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FIXED POINTS OF PLANAR HOMEOMORPHISMS OF THE FORM IDENTITY + CONTRACTION

GRZEGORZ GRAFF AND PIOTR NOWAK-PRZYGODZKI

ABSTRACT. Let f be a planar homeomorphism which has the form Identity + Contraction. We prove the existence of a fixed point of f under some geometrical condition on an orbit of f. The paper improves the result of Aarao and Martelli and provides an example which shows that, in the given setting, the theorem cannot be made stronger.

1. Introduction. The purpose of this paper is to study the existence of fixed points of maps $F : U \to \mathbf{R}^2$, where U is an open subset of $\mathbf{R}^2$, $F(x) = x + K(x)$ and K is contraction, i.e., $\|K(x)-K(y)\| \leq k\|x-y\|$, $0 < k < 1$. Each such map is an orientation preserving homeomorphism, cf. [**1**]. On the other hand, it is known that for orientation preserving homeomorphisms the presence of a periodic orbit forces the existence of a fixed point, which is an equivalent of one version of Brouwer's lemma on translation arcs, cf. [**2, 3**–**6**]. In [**1**] a stronger result is proved for the class of maps under consideration. Let for the rest of the paper $x_n = F^{n-1}(x)$ and $B(x, r)$ be a closed ball centered at x with the radius r. The theorem of Aarao and Martelli proved in [**1**] is the following:

Theorem 1.1. *Assume that there is a finite sequence $\{x_1, \dots , x_{n+1}\}$ such that its convex hull C is contained in U. Then there exists a point y in C such that $K(y) = 0$ provided that there is a $w \in [x_n, x_{n+1}]$ such that*:

$$\|w - x_1\| \leq \sqrt{1-k^2}\|K(x_1)\|. \tag{1.1}$$

Aarao and Martelli suspected, cf. [**1**, page 21], that the inequality (1.1) is optimal, i.e., that there are maps without fixed points with

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orbits as closed as desired to the inequality. They gave the example of fixed point free map with an orbit for which $\|w - x_1\|$ is about twice as large as $\sqrt{1-k^2}\|K(x_1)\|$.

However, in this paper, we show that a better estimate may be established. Namely, we prove a stronger theorem from which it follows that Theorem 1.1 remains true if we replace the inequality (1.1) by $\|w - x_1\| \leq 1/k\|K(x_1)\|$. We also provide an example showing that this inequality cannot be improved.

2 Preliminary results and definitions. Below we formulate two lemmas proved in [**1**] which will be used in the next section.

Lemma 2.1. *Assume that the line segment $[y, y+K(y)]$ intersects the line segment $[x, x+K(x)]$. Then the angle between the two oriented segments is acute.*

Lemma 2.2. *If for some $1 \leq i < j \leq n$ $[x_i, x_{i+1}] \cap [x_j, x_{j+1}] \neq \varnothing$, then F has a fixed point in the convex hull of $\{x_i, \dots, x_{j+1}\}$.*

Let us now recall the notion of the winding number. Let $F : [a, b] \to \mathbf{R}^2$ be a nonvanishing vector field such that $F(a) = F(b)$, and let $v : [a, b) \to \mathbf{R}^2$ be a positively oriented one-to-one parametrization of $S^1 = \{x \in \mathbf{R}^2 : \|x\| = 1\}$. We define $\widetilde{F} : S^1 \to S^1$ by the formula: $\widetilde{F}(x) = F(v^{-1}(x))/\|F(v^{-1}(x))\|$. Then, by $w(F)$, the winding number of F, we understand $\deg(\widetilde{F})$, which is an integer number.

Geometrically, it is the algebraic sum of the angles, divided by 2π, described by vector $F(t)$ when t changes from a to b. For further details the reader may consult [**7**].

The following well-known fact asserts the equality of the winding numbers for two vector fields which have different directions in each point.

Lemma 2.3 *Let F and G be two nonvanishing continuous vector fields, $F, G : [a, b] \to \mathbf{R}^2$ and $F(a) = F(b)$, $G(a) = G(b)$. Suppose that $F(x)/\|F(x)\| \neq G(x)/\|G(x)\|$ for each $x \in [a, b]$, then $w(F) = w(G)$.*

Definition 2.4. We say that a finite sequence of points in $\mathbf{R}^2$ $(V_1, \dots, V_n)$ forms a closed polygonal path P if the broken line made of segments $[V_1, V_2], \dots, [V_{n-1}, V_n], [V_n, V_1]$ is homeomorphic to S^1. We accompany with P the set of vectors $v_i = V_{i+1} - V_i$ for $i = 1, \dots, n-1$, $v_n = V_1 - V_n$, $v_{n+1} = v_1$.

Definition 2.5. Let $(V_1, \dots, V_n)$ form a closed polygonal path P. Put $V_{n+1} = V_1$. We will call $\gamma : [0, 2n] \to \mathbf{R}^2$ the parametrization of P if $\gamma|_{[2i-2,2i-1]}$ is the linear parametrization of the segment $[V_i, V_{i+1}]$ and $\gamma(t) = V_{i+1}$ for $t \in [2i-1, 2i]$, where $i = 1, \dots, n$.

Definition 2.6. Let $V_1, \dots, V_n$ form a closed polygonal path P. We define back vector field $G : [0, 2n] \to \mathbf{R}^2$ in the following way. Let $1 \le i \le n$,

(i) when $t \in [2i-2, 2i-1]$ set $G(t) = -v_i$,

(ii) when $t \in [2i-1, 2i]$ set $G(t) = -[(t - 2i + 1)v_{i+1} + (2i - t)v_i]$.

Lemma 2.7. *Let G be the back vector field; then $w(G) = \pm 1$.*

Proof. Vector field G never vanishes on P, is opposite to the direction of edges and at each vertex turns the corner. It is easy to observe that the algebraic sum of the angles described by $G(t)$ for t changing from 0 to $2n$ is $\pm 2\pi$, see also [**1**]. Thus $w(G) = \pm 1$. □

3. Main results. Let $F(x) = x + K(x)$, where K is a contraction with constant $k \in (0,1)$ in an open set U of the plane, and let $x_n = F^{n-1}(x)$. The main results of the paper are formulated in the two theorems below.

Theorem 3.1. *Assume that there is a finite sequence $\{x_1, \dots, x_{n+1}\}$ such that its convex hull C is contained in U, and let $r = \|K(x_1)\|/k$.*

If $x_n \notin B(x_1, r)$ and $[x_n, x_{n+1}] \cap B(x_1, r) \neq \varnothing$, then there exists a point y in C such that $K(y) = 0$.

Theorem 3.2. *Let $\{x_1, \dots, x_l\}$ be a sequence such that its convex hull is contained in U and $x_i \in B(x_1, r)$ for $i = 1, \dots, l$, where $r = \|K(x_1)\|/k$. Assume that x_1 is not a fixed point of F, then for each $i = 1, \dots, l$, there is*:

$$\|x_1 - x_i\| < \|x_1 - F(x_i)\|. \tag{3.2}$$

Remark 3.3. Theorems 3.1 and 3.2 taken together give a stronger result than Theorem 1.1. Theorem 1.1 can be reformulated in the following way. Let us denote by $\rho = \sqrt{1-k^2}\|K(x_1)\|$, as $\sqrt{1-k^2} < 1$, we have: $x_2 = x_1 + K(x_1) \notin B(x_1, \rho)$. Thus, the second element of the sequence $\{x_1, \dots, x_{n+1}\}$ leaves $B(x_1, \rho)$. Assume that for some n the segment $[x_n, x_{n+1}]$ intersects this ball nonempty. According to Theorem 3.2 some element of the sequence $\{x_1, \dots x_{n+1}\}$ must be outside a larger ball $B(x_1, r)$ $(1/k > 1 > \sqrt{1-k^2})$, so the condition $[x_n, x_{n+1}] \cap B(x_1, \rho) \neq \varnothing$ obviously implies $[x_i, x_{i+1}] \cap B(x_1, r) \neq \varnothing$, where $x_i \notin B(x_1, r)$ for some $i \leq n$.

Proof of Theorem 3.1. Let n be the first natural number satisfying the assumptions of the theorem. We will consider only such sequences that $[x_i, x_{i+1}] \cap [x_j, x_{j+1}] = \varnothing$ for $1 \leq i < j \leq n$, otherwise the theorem is valid by Lemma 2.2.

For convenience, let us take $x_1 = 0$, denote $K(0) = u$, and choose the coordinate system in the plane in such a way that $x_2 = (0, \|u\|)$. Let β be the closest point to x_n in the set $[x_n, x_{n+1}] \cap \partial B(0, r)$. The proof consists of two cases in dependence on the location of β. First we prove an important lemma which will be used in both parts, then in each case we first formulate and prove some lemmas and then give the proof of the case.

Lemma 3.4. *For every z in the interior of $B(0, r)$ we have $K(z)u > 0$; equivalently, the angle between $K(z)$ and u is acute. This result is still true if $z \in \partial B(0, r)$, provided $K(z) \neq 0$.*

Proof. Because K is a contraction and $x_1 = 0$ we have for $z \in \operatorname{Int} B(0,r)$:

$$\|K(z) - u\| \le k\|z\| < \|u\|, \quad \text{so}$$
$$\|K(z) - u\|^2 = K(z)^2 - 2K(z)u + u^2 < u^2;$$

thus, $2K(z)u > K(z)^2$. For $z \in \partial B(0,r)$ the inequality (3.3) does not have to be sharp, but we assume that $K(z) \neq 0$ on the boundary. □

Corollary 3.5. *If $z = (z_1, z_2)$ is in the interior of $B(0,r)$ and $F(z) = (z_3, z_4)$, then $z_2 < z_4$.*

Case 1. $\beta \in \partial B(0,r) \cap \{(x,y) \in \mathbf{R}^2 : y \le \|u\|\}$. We take the closed polygonal path P formed by the points $(x_2, x_3, \dots, x_n, \beta)$, then by Definition 2.4 there is: $v_1 = x_3 - x_2, \dots, v_{n-1} = \beta - x_n$, and $v_n = x_2 - \beta$.

Lemma needed in the proof of Case 1.

Lemma 3.6. *The angle between vectors v_{n-1} and v_n is acute.*

Proof. Taking into account that $\lambda v_{n-1} = x_{n+1} - x_n$, where $\lambda \ge 1$ and $v_{n-1} + v_n - u = x_1 - x_n = -x_n$ and that K is a contraction, we obtain:

$$\|\lambda v_{n-1} - u\| = \|K(x_n) - K(0)\| \le k\|v_{n-1} + v_n - u\|.$$

Taking squares, we get:

$$\lambda^2 v_{n-1}^2 - 2\lambda v_{n-1}u + u^2$$
$$\le k^2 v_{n-1}^2 + k^2 v_n^2 + k^2 u^2 + 2k^2 v_{n-1}v_n - 2k^2 v_{n-1}u - 2k^2 v_n u,$$

which is equivalent to:

(3.4)
$$(\lambda^2 - k^2)v_{n-1}^2 + 2(k^2 - \lambda)v_{n-1}u + (1-k^2)u^2 \le k^2 v_n^2 + 2k^2 v_{n-1}v_n - 2k^2 v_n u.$$

As $\beta \in B(0,r)$, $r = \|u\|/k$ and $v_n - u = x_1 - \beta = -\beta$ we have:

$$k\|v_n - u\| \le \|u\|,$$

which gives:

$$-k^2v_n^2 + 2k^2v_nu \geq u^2(k^2 - 1). \tag{3.5}$$

Notice that $x_n \notin B(0, r)$, thus:

$$k\|v_{n-1} + v_n - u\| > \|u\|,$$

or equivalently:

$$k^2v_{n-1}^2 + k^2v_n^2 + 2k^2v_{n-1}v_n - 2k^2v_{n-1}u - 2k^2v_nu > u^2 - k^2u^2. \tag{3.6}$$

Now, adding to (3.4) the sides of (3.5) we obtain:

$$(\lambda^2 - k^2)v_{n-1}^2 + 2(k^2 - \lambda)v_{n-1}u \leq 2k^2v_{n-1}v_n. \tag{3.7}$$

Adding sides of (3.5) to the inequality (3.6) we get:

$$(v_{n-1}^2 - 2v_{n-1}u) + 2v_{n-1}v_n > 0. \tag{3.8}$$

By the above inequality we have two possibilities:
either $v_{n-1}^2 - 2v_{n-1}u \leq 0$, then $2v_{n-1}v_n > 0$ which is what we need; or $v_{n-1}^2 - 2v_{n-1}u > 0$ but then, as $\lambda \geq 1$:

$$(\lambda^2 - k^2)v_{n-1}^2 + 2(k^2 - \lambda)v_{n-1}u \geq (\lambda - k^2)(v_{n-1}^2 - 2v_{n-1}u) > 0,$$

and the needed inequality follows from (3.7). □

Proof of Case 1. Let $\gamma : [0, 2n] \to \mathbf{R}^2$ be the parametrization of the closed polygonal path P formed by $(x_2, x_3, \dots, x_n, \beta)$. We consider two vector fields $V, G : [0, 2n] \to \mathbf{R}^2$, where $V(t) = K(\gamma(t))$ and $G(t)$ is a back vector field. We show that, for each $t \in [0, 2n]$, the assumption of Lemma 2.3 is satisfied; as a result, we will get by Lemmas 2.3 and 2.7 that $w(V) = \pm 1$ which will prove this part of the theorem.

Along the edges of the polygonal path P, apart from $[\beta, x_2]$, the angle between $V(t)$ and $-G(t)$ is acute by Lemma 2.1.

It is also acute in each vertex $x_3, \dots, x_n$, because for $i = 3, \dots, n$ we have: $K(x_i) = v_{i-1}$ and $v_{i-2}v_{i-1} > 0$; thus, for each $s \in [0,1]$ there is $K(x_i)(sv_{i-1} + (1-s)v_{i-2}) > 0$.

In x_2, i.e., for $t \in [0,1]$, by Lemma 3.4 the ending point of the vector $K(\gamma(t)) = v_1$ lies in the half-plane defined by the inequality $y > \|u\|$ and $G(t)$ changes from $-v_n$ to $-v_1$, both located in the half-plane defined by the inequality $y \leq \|u\|$.

The same situation we have for the points of the segment (β, x_2). Let L be the line parallel to $y = \|u\|$ which crosses a given point $x \in (\beta, x_2)$. Then in x, by Lemma 3.4, $K(\gamma(t)) = K(x)$ is situated in the upper half-plane bounded by L and $G(t) = -v_n = \beta - x_2$ in the lower closed half-plane.

Finally in β, by Lemma 3.6 $(-v_{n-1})(-v_n) > 0$; on the other hand, $G(t)$ changes in β from $-v_{n-1}$ to $-v_n$, thus $G(t)(-v_{n-1}) > 0$. As by Lemma 2.1 $K(\gamma(t))v_{n-1} > 0$, we see that $G(t)$ and $K(\gamma(t))$ for each t have different directions.

This ends the proof of Case 1. □

Case 2. Let (x_β, y_β) denote the coordinates of β. We assume now that $y_\beta > \|u\|$ and $x_\beta \leq 0$ (the similar arguments, by symmetry, apply if $x_\beta > 0$).

Let $\alpha = [0, x_n] \cap \partial B(0, r)$. We denote by (x_α, y_α) the coordinates of α.

Lemmas needed in the proof of Case 2.

Lemma 3.7.

$$(*) \qquad x_\beta \neq 0,$$
$$(**) \qquad y_\alpha < y_\beta \quad \text{and} \quad x_\alpha < 0.$$

Proof. The fact $(*)$ is a consequence of the reasoning given in (B) with $\beta = \beta'$. Denote by L the line which crosses points 0 and β. We prove the condition equivalent to $(**)$: x_n cannot be situated in the upper closed half-plane bounded by L.

To the contrary, assume that the assumption is not satisfied, which means that x_n lies above L and on or above the line tangent to $\partial B(0,r)$ in β. We consider two possible subcases.

(A) Let x_n be in the second open quadrant of the chosen coordinate system. Let $v = \alpha - x_n$, $\widetilde{v_n} = x_{n+1} - x_n$, and let v' be the projection of $\widetilde{v_n}$ on the line containing segment $[0, x_n]$. Observe that the vectors u and $\widetilde{v_n}$ directs different half planes determined by the line $0x_n$; thus $\|u - \widetilde{v_n}\| \geq \|u - v'\| \geq \|u - v\|$, where the last inequality results from the fact that $\|v\| \leq \|v'\|$ and the angle between v and u is greater than $\pi/2$. On the other hand, as K is a contraction, $k\|x_n\| \geq \|u - \widetilde{v_n}\|$. Now, to obtain a contradiction, it is enough to show that $\|u - v\| > k\|x_n\|$. As $\|\alpha\| = r = 1/k\|u\|$ we have: $k\|x_n\| = k(1/k\|u\| + \|v\|)$, so the previous inequality is equivalent to $\|u - v\| > \|u\| + k\|v\|$ or $u^2 - 2uv + v^2 > \|u\|^2 + 2\|u\|\|v\|k + k^2\|v\|^2$. This is satisfied if $-\cos\phi > k$, where ϕ is the angle between vectors u and v, or equivalently $\cos\phi' > k$, for ϕ' the angle between u and $-v$. From the triangle $0x_2p$, where $p = \partial B(0,r) \cap \{(x,y) \in \mathbf{R}^2 : y = \|u\|$ and $x < 0\}$, we find that $\cos\psi = k$, where ψ is the angle between vectors u and $p - 0$. Inequality $\psi > \phi'$ ends the proof of subcase (A).

(B) Let x_n be in the closed first quadrant of the chosen coordinate system (and on or above the line tangent to $\partial B(0,r)$ in β). Let β' be the projection of β on the line $x = 0$, define vectors: $z = \beta - x_n$, $\bar{z} = \beta' - x_n$. Notice that vector $\beta' - 0 = tu$, where $t \leq 1/k$. Thus $\|0 - x_n\| = \|tu - \bar{z}\|$. The location of x_n implies that the projection of vector $\bar{z}$ on the y-axis is equal to the projection of vector z on this axis, but the projection of $\bar{z}$ on the x-axis is shorter than the projection of z on this axis. Taking into account that u is a vector which lies on the y-axis, we get that: $\|u - \bar{z}\| \leq \|u - z\|$. The fact that K is a contraction gives the following:

$$\|u - \bar{z}\| \leq \|u - z\| \leq \|K(0) - K(x_n)\| = \|0 - x_n\| \leq k\|tu - \bar{z}\|.$$

After taking squares, we get: $u^2 - 2u\bar{z} + \bar{z}^2 \leq k^2t^2u^2 - 2k^2tu\bar{z} + k^2\bar{z}^2$, which is equivalent to:

$$(k^2t^2 - 1)u^2 + (k^2 - 1)\bar{z}^2 + 2(1 - k^2t)u\bar{z} \geq 0.$$

On the other hand, $k^2t^2 - 1 \leq 0$, $k^2 - 1 < 0$ and $2(1 - k^2t)u\bar{z} \leq 0$ because the angle between u and $\bar{z}$ is greater than or equal to $\pi/2$. This leads to a contradiction. $\square$

In the forthcoming lemmas we introduce the following notation. For $i \in \{1, 2, \dots, i_0\}$, $x_i \in B(0, r)$ and for $i \in \{i_0 + 1, i_0 + 2, \dots, n\}$, $x_i \notin B(0, r)$.

Lemma 3.8. *For each* $2 \leq i \leq i_0$*, there is*:

$$[x_i, x_{i+1}] \cap [0, x_n] = \varnothing.$$

Assume that i is the minimal number in $\{2, \dots, i_0\}$ such that $[x_i, x_{i+1}] \cap [0, x_n] \neq \varnothing$ (so x_i is above the line determined by $[0, x_n]$), with this assumption we show two helpful facts.

Fact 3.9. If x_i' is the projection of the point x_i on the line determined by the segment $[0, x_n]$, then $x_i' \in [0, \alpha]$.

Proof. Assume $x_i' \notin [0, \alpha]$; on the other hand, x_i' must lie on the diameter determined by α. Let $\bar{x}_i$ be the projection of x_i on the x-axis in the direction determined by the direction of $[0, \alpha]$. Let $0'$ be the orthogonal projection of 0 onto the line determined by the segment $[x_i, \bar{x}_i]$. There are two perpendicular triangles: $00'\bar{x}_i$ and $00'x_i$ from which we deduce that $\|\bar{x}_i\| \geq \|x_i\|$.

We move the vector $K(x_i)$ along the line $0'x_i$ from x_i to $\bar{x}_i$. Observe that, by Lemma 3.4, during this movement $K(x_i)$ have still the endpoint below the line determined by $[0, \alpha]$. As a result $K(x_i)$ as the vector with the starting point at $\bar{x}_i$ have the endpoint situated in the second quadrant. Notice that $K(0)$ lies on the y-axis, which implies that x-coordinate of $K(x_i) - K(0)$ is equal to the x-coordinate of $K(x_i)$. The above remarks give:

$$\|K(x_i) - K(0)\| \geq \|\bar{x}_i\|.$$

Finally, we obtain:

$$\|K(x_i) - K(0)\| \geq \|\bar{x}_i\| > k\|\bar{x}_i\| \geq k\|x_i\|.$$

Contradiction with the fact that K is a contraction. □

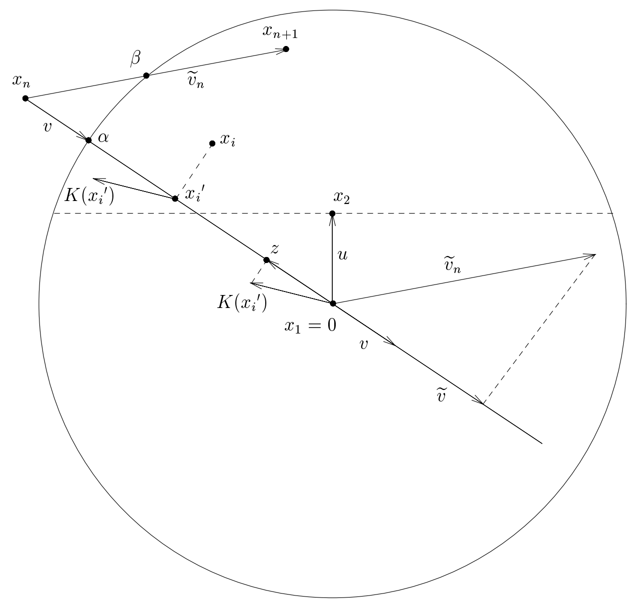

FIGURE 1. Illustration to the proof of Lemma 3.8.

Define $A = \{(x, y) \in \mathbf{R}^2 : y < (y_\alpha/x_\alpha)x\}$, A' the complement of A. Let us denote $x' = x_i' + K(x_i')$.

Fact 3.10. $x' \in A$.

Proof. By Fact 3.9, $x_i' \in [0, \alpha]$. To the contrary assume that $x' \in A'$. Then $\|K(x_i) - K(x_i')\| = \|(x_{i+1} - x_i) - (x' - x_i')\| = \|(x_{i+1} - x') - (x_i - x_i')\| > \|x_i - x_i'\|$, the last inequality results from the fact that the angle between vectors $(x_{i+1} - x')$ and $(x_i - x_i')$ is greater than $\pi/2$. We obtained the contradiction because K is a contraction. $\square$

Proof of Lemma 3.8. Assume that for some i there is $[x_i, x_{i+1}] \cap [0, x_n] \neq \varnothing$. We remind the reader that $u = K(0)$, $v = \alpha - x_n$, $\widetilde{v_n} = x_{n+1} - x_n = K(x_n)$. We have: $\|x_n\| = \|v\| + r = \|v\| + 1/k\|u\|$. Thus:

$$\begin{aligned} \|u - K(x_i')\| + \|K(x_i') - \widetilde{v_n}\| &\leq k\|0 - x_i'\| + k\|x_i' - x_n\| \\ &= k\|x_n\| = k\|v\| + \|u\|. \end{aligned} \tag{3.9}$$

On the other hand, consider vectors $v, u, \widetilde{v_n}, K(x_i')$ as having the common starting point at 0, see Figure 1. By Fact 3.10 $K(x_i')$ is situated under the line determined by $[0, x_n]$. Let z be the projection of $K(x_i')$ on $[0, x_n]$, and let $\tilde{v}$ be the projection of $\widetilde{v_n}$ on the line containing $[0, x_n]$. Observe that $\|\tilde{v}\| > \|v\|$. There is:

$$\begin{aligned} \|u - K(x_i')\| + \|K(x_i') - \widetilde{v_n}\| &\geq \|u - z\| + \|z - \tilde{v}\| \\ &= \|u - z\| + \|z\| + \|\tilde{v}\| \\ &> \|u\| + \|v\|. \end{aligned}$$

This contradicts (3.9), which ends the proof of Lemma 3.8. □

Lemma 3.11. *For each $2 \leq i \leq i_0$, there is: $[x_i, x_{i+1}] \cap [\alpha, \beta] = \varnothing$.*

Proof. Assume $[x_i, x_{i+1}] \cap [\alpha, \beta] \neq \varnothing$. Let us take the point $s \in \mathbf{R}^2$ such that we have the equality of the vectors $x_{i+1} - x_i = x_{n+1} - s$. Then $\|s - x_n\| = \|(x_{n+1} - x_n) - (x_{n+1} - s)\| = \|(x_{n+1} - x_n) - (x_{i+1} - x_i)\| = \|K(x_n) - K(x_i)\|$. On the other hand, we will show that $\|x_n - s\| \geq \|x_n - x_i\|$, which will give a contradiction to the fact that K is a contraction.

For the sake of the proof of this fact, let us choose the coordinate system in such a way that 0, the center, is situated in the point x_n, the x-axis is determined by the segment $[x_n, \alpha]$, the y-axis is the line perpendicular to the x-axis. There is:

(1) by Lemma 3.8 both coordinates of x_i are positive (if x_i has a nonpositive y-coordinate, then $[x_j, x_{j+1}] \cap [0, x_n] \neq \varnothing$ for some $j < i$ which contradicts Lemma 3.8).

(2) By Lemma 3.8 $[x_i, x_{i+1}]$ does not cross $[0, x_n]$. On the other hand, if $[x_i, x_{i+1}]$ crosses the segment $[x_n, \beta]$, then by Lemma 2.2 the theorem

is proved. As a consequence, we may assume that x_{i+1} is in the triangle $\alpha x_n \beta$.

(3) The angle between segments $[x_n, \alpha]$ and $[\alpha, \beta]$ is greater than $\pi/2$; thus, the coordinates of x_{i+1} (and all points in the triangle $\alpha x_n \beta$) are not greater than the respective coordinates of β, which are not greater than the coordinates of x_{n+1}.

(4) Because vectors $(x_{i+1} - x_i)$ and $(x_{n+1} - s)$ are equal, (3) implies that the coordinates of s are not less than the respective coordinates of x_i.

(5) The distance of the point s from the center, i.e., from x_n, is not less than the distance of x_i from the center, which ends the proof of the fact that $\|x_n - s\| \geq \|x_n - x_i\|$ and the whole proof of Lemma 3.11.

Proof of Case 2. We cannot repeat literally the same reasoning as in Case 1 because the direction of the back vector field may now be the same as $K(\gamma(t))$ on the segment $[x_n, \beta]$. However, due to Lemmas 3.8 and 3.11, we have control over the behavior of F in the area bounded by the convex (Lemma 3.7) quadrangle $0\alpha\beta x_2$. Namely, we know that elements of the orbit of 0 may leave this area only in such a way that the segment $[x_i, x_{i+1}]$ crosses $[x_2, \beta]$. This enables us to use slightly modified back vector field to prove the theorem.

Let $v_1, \ldots, v_n$ be the same as in Definition 2.6. We define *modified back function* $\widetilde{G}$ for $v_1, \ldots, v_n$ in the following way: $\widetilde{G}(t) = G(t)$ for $t \in [0, 2n-3]$, $\widetilde{G}(t) = -v_1$ for $t \in [2n-2, 2n]$. For $t \in [2n-3, 2n-2]$, we define $\widetilde{G}$ as a counterclockwise deformation between vectors $-v_{n-1}$ and $-v_1$. It is easy to see that $w(\widetilde{G}) = \pm 1$.

By Lemmas 3.8 and 3.11 we see that for some i the segment $[x_i, x_{i+1}]$ crosses $[x_2, \beta]$. Define $p = \max\{2 \leq j < n : [x_j, x_{j+1}] \cap [x_2, \beta] \neq \varnothing\}$ and let $g = [x_p, x_{p+1}] \cap [x_2, \beta]$.

Now we define the closed polygonal path $(g, x_{p+1}, \ldots, x_{n-1}, x_n, \beta)$ and consider:

(a) modified back function $\widetilde{G}$ for $(g, x_{p+1}, \ldots, x_{n-1}, x_n, \beta)$, $\widetilde{G} : [0, 2n_1] \to \mathbf{R}^2$, where $n_1 = n - p + 2$.

(b) $\tilde{\gamma}$, the parametrization of $(g, x_{p+1}, \ldots, x_{n-1}, x_n, \beta)$ and $\widetilde{V} : [0, 2n_1] \to \mathbf{R}^2$, given by the formula $\widetilde{V}(t) = K(\tilde{\gamma}(t))$.

As in Case 1 we show that for each $t \in [0, 2n_1]$ the assumption of Lemma 2.3 is satisfied for $\widetilde{V}$ and $\widetilde{G}$, which will prove the theorem.

By the same reasons as in Case 1 we see that $\widetilde{V}(t)/\|\widetilde{V}(t)\| \neq \widetilde{G}(t)/\|\widetilde{G}(t)\|$ for $t \in [0, 2n_1 - 3]$.

On the segment $(\beta, g]$, i.e., for $t \in [2n_1 - 2, 2n_1]$ by Lemma 3.4 there is: $\widetilde{V}(t)u > 0$ and $\widetilde{G}(t)u = -v_1 u < 0$, which gives the desired conclusion.

Finally, in the point β, i.e., for $t \in [2n_1 - 3, 2n_1 - 2]$, we consider two cases.

If x_n is situated on or under the line $y = y_\beta$, then by Lemma 3.4 both vectors $-v_{n-1}$ and $-v_1$ and so $\widetilde{G}(t)$ as a deformation between them lie on or under the line $y = y_\beta$, but, again by Lemma 3.4, the end of the vector $\widetilde{V}(t)$ lies over this line.

If x_n is situated over the line $y = y_\beta$, then from the definition of g and β one easily deduces that vectors $-v_{n-1}$ and $-v_1$ and so $\widetilde{G}(t)$ for each t lie in the lower half-plane bounded by the line L which has the direction of $v_n = g - \beta$, see Figure 2. On the other hand, the relations $\widetilde{V}(t)u > 0$ and $\widetilde{V}(t)v_{n-1} > 0$ imply that $\widetilde{V}(t)$ is situated in the upper half-plane bounded by L.

This ends the proof of Case 2 and the whole proof of Theorem 3.1. $\square$

Proof of Theorem 3.2. We use the notation from the proof of Theorem 3.1, in particular $x_1 = 0$. Assume that the inequality (3.2) does not hold. Let m be the first natural number such that $\|x_m\| \geq \|x_{m+1}\|$. Notice that the y-coordinate of x_m is less than that of x_{m+1} by Corollary 3.5. Let us take the ball $B(0, r')$ with $r' = \|x_{m+1}\|$. We define $\alpha' = [0, x_m] \cap \partial B(0, r')$. Let $\beta' = (x_{\beta'}, y_{\beta'})$ be the closest point to x_m in the set $[x_m, x_{m+1}] \cap \partial B(0, r')$. Repeating the same arguments as in Lemmas 3.8 and 3.11, we obtain that any $[x_i, x_{i+1}]$ for $i = 2, \dots, m-1$ does not cross the sum of segments $[0, \alpha'] \cup [\alpha', \beta']$. Thus, the sequence of segments $[x_i, x_{i+1}]$ must reach to x_m crossing the horizontal line $y = y_{\beta'}$ but this is impossible because, by Corollary 3.5, y-coordinates of the elements of the orbit must increase. $\square$

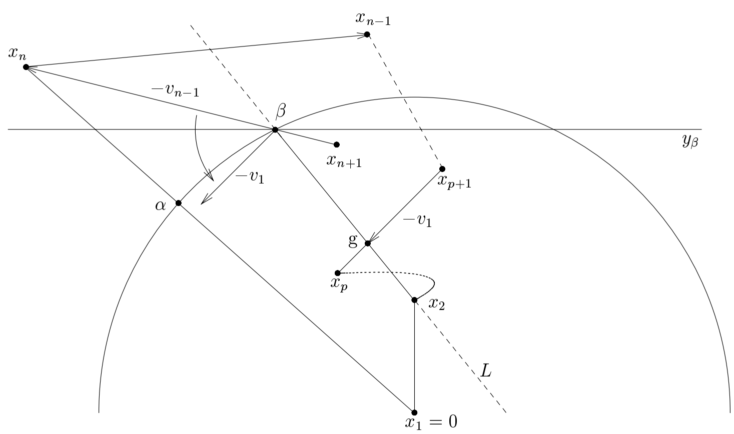

FIGURE 2. Illustration to the proof of Case 2.

4. Optimality of Theorem 3.1. In this final section we construct an example of a family of planar discrete dynamical systems, $F_\varepsilon(x) = x + \widehat{H_\varepsilon}(x)$, where $\widehat{H_\varepsilon}(x)$ is a contraction. Each F_ε is fixed point free and, for any arbitrary small $\delta > 0$, there is an $\varepsilon = \varepsilon(\delta)$ such that there exists a point x_1 whose orbit has an element which is closer than δ to the ball $B(x_1, \|\widehat{H_\varepsilon}(x_1)\|/k)$.

For any ε, which is much less than k, we define a broken line $ABCD$, where $A = (0, k-3\varepsilon)$, $B = (\varepsilon, k-3\varepsilon)$, $C = (\varepsilon, -\varepsilon)$, $D = (0, -\varepsilon)$.

Let $g : [0,1] \to [0,k]$ be given by the formula $g(x) = kx$. Notice that the length of $ABCD$ is equal to k. Define $h_\varepsilon : [0,k] \to ABCD$ with $h_\varepsilon(0) = A$, $h_\varepsilon(k) = D$ as the function which bends $[0,k]$ forming $ABCD$, i.e., h_ε is a piecewise linear map which transforms linearly the segment $[0,\varepsilon]$ onto AB, $[\varepsilon, k-\varepsilon]$ onto BC and $[k-\varepsilon, k]$ onto CD.

Now we define the function $H_\varepsilon : [0,1] \to \mathbf{R}^2$ by $H_\varepsilon(x) = h_\varepsilon g(x)$. Next define $\widehat{H_\varepsilon} : \mathbf{R}^2 \to \mathbf{R}^2$ by the formula:

$$\widehat{H_\varepsilon}(x,y) = \begin{cases} H_\varepsilon(x) & \text{if } x \in [0,1], \\ H_\varepsilon(0) & \text{if } x \in (-\infty, 0], \\ H_\varepsilon(1) & \text{if } x \in [1, \infty). \end{cases}$$

Lemma 4.1. $\widehat{H_\varepsilon}$ *is a contraction.*

Proof. First observe that H_ε and so $\widehat{H_\varepsilon}$ on $[0,1]\times\{0\}$ is a contraction. This is a consequence of the fact that, for every x, y in the domain of h_ε there is: $\|h_\varepsilon(x)-h_\varepsilon(y)\| \le \|x-y\|$ and that g is a contraction (with constant $k<1$).

For $p, q \in [0,1]\times(\mathbf{R}\setminus\{0\})$ we take a projection of these points: p', q' on the line $y=0$ and see that $\|p'-q'\| \le \|p-q\|$ but $\widehat{H_\varepsilon}(p) = \widehat{H_\varepsilon}(p') = H_\varepsilon(p')$, $\widehat{H_\varepsilon}(q) = \widehat{H_\varepsilon}(q') = H_\varepsilon(q')$, so the thesis is a consequence of the previous case.

It is obvious that $\widehat{H_\varepsilon}$ is a contraction in the remaining cases. $\square$

Consider the family of maps $F_\varepsilon(x) = x + \widehat{H_\varepsilon}(x)$, and take $x_1 \in [0,\varepsilon/k]\times\{0\}$. Elements of the orbit of x_1, $\{F_\varepsilon^j(x_1)\}_{j=1}^\infty$ go away from the line $y=0$ until the projection of $F_\varepsilon^j(x_1)$ on this line lies in $[0,1-2\varepsilon/k]$ and then start to approach it. For some large enough i the segment which joins two consecutive elements of the orbit crosses the x-axis. Because $\varepsilon \ll k$ it happens for the segment $[1-\varepsilon/k,1]$: $[F_\varepsilon^i(x_1), F_\varepsilon^{i+1}(x_1)] \cap ([1-\varepsilon/k,1]\times\{0\}) \ne \varnothing$.

Let $\delta>0$ be an arbitrary small real number. We want to show that there are ε and x_n such that $\|x_1-x_n\| - \delta < \|\widehat{H_\varepsilon}(x_1)\|/k$, or equivalently:

$$k - k\delta/\|x_1-x_n\| < \|\widehat{H_\varepsilon}(x_1)\|/\|x_1-x_n\|. \tag{4.10}$$

Let us take $x_n = x_i$. Then, by the construction of F_ε, we see that x_n is situated inside the triangle made of vertex: x_1, $(1,0)$, $(1,\varepsilon)$, thus $\|x_1-x_n\| < 1+\varepsilon$. On the other hand, $\|\widehat{H_\varepsilon}(x_1)\| > \|\widehat{H_\varepsilon}(0)\| = k-3\varepsilon$. Finally we obtain the inequality

$$\frac{k-3\varepsilon}{1+\varepsilon} < \|\widehat{H_\varepsilon}(x_1)\|/\|x_1-x_n\|.$$

For $\varepsilon \to 0^+$ the fraction on the lefthand side of the above formula converges to k, taking values which are less than k; thus, for appropriate ε and x_n defined as above, we get the needed inequality (4.10).

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Faculty of Applied Physics and Mathematics, Gdansk University of Technology, ul. Narutowicza 11/12, 80-952 Gdansk, Poland
Email address: graff@mifgate.pg.gda.pl

Email address: piotrnp@wp.pl