar{v} \le z_0(t)$, $t \in J$,

(H_6) *condition (5) holds for $L^*(t) = L(t)M(t)N(t) + L(t)$, where $f(t,w)$ is bounded by $N(t)$ for $t \in J$, $y_0 \le w \le z_0$.*

Then problem (1) has, in the sector $[y_0, z_0]_*$, a unique solution.

Proof. From Theorem 1 we know that $y, z \in [y_0, z_0]_*$, and $y(t) \le z(t)$, $t \in J$. We need to show that $y = z$. Put $q = z - y$, so $p(0) = \lambda p(T)$ and

$$\begin{aligned} p'(t) &= F(y,y)(t) - F(z,z)(t) \le L(t)[z(\beta(t,z(t))) - y(\beta(t,y(t)))] \\ &= L(t)[p(\beta(t,z(t))) + y(\beta(t,z(t))) - y(\beta(t,y(t)))] \\ &\le K(t)p(t) + L(t)p(\beta(t,z(t))) \quad \text{for } K(t) = L(t)M(t)N(t). \end{aligned}$$

This and Lemma 1 show that $z(t) \le y(t)$, $t \in J$. It means that $y = z$. ■

Example. We consider the following boundary value problem

$$\begin{cases} x'(t) = \gamma_1 \mathrm{e}^{-\gamma_2 x(\delta t x(t))}, & t \in J = [0,1], \\ x(0) = \lambda x(1) + k, & \lambda \ge 0, \end{cases} \tag{7}$$

where $0 < \delta \le \frac{1}{2}$, $0 < \gamma_1 \le 1$, $\gamma_2 > 0$. Here $\beta(t,u) = \delta t u$.

Take $y_0(t) = 0$, $z_0(t) = t + 1$, $t \in J$ and $0 \le k \le 2\lambda + k \le 1$. We see that $0 \le \beta(t,u) \le t$ for $y_0(t) \le u \le z_0(t)$, $t \in J$. Note that

$$F(z_0,z_0)(t) = \gamma_1 \mathrm{e}^{-\gamma_2(1+\delta t(1+t))} > 0 = y_0'(t), \qquad \lambda y_0(1) + k = k \ge 0 = y_0(0),$$
$$F(y_0,y_0)(t) = \gamma_1 \le 1 = z_0'(t), \qquad \lambda z_0(1) + k = 2\lambda + k \le 1 = z_0(0)$$

It proves that the pair (y_0, z_0) is a lower–upper solution of problem (7).

Moreover, $L(t) = \gamma_1\gamma_2$, $M(t) = \delta t$, $N(t) = \gamma_1$. In addition assume that

$$\lambda + \gamma_1\gamma_2\left(1 + \frac{1}{2}\gamma_1\delta\right) < 1. \tag{8}$$

Then problem (7) has, in the sector $[y_0, z_0]_*$, a unique solution, by Theorem 2. For example, if we take $\gamma_1 = \delta = \frac{1}{2}$, $\gamma_2 = 1$, the condition (8) holds for $\lambda < \frac{7}{16}$.

Now we consider the case when function β is nonincreasing with respect to the second variable. We have

Theorem 3. *Assume that assumptions* (H_1), (H_2'), (H_3'), (H_4'), (H_5'), (H_6) *are satisfied where*

(H_2') $\lambda \ge 0$, $u_0, w_0 \in C^1(J, \mathbb{R})$, $u_0(t) \le w_0(t)$, $t \in J$ *and*

$$\begin{cases} u_0'(t) \le F(w_0,u_0)(t), & t \in J, & u_0(0) \le \lambda u_0(T) + k, \\ w_0'(t) \ge F(u_0,w_0)(t), & t \in J, & w_0(0) \ge \lambda w_0(T) + k, \end{cases}$$

(H_3') $\beta : \bar{\Omega} \to J$, $\beta(t,u)$ *is nonincreasing with respect to u for $t \in J$, $u_0 \le u \le w_0$, $t \in J$, where* $\bar{\Omega} = \{(t,u) : u_0(t) \le u \le w_0(t), t \in J\}$,

(H_4') *u_0, w_0 are nondecreasing on J and $f(t,u) \ge 0$ for $t \in J$, $u_0 \le u \le w_0$,*
(H_5') *there exist functions $L, M \in C(J, R_+)$, such that*

$$f(t,u) - f(t,\bar{u}) \le L(t)(\bar{u} - u),$$
$$\beta(t,v) - \beta(t,\bar{v}) \le M(t)(\bar{v} - v)$$

if $u_0(t) \le u \le \bar{u} \le w_0(t)$, $u_0(t) \le v \le \bar{v} \le w_0(t)$, $t \in J$.

Then, problem (1) has, in the sector $[u_0, w_0]_*$, a unique solution.

Proof. Let us define the sequences $\{u_n, w_n\}$ be relations

$$\begin{cases} u_{n+1}'(t) = F(w_n, u_n)(t), & t \in J, \qquad u_{n+1}(0) = \lambda u_n(T) + k, \\ w_{n+1}'(t) = F(u_n, w_n)(t), & t \in J, \qquad w_{n+1}(0) = \lambda w_n(T) + k \end{cases}$$

for $n = 0, 1, \ldots$. The proof of this theorem is similar to the proof of Theorems 1 and 2, and therefore it is omitted. ■

3. Case $\lambda < 0$

A pair $u, v \in C^1(J, \mathbb{R})$ is called a lower–upper solution of problem (1) for $\lambda < 0$ if

$$\begin{cases} u'(t) \le F(v,v)(t), & t \in J, \qquad u(0) \le \lambda v(T) + k, \\ v'(t) \ge F(u,u)(t), & t \in J, \qquad v(0) \ge \lambda u(T) + k. \end{cases}$$

Theorem 4. *Let all assumptions of Theorems 1 and 2 be satisfied with (H_2'') instead of (H_2), where*

(H_2'') *a pair $y_0, z_0 \in C^1(J, \mathbb{R})$ is a lower–upper solution of problem (1) for $\lambda < 0$, and $y_0(t) \le z_0(t)$ on J.*

Then the assertion of Theorem 2 holds.

Proof. For $n = 0, 1, \ldots$, let us define the sequences $\{y_n, z_n\}$ by relations

$$\begin{cases} y_{n+1}'(t) = F(z_n, z_n)(t), & t \in J, \qquad y_{n+1}(0) = \lambda z_n(T) + k, \\ z_{n+1}'(t) = F(y_n, y_n)(t), & t \in J, \qquad z_{n+1}(0) = \lambda y_n(T) + k. \end{cases}$$

Repeating the proof of Theorems 1 and 2, we have the assertion of Theorem 4. ■

Theorem 5. *Let all assumptions of Theorem 3 be satisfied with (H'''_2) instead of (H_2'),*

(H_2''') $\lambda < 0$, $u_0, w_0 \in C^1(J, \mathbb{R})$, $u_0(t) \le w_0(t)$, $t \in J$, *and*

$$\begin{cases} u_0'(t) \le F(w_0, u_0)(t), & t \in J, \qquad u_0(0) \le \lambda w_0(T) + k, \\ w_0'(t) \ge F(u_0, w_0)(t), & t \in J, \qquad w_0(0) \ge \lambda u_0(T) + k. \end{cases}$$

Then the assertion of Theorem 3 hold.

In the proof use the sequences $\{u_n, w_n\}$ defined by relations.

$$\begin{cases} u_{n+1}'(t) = F(w_n, u_n)(t), & t \in J, \qquad u_{n+1}(0) = \lambda w_n(T) + k, \\ w_{n+1}'(t) = F(u_n, w_n)(t), & t \in J, \qquad w_{n+1}(0) = \lambda u_n(T) + k \end{cases}$$

for $n = 0, 1, \ldots$.

4. General case

Now we consider the problem

$$\begin{cases} x'(t) = f(t, x(\beta(t, x(t))), x(\gamma(t, x(t)))) \equiv \mathcal{F}(x,x,x,x)(t), & t \in J, \\ x(0) = \lambda x(T) + k, \end{cases} \tag{9}$$

where

$$\mathcal{F}(x, y, u, w)(t) = f(t, x(\beta(t, y(t))), u(\gamma(t, w(t)))) \tag{10}$$

and $J = [0, T]$, $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\beta, \gamma \in C(J \times \mathbb{R}, \mathbb{R})$, $\lambda, k \in \mathbb{R}$.

Theorem 6. *Assume that*

(A_1) $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\beta, \gamma \in C(J \times \mathbb{R}, \mathbb{R})$, *and* f *is nonincreasing with respect to the last two variables,*

(A_2) $\lambda \geq 0$, *and* $y_0, z_0 \in C^1(J, \mathbb{R})$ *satisfy the system*

$$\begin{cases} y_0'(t) \leq \mathcal{F}(z_0, z_0, z_0, y_0)(t), & t \in J, \quad y_0(0) \leq \lambda y_0(T) + k, \\ z_0'(t) \geq \mathcal{F}(y_0, y_0, y_0, z_0)(t), & t \in J, \quad z_0(0) \geq \lambda z_0(T) + k \end{cases}$$

and $y_0(t) \leq z_0(t)$, $t \in J$,

(A_3) $\beta, \gamma : \Omega \to J$, $\beta(t, u)$ *is nondecreasing, and* $\gamma(t, u)$ *is nonincreasing, with respect to* u *for* $y_0(t) \leq u \leq z_0(t)$, $t \in J$,

(A_4) y_0, z_0 *are nondecreasing on* J, $f(t, u, v) \geq 0$ *for* $t \in J$, $y_0 \leq u \leq z_0$, $y_0 \leq v \leq z_0$, $t \in J$.

(A_5) *There exist functions* $L_1, L_2, M_1, M_2 \in C(J, R_+)$, *such that*

$$\begin{aligned} &f(t, u, v) - f(t, \bar{u}, \bar{v}) \leq L_1(t)(\bar{u} - u) + L_2(t)(\bar{v} - v), \\ &\beta(t, \bar{v}) - \beta(t, v) \leq M_1(t)(\bar{v} - v) \\ &\gamma(t, w) - \gamma(t, \bar{w}) \leq M_2(t)(\bar{w} - w) \end{aligned}$$

if $y_0(t) \leq u \leq \bar{u} \leq z_0(t)$, $y_0(t) \leq v \leq \bar{v} \leq z_0(t)$, $y_0(t) \leq w \leq \bar{w} \leq z_0(t)$, $t \in J$.

(A_6) *Condition* (5) *holds for* $L^*(t) = N(t)[L_1(t)M_1(t) + L_2(t)M_2(t)] + L_1(t) + L_2(t)$, *where* $f(t, u.w)$ *is bounded by* $N(t)$ *for* $t \in J$, $y_0 \leq u \leq z_0$, $y_0 \leq v \leq z_0$.

Then problem (H_9) *has, in the sector* $[y_0, z_0]_*$, *a unique solution.*

In the proof, use the sequences $\{y_n, z_n\}$ defined by:

$$\begin{cases} y_{n+1}'(t) = \mathcal{F}(z_n, z_n, z_n, y_n)(t), & t \in J, \quad y_{n+1}(0) = \lambda y_n(T) + k, \\ z_{n+1}'(t) = \mathcal{F}(y_n, y_n, y_n, z_n)(t), & t \in J, \quad z_{n+1}(0) = \lambda z_n(T) + k \end{cases}$$

for $n = 0, 1, \ldots$.

Theorem 7. *Let all assumptions of Theorem 6 be satisfied with assumption* (A_2') *instead of* (A_2), *where*

(A_2') $\lambda < 0$, *and* $y_0, z_0 \in C^1(J, \mathbb{R})$ *satisfy the system*

$$\begin{cases} y_0'(t) \leq \mathcal{F}(z_0, z_0, z_0, y_0)(t), & t \in J, \quad y_0(0) \leq \lambda z_0(T) + k, \\ z_0'(t) \geq \mathcal{F}(y_0, y_0, y_0, z_0)(t), & t \in J, \quad z_0(0) \geq \lambda y_0(T) + k \end{cases}$$

and $y_0(t) \leq z_0(t)$, $t \in J$.

Then the assertion of Theorem 6 holds.

Now, the sequences $\{y_n, z_n\}$ are defined by:

$$\begin{cases} y_{n+1}'(t) = \mathcal{F}(z_n, z_n, z_n, y_n)(t), & t \in J, \quad y_{n+1}(0) = \lambda z_n(T) + k, \\ z_{n+1}'(t) = \mathcal{F}(y_n, y_n, y_n, z_n)(t), & t \in J, \quad z_{n+1}(0) = \lambda y_n(T) + k \end{cases}$$

for $n = 0, 1, \ldots$.

Remark 2. There is no problem to formulate corresponding existence results for problems having more arguments of type β and γ.

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