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DEGREE OF T-EQUIVARIANT MAPS IN $\mathbb{R}^n$

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Abstract. A special case of G-equivariant degree is defined, where $G = \mathbb{Z}_2$, and the action is determined by an involution $T : \mathbb{R}^p \oplus \mathbb{R}^q \to \mathbb{R}^p \oplus \mathbb{R}^q$ given by $T(u, v) = (u, -v)$. The presented construction is self-contained. It is also shown that two T-equivariant gradient maps $f, g : (\mathbb{R}^n, S^{n-1}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ are T-homotopic iff they are gradient T-homotopic. This is an equivariant generalization of the result due to Parusiński.

1. Introduction. Let Ω be an open bounded subset of $\mathbb{R}^n$. Consider a continuous map $f : \overline{\Omega} \to \mathbb{R}^n$ such that f is not equal to 0 at any point on the boundary of Ω. Then an integer $\deg(f, \Omega)$ called the Brouwer degree can be associated to f. The classical works on this subject are [3], [12], [13], and a modern one is [11]. It is well known that the Brouwer degree is an invariant of homotopy. This means that if $h : \overline{\Omega} \times [0, 1] \to \mathbb{R}^n$ is a homotopy nowhere vanishing on $\partial\Omega \times [0, 1]$ then $\deg(h_t, \Omega) = \deg(h_0, \Omega)$ for all $t \in [0, 1]$, where $h_t(x) = h(x, t)$.

Let G be a compact Lie group. Assume that V is a real finite-dimensional representation of G and $n = \dim V$. Take $\Omega \subset V$ and $f : \overline{\Omega} \to V$ as above. In addition, suppose that Ω is G-invariant ($gx \in \Omega$ for all $x \in \Omega$, $g \in G$) and f is G-equivariant ($f(gx) = gf(x)$

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for all $x \in \overline{\Omega}$, $g \in G$). In this case, the G-equivariant degree $\mathrm{Deg}_G(f, \Omega) \in B(G)$ is defined, where $B(G)$ stands for the Burnside ring of G. This degree was introduced by Ize, Massabó and Vignoli in [10]. Up till now, it was considered by many authors. See for instance [6], [11] and [15]. Of course, G-equivariant degree is an invariant of G-equivariant homotopy ($h(gx, t) = gh(x, t)$ for all $x \in \overline{\Omega}$, $t \in [0, 1]$, $g \in G$).

Let G be equal to $\mathbb{Z}_2$. The action of $\mathbb{Z}_2$ on $\mathbb{R}^n$ is determined by a decomposition of $\mathbb{R}^n$ onto the direct sum $\mathbb{R}^p \oplus \mathbb{R}^q$ and the involution $T(u, v) = (u, -v)$, where $n = p + q$, $p, q \in \mathbb{N} \cup \{0\}$ and $u \in \mathbb{R}^p$, $v \in \mathbb{R}^q$. In fact, to define the $\mathbb{Z}_2$-equivariant degree we do not need to use the representation theory. In this work we would like to describe a construction of this degree. We will call it T-equivariant degree.

Our approach is alternative to the one by Granas and Dugundji in [8]. There are two basic differences between our and their approach. Contrary to Granas and Dugundji, from the beginning we work with the family of T-equivariant maps (see Sec. 6, §20, Theorem 1.2, pp. 551–552 in [8], and Lemma 3.2, Conclusion 3.3 here). We also introduce notions of T-equivariant normal maps and homotopies, which are different from ones in [8]. Moreover, the proofs of all lemmas and propositions needed to define the degree are complete.

Our construction is divided into five main steps. Each step is a separate section.

In [14], Parusiński showed that if we have two gradient vector fields on the unit ball in $\mathbb{R}^n$ and nowhere vanishing on the sphere, then they are homotopic if and only if they are gradient homotopic. In the last section we will prove this theorem in T-equivariant case. Namely, consider two T-equivariant gradient vector fields f and g on the unit ball in $\mathbb{R}^n$ and nowhere vanishing on the sphere. It is shown that if there is a T-equivariant homotopy joining f to g then there is a T-equivariant gradient homotopy joining f to g. Our result suggests that there is no interesting generalization of T-equivariant degree on gradient vector fields. The proof is based on the latest results by Ferrario (see [4]) and Dancer, Gęba and Rybicki (see [1]).

2. T-equivariant maps and homotopy. Let $\mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^q$, where $n \in \mathbb{N}$, $p, q \in \mathbb{N} \cup \{0\}$ and $n = p + q$. For every $x \in \mathbb{R}^n$ we write $x = (u, v)$, where $u \in \mathbb{R}^p$ and $v \in \mathbb{R}^q$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be given by

$$T(u, v) := (u, -v).$$

The map T is a linear isomorphism and an involution, i.e. $T^2 = \mathrm{Id}_{\mathbb{R}^n}$.

DEFINITION 2.1. A set $X \subset \mathbb{R}^n$ is *T-invariant* if $T(X) \subset X$.

If a set X is T-invariant then $T(X) = X$ and $T_{|X} : X \to X$ is an involution onto X.

DEFINITION 2.2. Let $X \subset \mathbb{R}^n$ be T-invariant.

1. A map $f : X \to \mathbb{R}^n$ is called *T-equivariant* if $f(Tx) = Tf(x)$ for all $x \in X$.
2. $h : X \times [0, 1] \to \mathbb{R}^n$ is called *a T-equivariant homotopy* if it is continuous and $h(Tx, t) = Th(x, t)$ for all $x \in X$ and $t \in [0, 1]$.
3. A function $\tau : X \to \mathbb{R}$ is called *T-equivariant* if $\tau(Tx) = \tau(x)$ for all $x \in X$.

Remark that if $f = (f_1, f_2)$, where $f_1 : X \to \mathbb{R}^p$ and $f_2 : X \to \mathbb{R}^q$, then f is T-equivariant if and only if $f_1(u, -v) = f_1(u, v)$ and $f_2(u, -v) = -f_2(u, v)$ for all $(u, v) \in X$.

From now on, every open bounded T-invariant subset of $\mathbb{R}^n$ is said to be T-*admissible*.

Assume that $\Omega \subset \mathbb{R}^n$ is T-admissible. It is obvious that $\overline{\Omega}$ is T-invariant. We say that $f : \overline{\Omega} \to \mathbb{R}^n$ is T-*admissible* if f is continuous, T-equivariant and $f(x) \neq 0$ for all $x \in \partial\Omega$. We will denote by $\mathcal{A}_T(\Omega)$ the family of all T-admissible maps from $\overline{\Omega}$ into $\mathbb{R}^n$. In the same spirit we generalize the notion of homotopy. We say that a homotopy $h : \overline{\Omega} \times [0, 1] \to \mathbb{R}^n$ is T-*admissible* if h is T-equivariant and $h(x, t) \neq 0$ for all $x \in \partial\Omega$ and $t \in [0, 1]$. We will denote by $\mathcal{HA}_T(\Omega)$ the family of all T-admissible homotopies from $\overline{\Omega} \times [0, 1]$ into $\mathbb{R}^n$.

Definition 2.3. We say that f is *homotopic to g in $\mathcal{A}_T(\Omega)$* and write $f \sim g$ in $\mathcal{A}_T(\Omega)$ if there exists $h \in \mathcal{HA}_T(\Omega)$ joining f to g, i.e. $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for all $x \in \overline{\Omega}$.

It is easy to check that $\sim$ is an equivalence relation in $\mathcal{A}_T(\Omega)$. The homotopy class of $f \in \mathcal{A}_T(\Omega)$ under $\sim$ will be denoted by $[f]$. Finally, the set of all homotopy classes of the relation $\sim$ will be denoted by $\mathcal{A}_T[\Omega]$.

3. T-equivariant generic maps. Let $\Omega \subset \mathbb{R}^n$ be T-admissible. Here and subsequently,

$$\mathcal{A}_T^\infty(\Omega) := \{f \in \mathcal{A}_T(\Omega) : f_{|\Omega} \text{ is smooth}\},$$
$$\mathcal{HA}_T^\infty(\Omega) := \{h \in \mathcal{HA}_T(\Omega) : h_{t|\Omega} \text{ is smooth for } t \in [0, 1]\},$$

where $h_t : \overline{\Omega} \to \mathbb{R}^n$ is defined by $h_t(x) := h(x, t)$.

Definition 3.1. We say that f is *homotopic to g in $\mathcal{A}_T^\infty(\Omega)$* and write $f \simeq g$ in $\mathcal{A}_T^\infty(\Omega)$ if there exists $h \in \mathcal{HA}_T^\infty(\Omega)$ joining f to g.

The relation $\simeq$ is easily seen to be an equivalence relation in $\mathcal{A}_T^\infty(\Omega)$.

A map $f \in \mathcal{A}_T^\infty(\Omega)$ is said to be *generic* if $0 \in \mathbb{R}^n$ is a regular value of $f_{|\Omega}$, i.e. the derivative $Df(x) : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism for all $x \in f^{-1}(\{0\})$.

In this section we show that under some restrictions on Ω, every homotopy class in $\mathcal{A}_T[\Omega]$ possesses a T-equivariant generic map. For this purpose we prove now a few lemmas.

Let K be a compact subset of $\mathbb{R}^n$. A map $f : K \to \mathbb{R}^n$ is called *smooth* if there exists an open set $X \subset \mathbb{R}^n$ such that $K \subset X$ and there exists a smooth map $\tilde{f} : X \to \mathbb{R}^n$ such that $\tilde{f}_{|K} = f$. Let $\{U_i\}_{i=1}^k$ be an open covering of K. Set $U = \bigcup_{i=1}^k U_i$. We call a family of smooth functions $\lambda_i : U \to [0, 1]$, where $i = 1, 2, \ldots, k$, *a smooth partition of unity* subordinate to the covering $\{U_i\}_{i=1}^k$, if this family satisfies the following conditions:

- $\operatorname{supp} \lambda_i = \overline{\{x \in \mathbb{R}^n : \lambda_i(x) \neq 0\}} \subset U_i$ for every $i = 1, 2, \ldots, k$,
- $\sum_{i=1}^k \lambda_i(y) = 1$ for every $y \in K$.

It is well known that such a partition exists (see [16]). Additionally, if K and every U_i are T-invariant sets and every λ_i is a T-equivariant function then we say that $\{U_i\}_{i=1}^k$ is a T-invariant covering of K and $\{\lambda_i\}_{i=1}^k$ is a T-equivariant partition of unity.

LEMMA 3.1. *Assume that $K \subset \mathbb{R}^n$ is compact and T-invariant, and $\{U_i\}_{i=1}^k$ is an open T-equivariant covering of K. Then there exists a smooth T-equivariant partition of unity subordinate to the covering $\{U_i\}_{i=1}^k$.*

Proof. Let $\{\lambda_i\}_{i=1}^k$ be a smooth partition of unity subordinate to the covering $\{U_i\}_{i=1}^k$ of K. For every $i = 1, 2, \ldots, k$, let $\widehat{\lambda_i}$ be given by $\widehat{\lambda_i} = \frac{1}{2}(\lambda_i + \lambda_i T)$. It is obvious that each function $\widehat{\lambda_i}$ is smooth and T-equivariant. The family $\{\widehat{\lambda_i}\}_{i=1}^k$ is a desired one. ∎

Let $K \subset \mathbb{R}^n$ be compact and T-invariant. We say that T *acts freely on* K if $Tx \neq x$ for every $x \in K$, i.e. $K \cap \mathbb{R}^p = \emptyset$. Then

$$\operatorname{dist}(K, \mathbb{R}^p) := \inf\{|x - y| : x \in K,\ y \in \mathbb{R}^p\}$$

is a positive number.

LEMMA 3.2. *Let $K \subset \mathbb{R}^n$ be a compact T-invariant set such that T acts freely on K. If a map $f : K \to \mathbb{R}^n$ is continuous and T-equivariant then for every $\varepsilon > 0$ there is a smooth T-equivariant map $g : K \to \mathbb{R}^n$ such that*

$$\sup_{x \in K} |f(x) - g(x)| < \varepsilon.$$

From now on, $B(a, r)$ stands for an open ball of radius r, centered at a point $a \in \mathbb{R}^n$.

Proof. Fix $\varepsilon > 0$. Since K is compact, f is uniformly continuous. Hence, there is $\delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. Set $\delta' = \min\{\delta, \operatorname{dist}(K, \mathbb{R}^p)\}$, and $U_x = B(x, \delta')$ for every $x \in K$. Then

- $TU_x = B(Tx, \delta')$,
- $U_x \cap TU_x = \emptyset$,
- $K \subset \bigcup_{x \in K}(U_x \cup TU_x)$.

The compactness of K implies that there exist points $x_1, x_2, \ldots, x_k \in K$ such that $K \subset \bigcup_{i=1}^k (U_i \cup TU_i)$, where $U_i = U_{x_i}$. Consider a smooth T-equivariant partition of unity $\{\lambda_i\}_{i=1}^k$ subordinate to the covering $\{U_i \cup TU_i\}_{i=1}^k$ of K. Set $U = \bigcup_{i=1}^k (U_i \cup TU_i)$. For every $i \in \{1, 2, \ldots, k\}$, let $\pi_i : U \to \mathbb{R}^n$ be a map such that $\pi_i(U_i) = \{x_i\}$ and $\pi_i(TU_i) = \{Tx_i\}$. The function $g : U \to \mathbb{R}^n$ is defined by

$$g(x) = \sum_{i=1}^k \lambda_i(x) f(\pi_i(x)).$$

Take $x \in U$. If $x \in U_i \cup TU_i$ then in a sufficiently small neighbourhood of x the map $f \circ \pi_i$ is constant. If $x \in \partial U_i \cup \partial TU_i$ then in a sufficiently small neighbourhood of x the function λ_i is equal to 0. Hence g is smooth.

Take $x \in K$. If $x \in U_i$ then $\pi_i(x) = x_i$ and $|\pi_i(x) - x| < \delta$. If $x \in TU_i$ then $\pi_i(x) = Tx_i$ and $|\pi_i(x) - x| < \delta$. Finally, if $x \notin U_i \cup TU_i$ then $\lambda_i(x) = 0$. From this it follows that

$$|g(x) - f(x)| = \Big|\sum_{i=1}^k \lambda_i(x) f(\pi_i(x)) - \sum_{i=1}^k \lambda_i(x) f(x)\Big| \le \sum_{i=1}^k \lambda_i(x) |f(\pi_i(x)) - f(x)| < \varepsilon.$$

Moreover,

$$g(Tx) = \sum_{i=1}^{k} \lambda_i(Tx) f(\pi_i(Tx)) = \sum_{i=1}^{k} \lambda_i(x) f(T\pi_i(x)) = \sum_{i=1}^{k} \lambda_i(x) T f(\pi_i(x)) = Tg(x),$$

which completes the proof. ∎

CONCLUSION 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a T-admissible set such that T acts freely on $\overline{\Omega}$. Then for every $f \in \mathcal{A}_T(\Omega)$ there exists $g \in \mathcal{A}_T^{\infty}(\Omega)$ such that $f \sim g$ in $\mathcal{A}_T(\Omega)$.*

Proof. By the assumption, $\overline{\Omega}$ is a T-invariant compact set and T acts freely on $\overline{\Omega}$. Set $d = \inf\{|f(x)| : x \in \partial\Omega\}$. From Lemma 3.2 it follows that there exists $g \in \mathcal{A}_T^{\infty}(\Omega)$ such that

$$\sup_{x \in \overline{\Omega}} |f(x) - g(x)| < d.$$

Consider the linear homotopy $h : \overline{\Omega} \times [0,1] \to \mathbb{R}^n$ joining f to g, i.e.

$$h(x,t) = tg(x) + (1-t)f(x).$$

It is trivial that h is continuous and $h(Tx,t) = Th(x,t)$ for all $x \in \overline{\Omega}$ and $t \in [0,1]$. Take $x \in \partial\Omega$ and $t \in [0,1]$. Then

$$|h(x,t)| = |f(x) - t(f(x) - g(x))| \geq |f(x)| - t|f(x) - g(x)| \geq |f(x)| - |f(x) - g(x)| > 0.$$

Hence $h \in \mathcal{HA}_T(\Omega)$. ∎

Let $U \subset \mathbb{R}^n$ be an open bounded set, and $K \subset U$ be compact. It is well known that there exists a smooth function $\eta : \mathbb{R}^n \to [0,1]$ such that

$$\eta(x) = \begin{cases} 1 & \text{for } x \in K, \\ 0 & \text{for } x \in \mathbb{R}^n \setminus U. \end{cases}$$

In the mathematical literature, η is called *the Urysohn function* (see [8]).

LEMMA 3.4. *Let U and U_0 be T-admissible subsets of $\mathbb{R}^n$. Assume that $\overline{U}_0 \subset U$. Then there exists a smooth T-equivariant function $\tilde{\eta} : \mathbb{R}^n \to [0,1]$ such that $\tilde{\eta}(x) = 1$ for every $x \in \overline{U}_0$ and $\tilde{\eta}(x) = 0$ for every $x \in \mathbb{R}^n \setminus U$.*

The proof is similar to that of Lemma 3.1. We leave it to the reader.

LEMMA 3.5. *Let Ω_0 and Ω be T-admissible subsets of $\mathbb{R}^n$ such that $\Omega_0 \subset \Omega$. Suppose that $f_0 \simeq g_0$ in $\mathcal{A}_T^{\infty}(\Omega_0)$ and there is an $f \in \mathcal{A}_T^{\infty}(\Omega)$ such that $f_{|\Omega_0} = f_0$. Then there exist a map $g \in \mathcal{A}_T^{\infty}(\Omega)$ and a T-admissible set $U_0 \subset \Omega_0$ satisfying the following conditions*:

1. *$f \simeq g$ in $\mathcal{A}_T^{\infty}(\Omega)$,*
2. *$g(x) = f(x)$ for every $x \in \overline{\Omega} \setminus \Omega_0$,*
3. *$g(x) = g_0(x)$ for every $x \in U_0$,*
4. *$g_0^{-1}(\{0\}) \cap \Omega_0 = g^{-1}(\{0\}) \cap \Omega_0 \subset U_0$.*

Proof. Let $\bar{h} \in \mathcal{HA}_T^{\infty}(\Omega_0)$ be a homotopy joining f_0 to g_0. Take an open T-invariant subset U_0 of $\mathbb{R}^n$ such that $\overline{U}_0 \subset \Omega_0$ and $\bar{h}(x,t) \neq 0$ for every $(x,t) \in (\overline{\Omega}_0 \setminus U_0) \times [0,1]$. Consider an open T-invariant subset U of $\mathbb{R}^n$ such that $\overline{U}_0 \subset U \subset \overline{U} \subset \Omega_0$. Let $\eta : \mathbb{R}^n \to [0,1]$ be a smooth T-equivariant Urysohn function for the pair of sets U_0 and U, i.e. $\eta(x) = 1$ for every $x \in \overline{U}_0$ and $\eta(x) = 0$ for every $x \in \mathbb{R}^n \setminus U$. Let $h : \overline{\Omega} \times [0,1] \to \mathbb{R}^n$ be defined by

$$h(x,t) = \begin{cases} f(x) & \text{for } x \in \overline{\Omega} \setminus \overline{U}, \\ \bar{h}(x, t\eta(x)) & \text{for } x \in \Omega_0. \end{cases}$$

We check that $h \in \mathcal{HA}_T^\infty(\Omega)$ and $g(x) := h(x,1)$, $x \in \overline{\Omega}$, satisfies the claim of our lemma.

Remark that $(\overline{\Omega} \setminus \overline{U}) \cap \Omega_0 = \Omega_0 \setminus \overline{U}$. If $x \in \Omega_0 \setminus \overline{U}$ then $\eta(x) = 0$, and hence $\bar{h}(x, t\eta(x)) = \bar{h}(x,0) = f(x)$ for all $t \in [0,1]$. In consequence, h is smooth and $h(x,0) = f(x)$ for every $x \in \overline{\Omega}$. If $x \in \partial\Omega$ then $h(x,t) = f(x) \neq 0$ for all $t \in [0,1]$. Moreover, for $x \in \Omega_0$ and $t \in [0,1]$ we have $h(Tx,t) = \bar{h}(Tx, t\eta(Tx)) = \bar{h}(Tx, t\eta(x)) = T\bar{h}(x, t\eta(x)) = Th(x,t)$. Thus h is T-equivariant. Summarizing, $h \in \mathcal{HA}_T^\infty(\Omega)$ and it joins f to g.

Take $x \in \overline{\Omega} \setminus \Omega_0$. Since $\overline{\Omega} \setminus \Omega_0 \subset \overline{\Omega} \setminus U$, we get $g(x) = h(x,1) = f(x)$.

If $x \in U_0$ then $\eta(x) = 1$ and $g(x) = h(x,1) = \bar{h}(x,1) = g_0(x)$.

Finally, fix $x \in \Omega_0$. If $x \in \Omega_0 \setminus \overline{U}$ then $g(x) = \bar{h}(x,0) = f_0(x)$. If $x \in \overline{U}$ then $g(x) = \bar{h}(x, \eta(x))$. Since $\{x \in \overline{\Omega}_0 : \bar{h}(x,t) = 0 \text{ for any } t \in [0,1]\} \subset U_0$, we have $g_0^{-1}(\{0\}) \cap \Omega_0 = g^{-1}(\{0\}) \cap \Omega_0 \subset U_0$, which completes the proof. ∎

Let $K \subset \mathbb{R}^n$ be nonempty, compact and T-admissible. Set $k \in \mathbb{N}$. We call a family of open sets $\{U_i\}_{i=1}^k$ a (T,k)-*simple covering* of K if it satisfies the following conditions:

1. $U_i \cap TU_i = \emptyset$ for every $i \in \{1, 2, \ldots, k\}$,
2. $K \subset \bigcup_{i=1}^k (U_i \cup TU_i)$.

We say that K is *a* (T,k)-*simple set* if it possesses a (T,k)-simple covering. If $K = \emptyset$, it is said to be $(T,0)$-*simple*.

PROPOSITION 3.6. *Every nonempty compact T-invariant subset K of $\mathbb{R}^n$ such that T acts freely on K is (T,k)-simple for a certain $k \in \mathbb{N}$.*

Proof. Since T acts freely on K, $K \cap \mathbb{R}^p = \emptyset$. Set $l = \operatorname{dist}(K, \mathbb{R}^p)$. We have

$$K \subset \bigcup_{x \in K} B(x,l)$$

By compactness of K, there are $x_1, x_2, \ldots, x_k \in K$ such that

$$K \subset \bigcup_{i=1}^k B(x_i,l).$$

Let $U_i = B(x_i,l)$ for $i = 1, 2, \ldots, k$. It is evident that $U_i \cap TU_i = \emptyset$ and

$$K \subset \bigcup_{i=1}^k (U_i \cup TU_i).$$ ∎

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set. For every $f : \overline{\Omega} \to \mathbb{R}^n$ such that $f_{|\Omega}$ is C^r-smooth, where $r \geq 1$, and $f(x) \neq 0$ for all $x \in \partial\Omega$, set

$$R(f) = \{x \in f^{-1}(\{0\}) : Df(x) \in \mathrm{GL}(\mathbb{R}^n)\}.$$

LEMMA 3.7. *Assume that $\Omega \subset \mathbb{R}^n$ is T-admissible, T acts freely on $\overline{\Omega}$, which is (T,k)-simple for a certain $k \in \mathbb{N}$. Let $f \in \mathcal{A}_T^\infty(\Omega)$. Then there exists $g \in \mathcal{A}_T^\infty(\Omega)$ such that*

(i) *$f \simeq g$ in $\mathcal{A}_T^\infty(\Omega)$,*

(ii) *$g^{-1}(\{0\}) \setminus R(g)$ is $(T,k-1)$-simple.*

Proof. Let $\{U_i\}_{i=1}^k$ be a (T,k)-simple covering of $\overline{\Omega}$. Set

$$K = \overline{\Omega} \setminus \bigcup_{i=2}^{k} (U_i \cup TU_i), \quad K_1 = K \cap U_1.$$

Let us remark that K is T-invariant, $K \subset (U_1 \cup TU_1)$, K and K_1 are compact. Thus K is $(T,1)$-simple and $K = K_1 \cup TK_1$.

From the Sard theorem it follows that there exists a regular value y_0 of $f_{|\Omega\cap U_1}$ such that $|y_0| < \inf\{|f(x)| : x \in \partial\Omega\}$. Since $f \circ T = T \circ f$, we have $Df(Tx) = T \circ Df(x) \circ T$ for every $x \in \Omega$. Hence Ty_0 is also a regular value of $f_{|\Omega\cap TU_1}$. Moreover, $|Ty_0| = |y_0|$.

Let $\eta : \mathbb{R}^n \to [0,1]$ be a smooth function such that $\eta(x) = 1$ for all $x \in K_1$ and $\eta(x) = 0$ for all $x \in \mathbb{R}^n \setminus U_1$. Let $g : \overline{\Omega} \to \mathbb{R}^n$ be given by

$$g(x) = \begin{cases} f(x) - \eta(x)y_0 & \text{for } x \in U_1 \cap \overline{\Omega}, \\ f(x) - \eta(Tx)Ty_0 & \text{for } x \in TU_1 \cap \overline{\Omega}, \\ f(x) & \text{for } x \in \overline{\Omega} \setminus (U_1 \cup TU_1), \end{cases}$$

The map $g_{|\Omega}$ is easily seen to be smooth. Let

$$h(x,t) = f(x) + t(g(x) - f(x))$$

for all $(x,t) \in \overline{\Omega} \times [0,1]$. Take $x \in \overline{\Omega}$. If $x \in U_1 \cap \overline{\Omega}$ then $g(Tx) = f(Tx) - \eta(T^2x)Ty_0 = Tf(x) - \eta(x)Ty_0 = T(f(x) - \eta(x)y_0) = Tg(x)$. If $x \in TU_1 \cap \overline{\Omega}$ then $g(Tx) = f(Tx) - \eta(Tx)y_0 = Tf(x) - \eta(Tx)y_0 = Tf(x) - T^2\eta(Tx)y_0 = T(f(x) - \eta(Tx)Ty_0) = Tg(x)$. Finally, if $x \in \overline{\Omega} \setminus (U_1 \cup TU_1)$ then $g(Tx) = f(Tx) = Tf(x) = Tg(x)$. Consequently, g is T-equivariant.

Since $|g(x) - f(x)| < |y_0|$ for all $x \in \overline{\Omega}$, we conclude that h is a homotopy joining f to g in $\mathcal{A}_T^\infty(\Omega)$.

Remark that $g^{-1}(\{0\}) \setminus R(g)$ is a compact set and $g^{-1}(\{0\}) \subset \bigcup_{i=2}^k (U_i \cup TU_i) \cup K$. Take $x \in K$. If $x \in K_1$ then $g(x) = f(x) - y_0$. If $x \in TK_1$ then $g(x) = f(x) - Ty_0$. From this $K \cap g^{-1}(\{0\}) \subset R(g)$, and so $g^{-1}(\{0\}) \setminus R(g) \subset \bigcup_{i=2}^k (U_i \cup TU_i)$ is $(T,k-1)$-simple. ∎

LEMMA 3.8. *Let $\Omega \subset \mathbb{R}^n$ be a T-admissible set such that T acts freely on $\overline{\Omega}$. Assume that $f \in \mathcal{A}_T^\infty(\Omega)$ and $f^{-1}(\{0\}) \setminus R(f)$ is (T,k)-simple for a certain $k \in \mathbb{N}$. Then there exists a map $g \in \mathcal{A}_T^\infty(\Omega)$ such that*

(i) *$f \simeq g$ in $\mathcal{A}_T^\infty(\Omega)$,*

(ii) *$g^{-1}(\{0\}) \setminus R(g)$ is $(T,k-1)$-simple.*

Proof. Since $f^{-1}(\{0\}) \setminus R(f)$ is (T,k)-simple, there is an open and T-invariant subset Ω_0 of Ω such that

(a) $f^{-1}(\{0\}) \setminus R(f) \subset \Omega_0$,

(b) $R(f) \subset \Omega \setminus \overline{\Omega}_0$,

(c) $\overline{\Omega}_0$ is (T,k)-simple.

Set $f_0 = f_{|\Omega_0}$. Combining (a) with (b), we see that $f_0 \in \mathcal{A}_T^\infty(\Omega_0)$. By Lemma 3.7 it follows that there is $g_0 \in \mathcal{A}_T^\infty(\Omega_0)$ such that $f_0 \simeq g_0$ in $\mathcal{A}_T^\infty(\Omega_0)$ and $g_0^{-1}(\{0\}) \setminus R(g_0)$ is $(T, k-1)$-simple. From Lemma 3.5 we have that there is $g \in \mathcal{A}_T^\infty(\Omega)$ such that $f \simeq g$ in $\mathcal{A}_T^\infty(\Omega)$ and $g^{-1}(\{0\})\setminus R(g) = g_0^{-1}(\{0\})\setminus R(g_0)$. Thus $g^{-1}(\{0\})\setminus R(g)$ is $(T, k-1)$-simple. ∎

Applying the mathematical induction, Lemma 3.8 and Conclusion 3.3, one can immediately prove the next theorem.

Theorem 3.9. *Let $\Omega \subset \mathbb{R}^n$ be a T-admissible set such that T acts freely on $\overline{\Omega}$. If $f \in \mathcal{A}_T(\Omega)$ then there exists a generic map $g \in \mathcal{A}_T^\infty(\Omega)$ such that $f \sim g$ in $\mathcal{A}_T(\Omega)$.*

Conclusion 3.10. *Let $\Omega \subset \mathbb{R}^n$ be a T-admissible set such that T acts freely on $\overline{\Omega}$. If $f \in \mathcal{A}_T(\Omega)$ then* $\deg(f,\Omega) \in 2\mathbb{Z}$.

Here and subsequently, $\deg(f,\Omega)$ stands for the Brouwer degree of f on Ω.

Proof. Fix $f \in \mathcal{A}_T(\Omega)$. By Theorem 3.9 there is a generic map $g \in \mathcal{A}_T^\infty(\Omega)$ such that $f \sim g$ in $\mathcal{A}_T(\Omega)$. Hence $\deg(f,\Omega) = \deg(g,\Omega)$. Since $T\circ g = g\circ T$, we have $Dg(x) = T\circ Dg(Tx)\circ T$ for every $x \in \Omega$. From this

$$g^{-1}(\{0\}) \cap \Omega = \{x_1, x_2, \ldots, x_m\} \cup \{Tx_1, Tx_2, \ldots, Tx_m\}$$

and $\operatorname{sign}\det Dg(x_i) = \operatorname{sign}\det Dg(Tx_i)$ for $i = 1, 2, \ldots, m$. In consequence,

$$\deg(g,\Omega) = \sum_{i=1}^m \operatorname{sign}\det Dg(x_i) + \sum_{i=1}^m \operatorname{sign}\det Dg(Tx_i) = 2\sum_{i=1}^m \operatorname{sign}\det Dg(x_i),$$

which completes the proof. ∎

4. T-equivariant normal maps.

Let $\Omega \subset \mathbb{R}^n$ be T-admissible and let $\varepsilon > 0$. Define

$$\Omega(\varepsilon) = \{(u,v) \in \Omega : |v| < \varepsilon\}.$$

Definition 4.1. Let $f = (f_1, f_2) \in \mathcal{A}_T(\Omega)$, where $f_1 : \overline{\Omega} \to \mathbb{R}^p$, and $f_2 : \overline{\Omega} \to \mathbb{R}^q$.

1. A map f is said to be *ε-normal* if there exists $\varepsilon > 0$ such that

$$f(u,v) = (f_1(u,0), v)$$

for all $(u,v) \in \Omega(\varepsilon)$.

2. A map f is called *normal* if there exists $\varepsilon > 0$ such that f is ε-normal.

We will denote by $\mathcal{NA}_T(\Omega)$ the family of all normal maps from $\overline{\Omega}$ into $\mathbb{R}^n$.

Definition 4.2. Let $\Omega \subset \mathbb{R}^n$ be a T-admissible set.

1. A homotopy $h \in \mathcal{HA}_T(\Omega)$ is called *normal* if there exists $\varepsilon > 0$ such that $h_t : \overline{\Omega} \to \mathbb{R}^n$ is ε-normal for every $t \in [0,1]$.

2. We say that f is *homotopic to g in $\mathcal{NA}_T(\Omega)$* and write $f \approx g$ in $\mathcal{NA}_T(\Omega)$ if there exists a normal homotopy joining f to g.

We will denote by $\mathcal{HNA}_T(\Omega)$ the family of all normal homotopies from $\overline{\Omega} \times [0,1]$ into $\mathbb{R}^n$. The homotopy class of $f \in \mathcal{NA}_T(\Omega)$ under $\approx$ will be denoted by $[[f]]$. Finally, the set of all homotopy classes of the relation $\approx$ will be denoted by $\mathcal{NA}_T[\Omega]$.

The construction of the degree for maps in $\mathcal{A}_T(\Omega)$, which will be described in the next section, is based on the following theorem.

Theorem 4.1. *The map* $\tau : \mathcal{N}\mathcal{A}_T[\Omega] \to \mathcal{A}_T[\Omega]$, $[[f]] \mapsto [f]$ *is a bijection.*

Proof.

Step 1. We show that τ is a surjection.

Fix $f = (f_1, f_2) \in \mathcal{A}_T(\Omega)$. We show that there is $g \in \mathcal{N}\mathcal{A}_T(\Omega)$ such that $f \sim g$ in $\mathcal{A}_T(\Omega)$. Set $d = \inf\{|f(x)| : x \in \partial\Omega\}$. Since f is T-equivariant, $f_2(u, 0) = 0$ for every $(u, 0) \in \overline{\Omega}$. By the continuity of f, there is $0 < \varepsilon \leq d/12$ such that if $x, y \in \overline{\Omega}$ and $|x - y| < 2\varepsilon$ then $|f_i(x) - f_i(y)| < d/6$ for $i = 1, 2$.

Let $\eta : \mathbb{R} \to [0, 1]$ be a smooth function such that $\eta(t) = 1$ for every $|t| \leq \varepsilon$ and $\eta(t) = 0$ for every $|t| \geq 2\varepsilon$. Let $h : \overline{\Omega} \times [0, 1] \to \mathbb{R}^n$ be defined by

$$h(u, v, t) = (1 - t\eta(|v|))f(u, v) + t\eta(|v|)(f_1(u, 0), v).$$

Then

$$\begin{aligned} h(T(u, v), t) = h(u, -v, t) &= (1 - t\eta(|v|))f(u, -v) + t\eta(|v|)(f_1(u, 0), -v) \\ &= (1 - t\eta(|v|))Tf(u, v) + t\eta(|v|)T(f_1(u, 0), v) = Th(u, v, t), \end{aligned}$$

for every $(u, v) \in \overline{\Omega}$ and $t \in [0, 1]$.

Take $(u, v) \in \partial\Omega$ and $t \in [0, 1]$. If $|v| \geq 2\varepsilon$ then $h(u, v, t) = f(u, v) \neq 0$. If $|v| < 2\varepsilon$ then

$$\begin{aligned} |h(u, v, t)| &= |f(u, v) - t\eta(|v|)\big(f_1(u, v) - f_1(u, 0), f_2(u, v) - v\big)| \\ &\geq |f(u, v)| - t\eta(|v|)|f_1(u, v) - f_1(u, 0)| - t\eta(|v|)|f_2(u, v) - v| \\ &\geq |f(u, v)| - |f_1(u, v) - f_1(u, 0)| - |f_2(u, v) - v| \\ &\geq |f(u, v)| - \big(|f_1(u, v) - f_1(u, 0)| + |f_2(u, v)| + |v|\big) \\ &> d - 3\frac{d}{6} = \frac{d}{2} > 0. \end{aligned}$$

In consequence, $h \in \mathcal{H}\mathcal{A}_T(\Omega)$. Set $g := h_1$. If $|v| \leq \varepsilon$ then $g(u, v) = h(u, v, 1) = (f_1(u, 0), v)$. Thus g is normal.

Step 2. We show that τ is an injection.

Take $f = (f_1, f_2) \in \mathcal{N}\mathcal{A}_T(\Omega)$ and $g = (g_1, g_2) \in \mathcal{N}\mathcal{A}_T(\Omega)$ such that $f \sim g$ in $\mathcal{A}_T(\Omega)$. We prove that $f \approx g$ in $\mathcal{N}\mathcal{A}_T(\Omega)$. Let $h = (h_I, h_{II}) \in \mathcal{A}_T(\Omega)$ be a homotopy joining f to g in $\mathcal{A}_T(\Omega)$. Set $d = \inf\{|h(x, t)| : x \in \partial\Omega \wedge t \in [0, 1]\}$. Since h is T-equivariant, we get $h_{II}(u, 0, t) = 0$ for every $(u, 0) \in \overline{\Omega}$ and $t \in [0, 1]$. Take $0 < \varepsilon \leq d/12$ such that f, g are 2ε-normal, and if $x, y \in \overline{\Omega}$ and $|x - y| < 2\varepsilon$ then $|h(x, t) - h(y, t)| < d/6$. Let $\eta : \mathbb{R} \to [0, 1]$ be a smooth function such that $\eta(t) = 1$ for every $|t| \leq \varepsilon$ and $\eta(t) = 0$ for every $|t| \geq 2\varepsilon$. Let $\hat{h} : \overline{\Omega} \times [0, 1] \to \mathbb{R}^n$ be given by

$$\hat{h}(u, v, t) = (1 - \eta(|v|))h(u, v, t) + \eta(|v|)(h_I(u, 0, t), v).$$

We check at once that $\hat{h}$ is a normal homotopy joining f to g. ∎

5. T-equivariant degree. In this section we introduce the degree of T-equivariant maps in $\mathbb{R}^n$, called the T-equivariant degree. First we define this degree for T-equivariant normal maps, and next for all T-admissible ones.

Let $\Omega \subset \mathbb{R}^n$ be a T-admissible set and let $f = (f_1, f_2) \in \mathcal{NA}_T(\Omega)$, where $f_1 : \overline{\Omega} \to \mathbb{R}^p$, $f_2 : \overline{\Omega} \to \mathbb{R}^q$. Set $\Omega_0 = \Omega \cap \mathbb{R}^p$. Assume that $\Omega_0 \neq \emptyset$. The map $g_0 : \overline{\Omega}_0 \to \mathbb{R}^p$ is given by $g_0(u) = f_1(u, 0)$. Since $f(u, v) \neq 0$ for all $(u, v) \in \partial\Omega$ and $f_2(u, 0) = 0$ for all $(u, 0) \in \overline{\Omega}$, we conclude that $g_0(u, v) \neq 0$ for all $(u, v) \in \partial\Omega_0$. Define

$$d_0 = \begin{cases} \deg(g_0, \Omega_0) & \text{if } \Omega_0 \neq \emptyset, \\ 0 & \text{if } \Omega_0 = \emptyset. \end{cases}$$

Since f is normal, there is $\varepsilon > 0$ such that $f(x) \neq 0$ for all $x \in \partial\Omega(\varepsilon)$. Set $\Omega_1 = \Omega \setminus \overline{\Omega(\varepsilon)}$. Let us remark that T acts freely on $\overline{\Omega}_1$. Define

$$g_1(x) = f(x),$$

where $x \in \overline{\Omega}_1$. It is evident that $g_1 \in \mathcal{A}_T(\Omega_1)$. By Conclusion 3.10 there exists an integer d_1 such that $\deg(g_1, \Omega_1) = 2d_1$. The T-equivariant degree of f on Ω is given as follows:

$$\deg_T(f, \Omega) = (d_0, d_1) \in \mathbb{Z} \oplus \mathbb{Z}.$$

Let us denote by $\mathcal{N}$ the set of all pairs (f, Ω) such that $f \in \mathcal{NA}_T(\Omega)$ and $\Omega \subset \mathbb{R}^n$ is T-admissible.

THEOREM 5.1. *The map* $\deg_T : \mathcal{N} \to \mathbb{Z} \oplus \mathbb{Z}$, $(f, \Omega) \mapsto \deg_T(f, \Omega)$, *possesses the following properties*:

1. *Homotopy invariance*:
If $h \in \mathcal{HNA}_T(\Omega)$ *then* $\deg_T(h_t, \Omega) = \deg_T(h_0, \Omega)$ *for every* $t \in (0, 1]$.
2. *Excision*:
Assume that $\Omega_0 \subset \Omega$ *is* T*-invariant and* $f^{-1}(\{0\}) \cap \Omega \subset \Omega_0$. *Then*
$$\deg_T(f, \Omega) = \deg_T(f_{|\Omega_0}, \Omega_0).$$
3. *Additivity*:
Assume that Ω_1, Ω_2 *are disjoint open* T*-invariant subsets of* Ω *such that* $f^{-1}(\{0\}) \cap \Omega \subset \Omega_1 \cup \Omega_2$. *Then*
$$\deg_T(f, \Omega) = \deg_T(f_{|\Omega_1}, \Omega_1) + \deg_T(f_{|\Omega_2}, \Omega_2).$$
4. *Existence*:
If $\deg_T(f, \Omega) \neq 0$ *then there exists a point* $x \in \Omega$ *such that* $f(x) = 0$.

We call $\deg_T : \mathcal{N} \to \mathbb{Z} \oplus \mathbb{Z}$ *the* T*-equivariant degree* of normal maps. Its properties follow directly from the definition. It is worth pointing out that if $f \in \mathcal{NA}_T(\Omega)$ then there is the following dependence between $\deg(f, \Omega)$ and $\deg_T(f, \Omega)$:

$$\deg(f, \Omega) = d_0 + 2d_1, \qquad d_0 = \deg(f, \Omega(\varepsilon)).$$

Let $\mathcal{E}$ denote the family of all pairs (f, Ω) such that $f \in \mathcal{A}_T(\Omega)$ and Ω is T-admissible. Applying Theorem 4.1 one can extend the T-equivariant degree over $\mathcal{E}$. Consider $f \in \mathcal{A}_T(\Omega)$. There exists $g \in \mathcal{NA}_T(\Omega)$ such that $[g] = [f]$. Set

$$\mathrm{Deg}_T(f, \Omega) = \deg_T(g, \Omega).$$

From Theorem 4.1 it follows that the above formula does not depend on the choice of g.

DEFINITION 5.1. The map $\mathrm{Deg}_T : \mathcal{E} \to \mathbb{Z} \oplus \mathbb{Z}$, $(f, \Omega) \mapsto \mathrm{Deg}_T(f, \Omega)$, is called *the T-equivariant degree.*

The next theorem is a natural consequence of Definition 5.1 and Theorem 5.1.

THEOREM 5.2. *The T-equivariant degree possesses the following properties*:

1. *If $h \in \mathcal{HA}_T(\Omega)$ then $\mathrm{Deg}_T(h_t, \Omega) = \mathrm{Deg}_T(h_0, \Omega)$ for every $t \in (0, 1]$.*
2. *Assume that $\Omega_0 \subset \Omega$ is T-invariant and $f^{-1}(\{0\}) \cap \Omega \subset \Omega_0$. Then*
$$\mathrm{Deg}_T(f, \Omega) = \mathrm{Deg}_T(f_{|\Omega_0}, \Omega_0).$$
3. *Assume that Ω_1 and Ω_2 are disjoint open T-invariant subsets of Ω such that $f^{-1}(\{0\}) \cap \Omega \subset \Omega_1 \cup \Omega_2$. Then*
$$\mathrm{Deg}_T(f, \Omega) = \mathrm{Deg}_T(f_{|\Omega_1}, \Omega_1) + \mathrm{Deg}_T(f_{|\Omega_2}, \Omega_2).$$
4. *If $\mathrm{Deg}_T(f, \Omega) \neq 0$ then there exists a point $x \in \Omega$ such that $f(x) = 0$.*

6. T-homotopies versus gradient T-homotopies. In this section we prove the Parusiński theorem in T-invariant case.

From now on, we assume that $\mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^q$ and $p \geq 2$. Let B^n denote the open unit ball, S^{n-1} the unit sphere, and D^n the unit disc in $\mathbb{R}^n$ centered at 0. We have $D^n = B^n \cup S^{n-1}$. It is trivial that these sets are T-invariant. Set $D^p = D^n \cap \mathbb{R}^p$, $B^p = B^n \cap \mathbb{R}^p$ and $S^{p-1} = S^{n-1} \cap \mathbb{R}^p$.

Among many generalizations of the Brouwer degree there is the stable equivariant degree. It was considered by several authors (see [2, 4, 17] and the references given there). The stable equivariant degree of the T-equivariant continuous map $f : S^{n-1} \to S^{n-1}$ is the element $d_T(f) \in \mathbb{Z} \oplus \mathbb{Z}$ given by
$$d_T(f) = (\deg(f, S^{p-1}), \deg(f, S^{n-1})).$$
Let $[S^{n-1}, S^{n-1}]_T$ denote the set of all T-equivariant homotopy classes of T-equivariant continuous self-maps of S^{n-1}. Let $[f]_T$ stands for the T-equivariant homotopy class of $f : S^{n-1} \to S^{n-1}$. D. Ferrario proved that the stable equivariant degree d_T classifies T-equivariant continuous self-maps of S^{n-1} (see Theorem 7.1 in [4]). This means that the map
$$[S^{n-1}, S^{n-1}]_T \ni [f]_T \longmapsto d_T(f) \in \mathbb{Z} \oplus \mathbb{Z}$$
is an injection.

PROPOSITION 6.1. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a T-equivariant continuous map such that $f(S^{n-1}) \subset \mathbb{R}^n \setminus \{0\}$. Then there exist a T-equivariant continuous map $\hat{f} : \mathbb{R}^n \to \mathbb{R}^n$ and a T-equivariant homotopy $h : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ such that*

- $h_0 = f$, $h_1 = \hat{f}$,
- $\hat{f}(S^{n-1}) \subset S^{n-1}$,
- $\hat{f}(D^n) \subset D^n$,
- $h(S^{n-1} \times [0, 1]) \subset \mathbb{R}^n \setminus \{0\}$.

Proof. We check at once that $\hat{f} : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\hat{f}(x) = \begin{cases} |x| \left| f\left(\frac{x}{|x|}\right)\right|^{-1} f\left(\frac{x}{|x|}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}$$

and $h(x,t) = t\hat{f}(x) + (1-t)f(x)$ satisfy all the claims. ∎

Fix $f \in \mathcal{A}_T(B^n)$. Clearly, f can be extended to a T-equivariant continuous map over $\mathbb{R}^n$. Moreover, two different extensions of f are linear homotopic and the linear homotopy joining these extensions has no zeroes on $S^{n-1} \times [0,1]$. Therefore we identify f with its extension. Let $\hat{f}$ be a map as in Proposition 6.1. Then $\mathrm{Deg}_T(f, B^n) = \mathrm{Deg}_T(\hat{f}, B^n)$. Remark that there is one-to-one correspondence between $\mathrm{Deg}_T(f, B^n) = (d_0, d_1)$ and $d_T(\hat{f}) = (\deg(\hat{f}, S^{p-1}), \deg(\hat{f}, S^{n-1}))$. Namely,

$$d_0 = \deg(\hat{f}, S^{p-1}), \qquad d_1 = \frac{1}{2}\big(\deg(\hat{f}, S^{n-1}) - \deg(\hat{f}, S^{p-1})\big).$$

Therefore the map

$$\mathcal{A}_T[B^n] \ni [f] \longmapsto d_T(\hat{f}) \in \mathbb{Z} \oplus \mathbb{Z}$$

is an injection.

Conclusion 6.2. *The T-equivariant degree* $\mathrm{Deg}_T(f, B^n)$ *classifies T-admissible maps from D^n into $\mathbb{R}^n$.*

Another generalization of the Brouwer degree is the T-equivariant degree for gradient T-equivariant maps from $\mathbb{R}^n$ into $\mathbb{R}^n$. This degree was considered in [7, 5, 1].

Assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a T-equivariant continuous map. We say that f is ∇_T-admissible if $f(S^{n-1}) \subset \mathbb{R}^n \setminus \{0\}$ and there exists a T-equivariant C^1 function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $f = \nabla\varphi$. We will denote by $\nabla\mathcal{A}_T(B^n)$ the set of all ∇_T-admissible maps. In the same spirit we introduce the notion of ∇_T-admissible homotopy. Let $h : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ be a T-equivariant homotopy. We say that h is ∇_T-admissible if $h(S^{n-1} \times [0,1]) \subset \mathbb{R}^n \setminus \{0\}$ and there exists a T-equivariant C^1 function $\chi : \mathbb{R}^n \times [0,1] \to \mathbb{R}$ such that $h(x,t) = \nabla_x \chi(x,t)$ for all $x \in \mathbb{R}^n$ and $t \in [0,1]$.

f is homotopic to g in $\nabla\mathcal{A}_T(B^n)$, if there is a ∇_T-admissible homotopy h joining f to g. The ∇_T-admissible homotopy class of $f \in \nabla\mathcal{A}_T(B^n)$ will be denoted by $[f]_\nabla$. The set of all ∇_T-admissible homotopy classes in $\nabla\mathcal{A}_T(B^n)$ will be denoted by $\nabla\mathcal{A}_T[B^n]$.

The T-equivariant degree of $f \in \nabla\mathcal{A}_T(B^n)$ is the element $\nabla_T \deg(f, B^n) \in \mathbb{Z} \oplus \mathbb{Z}$. From the construction made by Gęba in [5] (see formula 3.5, Theorems 3.2 and 3.3), it follows that

$$\nabla_T \deg(f, B^n) = \mathrm{Deg}_T(f, B^n).$$

Dancer, Gęba and Rybicki proved that this degree classifies ∇_T-admissible maps. More precisely, the map

$$\nabla\mathcal{A}_T[B^n] \ni [f]_\nabla \longmapsto \nabla_T \deg(f, B^n) \in \mathbb{Z} \oplus \mathbb{Z}$$

is a bijection (see Corollary 4.1 and Remark 4.1 in [1]).

Conclusion 6.3. *The T-equivariant degree* $\mathrm{Deg}_T(f, B^n)$ *classifies ∇_T-admissible maps.*

From Conclusions 6.2 and 6.3 we get a nice theorem.

THEOREM 6.4. *Assume that $f, g \in \nabla\mathcal{A}_T(B^n)$. If f is homotopic to g in $\mathcal{A}_T(B^n)$ then f is homotopic to g in $\nabla\mathcal{A}_T(B^n)$.*

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