

Existence of solutions of boundary value problems for differential equations in which deviated arguments depend on the unknown solution

Tadeusz Jankowski

Gdansk University of Technology, Department of Differential Equations, 11/12 G.Narutowicz Str., 80-952 Gdańsk, Poland

Received 14 December 2006; accepted 22 January 2007

Abstract

This paper concerns differential equations with boundary conditions. Given are sufficient conditions under which such problems with deviated arguments have a unique solution in a corresponding sector. To obtain existence results we apply a monotone iterative method.

© 2007 Elsevier Ltd. All rights reserved.

Keywords: Monotone iterative technique; Equations with deviated arguments; Monotone sequences; Convergence; Existence of solutions

1. Introduction

In this paper, we deal with the following problem

$$\begin{cases} x'(t) = f(t, x(\beta(t, x(t)))) \equiv F(x, x)(t), & t \in J, \\ x(0) = \lambda x(T) + k, \end{cases} \quad (1)$$

where

$$F(x, y)(t) = f(t, x(\beta(t, y(t)))) \quad (2)$$

and $J = [0, T]$, $f \in C(J \times \mathbb{R}, \mathbb{R})$, $\beta \in C(J \times \mathbb{R}, \mathbb{R})$, $\lambda, k \in \mathbb{R}$.

If $\lambda = 1$ and $k = 0$, then we have the periodic boundary condition, if $\lambda = -1$ and $k = 0$, then we have the antiperiodic boundary condition, and if $\lambda = 0$, we have an initial condition as special cases of the boundary condition in (1).

To obtain existence results for differential problems someone may use the monotone iterative method, for details see for example [1]. There is a vast literature devoted to the applications of this method to differential equations with initial and boundary conditions. It can be applied to differential problems with deviated arguments, see for example the papers [2–8]. We also apply this technique to problem (1). It is important to indicate that (1) is different from

E-mail address: tjank@mifgate.pg.gda.pl.

corresponding problems investigated in the papers published earlier. Note that in problem (1) a deviated argument β depends on the unknown solution x . It is the first paper when the monotone iterative method is applied to problems of type (1).

The plan of this paper is as follows. Section 2 concerns the case when a parameter $\lambda \geq 0$, while in Section 3 we discuss problem (1) when $\lambda < 0$. In both sections, we formulate sufficient conditions when problem (1) has a unique solution in a corresponding sector. In Section 2, an example is added to illustrate imposed assumptions. A problem more general than (1) is discussed in Section 4.

2. Case $\lambda \geq 0$

Take $y_0, z_0 \in C^1(J, \mathbb{R})$ such that $y_0(t) \leq z_0(t), t \in J$. Let

$$\Omega = \{(t, u) : y_0(t) \leq u \leq z_0(t), t \in J\}.$$

A pair $u, v \in C^1(J, \mathbb{R})$ is called a lower–upper solution of problem (1) for $\lambda \geq 0$ if

$$\begin{cases} u'(t) \leq F(v, v)(t), & t \in J, & u(0) \leq \lambda u(T) + k, \\ v'(t) \geq F(u, u)(t), & t \in J, & v(0) \geq \lambda v(T) + k. \end{cases}$$

Let us define two sequences $\{y_n, z_n\}$ by relations:

$$\begin{cases} y'_{n+1}(t) = F(z_n, z_n)(t), & t \in J, & y_{n+1}(0) = \lambda y_n(T) + k, \\ z'_{n+1}(t) = F(y_n, y_n)(t), & t \in J, & z_{n+1}(0) = \lambda z_n(T) + k \end{cases} \quad (3)$$

for $n = 0, 1, \dots$. Functions y_0, z_0 will be defined later.

A pair $X, Y \in C^1(J, \mathbb{R})$ is called a quasi-solution of (1) if

$$\begin{cases} X'(t) = F(Y, Y)(t), & t \in J, & X(0) = \lambda X(T) + k, \\ Y'(t) = F(X, X)(t), & t \in J, & Y(0) = \lambda Y(T) + k. \end{cases}$$

A pair $\rho, \gamma \in C^1(J, \mathbb{R})$ is called the minimal and maximal quasi-solution of problem (1) if for any $U, V \in C^1(J, \mathbb{R})$ quasi-solution of (1) we have $\rho(t) \leq U(t), V(t) \leq \gamma(t)$ on J .

Theorem 1. Assume that

- (H₁) $f \in C(J \times \mathbb{R}, \mathbb{R}), \beta \in C(J \times \mathbb{R}, \mathbb{R})$, and f is nonincreasing with respect to the last variable,
- (H₂) a pair $y_0, z_0 \in C^1(J, \mathbb{R})$ is a lower–upper solution of problem (1) for $\lambda \geq 0$, and $y_0(t) \leq z_0(t)$ on J ,
- (H₃) $\beta : \Omega \rightarrow J, \beta(t, u)$ is nondecreasing with respect to u for $y_0(t) \leq u \leq z_0(t), t \in J$,
- (H₄) y_0, z_0 are nondecreasing on J and $f(t, u) \geq 0$ for $t \in J, y_0 \leq u \leq z_0$.

Then problem (1) has the minimal and maximal quasi-solution in the sector

$$[y_0, z_0]_* = \{u \in C^1(J, \mathbb{R}) : y_0(t) \leq u(t) \leq z_0(t), t \in J\}.$$

Proof. Note that $y_0(t) \leq y_1(t), z_1(t) \leq z_0(t)$ on J . Put $p = y_1 - z_1$. Then $p(0) \leq 0$, and $p'(t) = F(z_0, z_0)(t) - F(y_0, y_0)(t) \leq 0$ because

$$y_0(\beta(t, y_0(t))) \leq y_0(\beta(t, z_0(t))) \leq z_0(\beta(t, z_0(t))).$$

It shows that

$$y_0(t) \leq y_1(t) \leq z_1(t) \leq z_0(t), \quad t \in J.$$

Moreover, in view of assumptions (H₃), (H₄), we have

$$\begin{aligned} y'_1(t) &= F(z_0, z_0)(t) - F(z_1, z_1)(t) + F(z_1, z_1)(t) \leq F(z_1, z_1)(t), \\ z'_1(t) &= F(y_0, y_0)(t) - F(y_1, y_1)(t) + F(y_1, y_1)(t) \geq F(y_1, y_1)(t) \end{aligned}$$

because y_0, z_0 are nondecreasing and

$$y_0(\beta(t, z_0(t))) \geq z_1(\beta(t, z_1(t))), \quad y_0(\beta(t, y_0(t))) \leq y_1(\beta(t, y_1(t))).$$

By induction, we can show that

$$y_0(t) \leq y_1(t) \leq \dots \leq y_n(t) \leq z_n(t) \leq \dots \leq z_1(t) \leq z_0(t)$$

for $t \in J$ and $n = 0, 1, \dots$.

By the Arzeli theorem, $y_n \rightarrow y, z_n \rightarrow z$, where the pair $y, z \in C^1(J, \mathbb{R})$ is a quasi-solution of problem (1) and $y_0(t) \leq y(t) \leq z(t) \leq z_0(t), t \in J$. Now, we need to show that the pair y, z is the minimal and maximal quasi-solution of (1) in the sector $[y, z_0]_*$. Let $u, v \in [y_0, z_0]_*$ be any quasi-solution of problem (1). Put $p = y_1 - u, q = v - z_1$. Then $p(0) \leq 0, q(0) \leq 0$, and

$$\begin{aligned} p'(t) &= F(z_0, z_0)(t) - F(u, u)(t) \leq 0, \\ q'(t) &= F(v, v)(t) - F(y_0, y_0)(t) \leq 0 \end{aligned}$$

because

$$\begin{aligned} z_0(\beta(t, z_0(t))) &\geq z_0(\beta(t, u(t))) \geq u(\beta(t, u(t))), \\ y_0(\beta(t, y_0(t))) &\leq y_0(\beta(t, v(t))) \leq v(\beta(t, v(t))), \end{aligned}$$

Hence $y_1(t) \leq u(t), v(t) \leq z_1(t), t \in J$. By induction, we can prove that $y_n(t) \leq u(t)$ and $v(t) \leq z_n(t), t \in J, n = 0, 1, \dots$. If $n \rightarrow \infty$, then we have the assertion of Theorem 1. ■

It is easy to show the following.

Remark 1. Let all assumptions of Theorem 1 hold. If u is any solution of (1) such that $y_0(t) \leq u(t) \leq z_0(t), t \in J$, then

$$y_n(t) \leq u(t) \leq z_n(t), \quad t \in J, n = 0, 1, \dots$$

and $y(t) \leq u(t) \leq z(t), t \in J$, where y, z are from Theorem 1.

Now, we want to formulate sufficient conditions under which problem (1) has a unique solution. First we give the following.

Lemma 1. Assume that $\beta \in C(\Omega, J), K, L \in C(J, \mathbb{R}_+), R_+ = [0, \infty), p \in C^1(J, \mathbb{R})$ and

$$p'(t) \leq K(t)p(t) + L(t)p(\beta(t, w(t))), \quad t \in J, \quad p(0) = \lambda p(T), \quad \lambda \in [0, 1) \tag{4}$$

for $y_0(t) \leq w(t) \leq z_0(t), t \in J$. In addition assume that for $L^*(t) = K(t) + L(t)$ we have

$$\lambda + \int_0^T L^*(t)dt < 1. \tag{5}$$

Then $p(t) \leq 0, t \in J$.

Proof. Suppose that the assertion $p(t) \leq 0, t \in J$ is not true. Then, we can find $t_0 \in J$ such that $p(t_0) > 0$. Put

$$p(t_1) = \max_{t \in J} p(t) > 0.$$

Integrating the differential inequality in (4) we obtain

$$p(t) \leq p(0) + p(t_1) \int_0^T L^*(s)ds, \quad t \in J. \tag{6}$$

Then

$$p(0) = \lambda p(T) \leq \lambda \left[p(0) + p(t_1) \int_0^T L^*(s)ds \right].$$

This gives

$$p(0) \leq \frac{\lambda}{1-\lambda} p(t_1) \int_0^T L^*(s)ds.$$

This and (6) for $t = t_1$ yield

$$p(t_1) \left[1 - \frac{1}{1-\lambda} \int_0^{t_1} L^*(s) ds \right] \leq 0.$$

It contradicts the assumption that $p(t_1) > 0$. This shows that $p(t) \leq 0$ on J and the proof is complete. ■

Theorem 2. Let all assumptions of Theorem 1 hold. In addition assume that

(H₅) there exists functions $L, M \in C(J, \mathbb{R}_+)$, such that

$$\begin{aligned} f(t, u) - f(t, \bar{u}) &\leq L(t)(\bar{u} - u), \\ \beta(t, \bar{v}) - \beta(t, v) &\leq M(t)(\bar{v} - v) \end{aligned}$$

if $y_0(t) \leq u \leq \bar{u} \leq z_0(t)$, $y_0(t) \leq v \leq \bar{v} \leq z_0(t)$, $t \in J$,

(H₆) condition (5) holds for $L^*(t) = L(t)M(t)N(t) + L(t)$, where $f(t, w)$ is bounded by $N(t)$ for $t \in J$, $y_0 \leq w \leq z_0$.

Then problem (1) has, in the sector $[y_0, z_0]_*$, a unique solution.

Proof. From Theorem 1 we know that $y, z \in [y_0, z_0]_*$, and $y(t) \leq z(t)$, $t \in J$. We need to show that $y = z$. Put $q = z - y$, so $p(0) = \lambda p(T)$ and

$$\begin{aligned} p'(t) &= F(y, y)(t) - F(z, z)(t) \leq L(t)[z(\beta(t, z(t))) - y(\beta(t, y(t)))] \\ &= L(t)[p(\beta(t, z(t))) + y(\beta(t, z(t))) - y(\beta(t, y(t)))] \\ &\leq K(t)p(t) + L(t)p(\beta(t, z(t))) \quad \text{for } K(t) = L(t)M(t)N(t). \end{aligned}$$

This and Lemma 1 show that $z(t) \leq y(t)$, $t \in J$. It means that $y = z$. ■

Example. We consider the following boundary value problem

$$\begin{cases} x'(t) = \gamma_1 e^{-\gamma_2 x(\delta t x(t))}, & t \in J = [0, 1], \\ x(0) = \lambda x(1) + k, & \lambda \geq 0, \end{cases} \quad (7)$$

where $0 < \delta \leq \frac{1}{2}$, $0 < \gamma_1 \leq 1$, $\gamma_2 > 0$. Here $\beta(t, u) = \delta t u$.

Take $y_0(t) = 0$, $z_0(t) = t + 1$, $t \in J$ and $0 \leq k \leq 2\lambda + k \leq 1$. We see that $0 \leq \beta(t, u) \leq t$ for $y_0(t) \leq u \leq z_0(t)$, $t \in J$. Note that

$$\begin{aligned} F(z_0, z_0)(t) &= \gamma_1 e^{-\gamma_2(1+\delta t(1+t))} > 0 = y_0'(t), & \lambda y_0(1) + k &= k \geq 0 = y_0(0), \\ F(y_0, y_0)(t) &= \gamma_1 \leq 1 = z_0'(t), & \lambda z_0(1) + k &= 2\lambda + k \leq 1 = z_0(0) \end{aligned}$$

It proves that the pair (y_0, z_0) is a lower–upper solution of problem (7).

Moreover, $L(t) = \gamma_1 \gamma_2$, $M(t) = \delta t$, $N(t) = \gamma_1$. In addition assume that

$$\lambda + \gamma_1 \gamma_2 \left(1 + \frac{1}{2} \gamma_1 \delta \right) < 1. \quad (8)$$

Then problem (7) has, in the sector $[y_0, z_0]_*$, a unique solution, by Theorem 2. For example, if we take $\gamma_1 = \delta = \frac{1}{2}$, $\gamma_2 = 1$, the condition (8) holds for $\lambda < \frac{7}{16}$.

Now we consider the case when function β is nonincreasing with respect to the second variable. We have

Theorem 3. Assume that assumptions (H₁), (H'₂), (H'₃), (H'₄), (H'₅), (H₆) are satisfied where

(H'₂) $\lambda \geq 0$, $u_0, w_0 \in C^1(J, \mathbb{R})$, $u_0(t) \leq w_0(t)$, $t \in J$ and

$$\begin{cases} u_0'(t) \leq F(w_0, u_0)(t), & t \in J, & u_0(0) \leq \lambda u_0(T) + k, \\ w_0'(t) \geq F(u_0, w_0)(t), & t \in J, & w_0(0) \geq \lambda w_0(T) + k, \end{cases}$$

(H'₃) $\beta : \bar{\Omega} \rightarrow J$, $\beta(t, u)$ is nonincreasing with respect to u for $t \in J$, $u_0 \leq u \leq w_0$, $t \in J$, where $\bar{\Omega} = \{(t, u) : u_0(t) \leq u \leq w_0(t), t \in J\}$,

(H₄') u_0, w_0 are nondecreasing on J and $f(t, u) \geq 0$ for $t \in J, u_0 \leq u \leq w_0$,

(H₅') there exist functions $L, M \in C(J, \mathbb{R}_+)$, such that

$$f(t, u) - f(t, \bar{u}) \leq L(t)(\bar{u} - u),$$

$$\beta(t, v) - \beta(t, \bar{v}) \leq M(t)(\bar{v} - v)$$

if $u_0(t) \leq u \leq \bar{u} \leq w_0(t), u_0(t) \leq v \leq \bar{v} \leq w_0(t), t \in J$.

Then, problem (1) has, in the sector $[u_0, w_0]_*$, a unique solution.

Proof. Let us define the sequences $\{u_n, w_n\}$ be relations

$$\begin{cases} u'_{n+1}(t) = F(w_n, u_n)(t), & t \in J, & u_{n+1}(0) = \lambda u_n(T) + k, \\ w'_{n+1}(t) = F(u_n, w_n)(t), & t \in J, & w_{n+1}(0) = \lambda w_n(T) + k \end{cases}$$

for $n = 0, 1, \dots$. The proof of this theorem is similar to the proof of Theorems 1 and 2, and therefore it is omitted. ■

3. Case $\lambda < 0$

A pair $u, v \in C^1(J, \mathbb{R})$ is called a lower–upper solution of problem (1) for $\lambda < 0$ if

$$\begin{cases} u'(t) \leq F(v, v)(t), & t \in J, & u(0) \leq \lambda v(T) + k, \\ v'(t) \geq F(u, u)(t), & t \in J, & v(0) \geq \lambda u(T) + k. \end{cases}$$

Theorem 4. Let all assumptions of Theorems 1 and 2 be satisfied with (H₂') instead of (H₂), where

(H₂') a pair $y_0, z_0 \in C^1(J, \mathbb{R})$ is a lower–upper solution of problem (1) for $\lambda < 0$, and $y_0(t) \leq z_0(t)$ on J .

Then the assertion of Theorem 2 holds.

Proof. For $n = 0, 1, \dots$, let us define the sequences $\{y_n, z_n\}$ by relations

$$\begin{cases} y'_{n+1}(t) = F(z_n, z_n)(t), & t \in J, & y_{n+1}(0) = \lambda z_n(T) + k, \\ z'_{n+1}(t) = F(y_n, y_n)(t), & t \in J, & z_{n+1}(0) = \lambda y_n(T) + k. \end{cases}$$

Repeating the proof of Theorems 1 and 2, we have the assertion of Theorem 4. ■

Theorem 5. Let all assumptions of Theorem 3 be satisfied with (H''₂) instead of (H₂'),

(H''₂) $\lambda < 0, u_0, w_0 \in C^1(J, \mathbb{R}), u_0(t) \leq w_0(t), t \in J$, and

$$\begin{cases} u'_0(t) \leq F(w_0, u_0)(t), & t \in J, & u_0(0) \leq \lambda w_0(T) + k, \\ w'_0(t) \geq F(u_0, w_0)(t), & t \in J, & w_0(0) \geq \lambda u_0(T) + k. \end{cases}$$

Then the assertion of Theorem 3 hold.

In the proof use the sequences $\{u_n, w_n\}$ defined by relations.

$$\begin{cases} u'_{n+1}(t) = F(w_n, u_n)(t), & t \in J, & u_{n+1}(0) = \lambda w_n(T) + k, \\ w'_{n+1}(t) = F(u_n, w_n)(t), & t \in J, & w_{n+1}(0) = \lambda u_n(T) + k \end{cases}$$

for $n = 0, 1, \dots$.

4. General case

Now we consider the problem

$$\begin{cases} x'(t) = f(t, x(\beta(t, x(t))), x(\gamma(t, x(t)))) \equiv \mathcal{F}(x, x, x, x)(t), & t \in J, \\ x(0) = \lambda x(T) + k, \end{cases} \quad (9)$$

where

$$\mathcal{F}(x, y, u, w)(t) = f(t, x(\beta(t, y(t))), u(\gamma(t, w(t)))) \quad (10)$$

and $J = [0, T]$, $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\beta, \gamma \in C(J \times \mathbb{R}, \mathbb{R})$, $\lambda, k \in \mathbb{R}$.

Theorem 6. Assume that

- (A₁) $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\beta, \gamma \in C(J \times \mathbb{R}, \mathbb{R})$, and f is nonincreasing with respect to the last two variables,
 (A₂) $\lambda \geq 0$, and $y_0, z_0 \in C^1(J, \mathbb{R})$ satisfy the system

$$\begin{cases} y_0'(t) \leq \mathcal{F}(z_0, z_0, z_0, y_0)(t), & t \in J, & y_0(0) \leq \lambda y_0(T) + k, \\ z_0'(t) \geq \mathcal{F}(y_0, y_0, y_0, z_0)(t), & t \in J, & z_0(0) \geq \lambda z_0(T) + k \end{cases}$$

and $y_0(t) \leq z_0(t)$, $t \in J$,

- (A₃) $\beta, \gamma : \Omega \rightarrow J$, $\beta(t, u)$ is nondecreasing, and $\gamma(t, u)$ is nonincreasing, with respect to u for $y_0(t) \leq u \leq z_0(t)$, $t \in J$,
 (A₄) y_0, z_0 are nondecreasing on J , $f(t, u, v) \geq 0$ for $t \in J$, $y_0 \leq u \leq z_0$, $y_0 \leq v \leq z_0$, $t \in J$.
 (A₅) There exist functions $L_1, L_2, M_1, M_2 \in C(J, \mathbb{R}_+)$, such that

$$\begin{aligned} f(t, u, v) - f(t, \bar{u}, \bar{v}) &\leq L_1(t)(\bar{u} - u) + L_2(t)(\bar{v} - v), \\ \beta(t, \bar{v}) - \beta(t, v) &\leq M_1(t)(\bar{v} - v) \\ \gamma(t, w) - \gamma(t, \bar{w}) &\leq M_2(t)(\bar{w} - w) \end{aligned}$$

if $y_0(t) \leq u \leq \bar{u} \leq z_0(t)$, $y_0(t) \leq v \leq \bar{v} \leq z_0(t)$, $y_0(t) \leq w \leq \bar{w} \leq z_0(t)$, $t \in J$.

- (A₆) Condition (5) holds for $L^*(t) = N(t)[L_1(t)M_1(t) + L_2(t)M_2(t)] + L_1(t) + L_2(t)$, where $f(t, u, w)$ is bounded by $N(t)$ for $t \in J$, $y_0 \leq u \leq z_0$, $y_0 \leq v \leq z_0$.

Then problem (H₉) has, in the sector $[y_0, z_0]_*$, a unique solution.

In the proof, use the sequences $\{y_n, z_n\}$ defined by:

$$\begin{cases} y_{n+1}'(t) = \mathcal{F}(z_n, z_n, z_n, y_n)(t), & t \in J, & y_{n+1}(0) = \lambda y_n(T) + k, \\ z_{n+1}'(t) = \mathcal{F}(y_n, y_n, y_n, z_n)(t), & t \in J, & z_{n+1}(0) = \lambda z_n(T) + k \end{cases}$$

for $n = 0, 1, \dots$

Theorem 7. Let all assumptions of Theorem 6 be satisfied with assumption (A'₂) instead of (A₂), where

- (A'₂) $\lambda < 0$, and $y_0, z_0 \in C^1(J, \mathbb{R})$ satisfy the system

$$\begin{cases} y_0'(t) \leq \mathcal{F}(z_0, z_0, z_0, y_0)(t), & t \in J, & y_0(0) \leq \lambda z_0(T) + k, \\ z_0'(t) \geq \mathcal{F}(y_0, y_0, y_0, z_0)(t), & t \in J, & z_0(0) \geq \lambda y_0(T) + k \end{cases}$$

and $y_0(t) \leq z_0(t)$, $t \in J$.

Then the assertion of Theorem 6 holds.

Now, the sequences $\{y_n, z_n\}$ are defined by:

$$\begin{cases} y_{n+1}'(t) = \mathcal{F}(z_n, z_n, z_n, y_n)(t), & t \in J, & y_{n+1}(0) = \lambda z_n(T) + k, \\ z_{n+1}'(t) = \mathcal{F}(y_n, y_n, y_n, z_n)(t), & t \in J, & z_{n+1}(0) = \lambda y_n(T) + k \end{cases}$$

for $n = 0, 1, \dots$

Remark 2. There is no problem to formulate corresponding existence results for problems having more arguments of type β and γ .

References

- [1] G.S. Ladde, V. Lakshmikantham, A.S. Vatsala, *Monotone Iterative Techniques for Nonlinear Differential Equations*, Pitman, Boston, 1985.
- [2] T. Jankowski, Monotone iterative technique for differential equations with nonlinear boundary conditions, *Nonlinear Stud.* 8 (2001) 381–388.
- [3] T. Jankowski, Existence of solutions of boundary value problems for differential equations with delayed arguments, *J. Comput. Appl. Math.* 156 (2003) 239–252.
- [4] T. Jankowski, On delay differential equations with nonlinear boundary conditions, *Bound. Value Probl.* 2005 (2) (2005) 201–214.
- [5] T. Jankowski, Advanced differential equations with nonlinear boundary conditions, *J. Math. Anal. Appl.* 304 (2005) 490–503.
- [6] D. Jiang, J. Wei, Monotone method for first- and second-order periodic boundary value problems and periodic solutions of functional differential equations, *Nonlinear Anal.* 50 (2002) 885–898.
- [7] J.J. Nieto, R. Rodríguez-López, Existence and approximation of solutions for nonlinear functional differential equations with periodic boundary value conditions, *Comput. Math. Appl.* 40 (2000) 433–442.
- [8] J.J. Nieto, R. Rodríguez-López, Remarks on periodic boundary value problems for functional differential equations, *J. Comput. Appl. Math.* 158 (2003) 339–353.