This version of the article has been accepted for publication, after peer review (when applicable) and is subject to Springer Nature's AM terms of use, but is not the Version of Record and does not reflect post-acceptance improvements, or any corrections. The Version of Record is available online at: https://doi.org/10.1007/s10773-006-9199-8

Postprint of: Czachor M., Naudts J., Regularization as Quantization in Reducible Representations of CCR, INTERNATIONAL JOURNAL OF THEORETICAL PHYSICS, Vol. 46 (2007), pp. 70–101

Regularization as quantization in reducible representations of CCR

Marek Czachor[1,2] and Jan Naudts[2]

[1] *Katedra Fizyki Teoretycznej i Metod Matematycznych Politechnika Gdańska, 80-952 Gdańsk, Poland*

[2] *Departement Fysica, Universiteit Antwerpen, B2610 Antwerpen, Belgium*

A covariant quantization scheme employing reducible representations of canonical commutation relations with positive-definite metric and Hermitian four-potentials (an alternative to the Gupta-Bleuler method) is tested on the example of quantum electromagnetic fields produced by a classical current. The Heisenberg dynamics can be consistently formulated since the fields are given by operators and not operator-valued distributions. The scheme involves a Hamiltonian whose free part is modified but the minimal-coupling interaction is the standard one. Solving Heisenberg equations of motion under the assumption that the fields are free for times $t_0 = \pm\infty$ we arrive at retarded and advanced solutions. Once we have these solutions we can deduce the form of evolution of retarded and advanced fields between two arbitrary *finite* times. The appropriate unitary evolution operators are found and their generators are computed. Now the generators involve the same free part as before, but the interaction term turns out to be modified. For a pointlike charge localized on a world-line $z^a(t)$ we find the interaction term of the form $-q\vec{A}\big(z(t)\big)\cdot\vec{v}(t) - q\int d\vec{z}\cdot\vec{E}$ where the integration is along those parts of the charge world-line where the charge velocity is nonzero. There is no self-energy contribution. Next we compute photon statistics. Poisson statistics naturally results and infrared divergence can be avoided even for pointlike sources. Classical fields produced by classical sources can be obtained if one computes coherent-state averages of Heisenberg-picture operators. It is shown that the new form of representation automatically smears out pointlike currents. We discuss in detail Poincaré covariance of the theory and the role of Bogoliubov transformations for the issue of gauge invariance. The representation we employ is parametrized by a number that is related to Rényi's α. It is shown that the "Shannon limit" $\alpha \to 1$ plays here a role of a correspondence principle with the standard regularized formalism.

PACS numbers: 03.70.+k, 41.20.Jb, 42.50.-p

I. INTRODUCTION

Dynamics of observables is given in quantum mechanics by Heisenberg equations of motion. In quantum field theory one does not really work with Heisenberg equations but prefers the S-matrix formalism. The situation is caused mainly by the fact that quantum fields are operator valued distributions and their products taken at the same point in space-time are meaningless. Formal solutions, when inserted back into Heisenberg equations, typically lead to mathematical absurdities due to divergences. The difficulties are less visible and thus easier to live with at the S-matrix level. Another reason for the popularity of the S-matrix formalism is the question of gauge invariance. There are various 'proofs' in the literature that the S-matrix formalism of quantum electrodynamics (QED) is manifestly gauge invariant. However, as discussed in detail in [1], all the 'easy one-line proofs', including the celebrated Feynman one [2], are false if divergent expressions are encountered.

As often stressed by Dirac, the removal-of-divergences techniques practically mean a departure from the Heisenberg equation. Dirac regarded this as a serious drawback of quantum field theory. In his last published lecture [3], entitled 'The requirements of fundamental physical theory', and given to a gathering of Nobel Laureates at Lindau on July 1, 1982, he said:

"I feel that we have to insist on the validity of this Heisenberg equation. This is the whole basis of quantum theory. We have got to hold onto it whatever we do, and if the equation gives results which are not correct it means that we are using the wrong Hamiltonian. This is the point I want to emphasise (...). Heisenberg originally formulated these equations with the dynamical variables appearing as matrices. You can generalise this very much by allowing more general kinds of quantities for your dynamical variables. They can be any algebraic quantities such that you do not in general have commutative multiplication (...). Some day people will find the correct Hamiltonian and there will be some new degrees of freedom, something which we cannot understand according to classical ideas, playing a role in the foundations of quantum mechanics."

The talk appeared in a published form in 1984, the year of Dirac's death. Also in 1984 Dirac published his last paper [4], entitled 'The future of atomic physics' where he criticised, again from the perspective of the divergences, the dogma of irreducibility of representations of elementary quantum symmetries. Taken together, the two papers form a kind of scientific testament of this great physicist.

Quite recently, in a series of papers [5, 6, 7], one of us was advocating the idea that the occurrence of the divergences may be related to the fact that the standard quantization scheme is based on irreducible representations of canonincal (anti-)commutation relations. It was argued that there are physical reasons for the use of certain reducible representations of CCR (in [5, 6]) and CAR (in [7]). There is no problem with multiplying fields at the same space-time point because the new representation leads to fields that are operators and not operator-valued distributions. Preliminary analysis of interactions with charges discussed in these papers (two-level atoms [5], classical current [6], reducibly quantized fermionic fields [7]) was always pointing into the conclusion that the reducible representation *may* indeed remove divergences, although definite statements would have been far premature at that stage.

Particularly intriguinging seems the possibility, discussed in [8, 9], that the experiments on Rabi oscillations performed by the Paris group [10] may be more consistently interpreted in terms of fields quantized by means of the reducible representations. If this were the case, it would mean that the fundamental question which representation is more physical would be within the reach of present-day optical experiments. But then we have to go further with the analysis of the new representation. In particular, we have to control the relativistic and gauge properties of the formalism, a fact that justifies the origin of the present paper.

The goal of the present work is to apply a covariant analogue of the representation introduced in [6] to the problem of Heisenberg-picture evolution of electromagnetic fields interacting with a classical current. The issue of infinities is not our main concern here since any reliable discussion of ultraviolet divergences would require quantized currents, but various automatic regularizations due to our choice of representation can also be observed. In the present paper we are more interested in the problem of Poincaré covariance vs. gauge invariance, positive-definiteness of the scalar product, and unitarity of evolution. Interaction with classical currents does not grasp all the possible subtleties of QED, but it allows for exact solutions and thus is a natural playground for testing any new quantization paradigm.

The results are quite promising. As opposed to standard quantization schemes it makes in our representation perfect sense to speak of the Heisenberg-picture evolution of field operators. We begin with solving the equations under the assumption that the fields are free at timelike infinities. We make this condition precise by first taking solutions that are free at an arbitrary t_0, and afterwards taking the limit $t_0 \to \pm\infty$. The results of this limiting procedure are, respectively, retarded and advanced solutions of the Maxwell equations. Having the solutions for all times, we can find a unitary map $W_\pm(t, t_1)$ that maps retarded/advanced fields at time t_1 into retarded/advanced fields at time t. Once we have this unitary map, we can compute its generator. We show that it differs from the usual minimal-coupling Hamiltonian.

We study in more detail the solutions corresponding to pointlike classical sources. Taking a relatively general form of the charge world-line we compute the explicit form of the modified interaction Hamiltonian. The interaction term does not contain any self-energy contribution but we find in addition a term that describes the work performed by the charge against free electric field. We then show what kind of a *classical* field one arrives at if one computes coherent-state averages of our field operators. The resulting field looks as if it were produced by an extended current, even though the current is pointlike. We construct the evolution operator and consider its associated S-matrix. No infrared divergence occurs and there is no problem with computing photon statistics even for pointlike sources.

What is interesting, the different form of representation of CCR we work with implies also a slightly modified form of the *free* part of the Hamiltonian. The modification is subtle and is related to some additional degree of freedom characterized by a parameter N. The role of N for Rabi oscillations was discussed in detail in [8] with the conclusion that finite N are compatible with experiment. The "thermodynamic limit" $N \to \infty$ plays a role of a correspondence principle that maps the new theory into a *regularized* form of the standard one. Effectively, the limiting results are equivalent to the standard ones but with a cut-off. It has to be stressed, however, that there is no cut-off in the Hamiltonian, and the regularization is a result of the special form of the vacuum one employs in the new representation. We also show that the limit $N \to \infty$ is equivalent to Rényi's limit $\alpha \to 1$.

The modification of the formalism we discuss seems to satisfy standards which are not that far from the requirements proposed by Dirac: We have new types of noncommuting dynamical variables, they involve certain new degrees of freedom that have no counterpart in classical electrodynamics, and their dynamics is governed by a modified Hamiltonian. The dynamics is given by Heisenberg equations but their solutions simultaneously satisfy Maxwell's equations with a regularized current. The regularization is not introduced ad hoc, but follows from the quantum structure. As opposed to standard quantization schemes the reducibly quantized fields are less singular than their classical counterparts. One can also think of this effect as an example of David Finkelstein's idea of 'quantization as regularization' [11].

The paper is organized as follows. We choose the Penrose-Rindler spinor formalism since it naturally leads to null tetrads that are implicitly present in Lorenz-gauge potentials, and play a role of polarization vectors. We begin with clarifying links between the nonuniqueness of Lorenz-gauge *classical* 4-potentials and equivalence classes of spin frames associated with null 4-momenta. The spin frames are later used in construction of two types of momentum-dependent tetrads (associated with circular and linear polarizations of spin-1 fields and 'longitudinal' and 'timelike' polarizations of two additional scalar fields). As an intermediate step towards quantization we explain in Section V how a change of spin frame within an equivalence class is related to a Bogoliubov transformation of creation and annihilation operators. For classical fields we do not have to worry about changes within equivalence classes since anyway the result is a gauge transformation. However, when we quantize the 4-potential the changes of spin frames become visible in transformations of the two 'timelike' and 'longitudinal' fields and the formalism becomes ambiguous unless we compensate this modification by a unitary transformation of annihilation and creation operators — this is where we need a Bogoliubov transformation. In Section VI we introduce the reducible representation, parametrized by a natural number N, of canonical commutation relations and, in Section VII, we briefly explain why in thermodynamic limit $N\to\infty$ the representation will automatically introduce cut-offs at the level of averages. In Section VII we show how to quantize the 4-potential in a manifestly covariant way and without indefinite metric or non-Hermitian field operators. In Section IX we discuss the analogue of the Jordan-Pauli function and show that distributions typical of the standard formalism are here replaced by non-singular objects. Section X discusses Poincaré covariance of the theory. Sections XI and XII investigate various subtleties of the Lorenz condition. In Section XIII we introduce vacuum, multi-photon, and coherent states. The problem of fields produced by classical currents is discussed in Section XIV. We solve Heisenberg equations under the assumption that at t_0 the fields are free, and we conclude that the assumption makes physical sense only if t_0 is moved to $\pm\infty$. In these limits the solutions of Heisenberg equations are equivalent to retarded and advanced fields, but the effective current that occurs in the source term is not a 'c-number' but an operator that nevertheless commutes with all field operators (the operator can be nontrivial because the representation is reducible). Having these solutions we are in position to find an explicit unitary transformation that shifts a retarded/advanced solution in time by a finite Δt. We find both the transformation and its generator in Section XVI, and in Section XVII we check the result on a pointlike charge. In this way we have systematically derived the Hamiltonian that allows to solve the problem with retarded or advanced initial condition at $t=0$ instead of that with free field at $\pm\infty$. In Section XVIII we address the issue of S-matrix and the associated photon statistics. Finally, having the operator solutions we compute their coherent-state averages and discuss the implications of the formalism for classical electrodynamics.

II. EQUIVALENCE CLASSES OF SPIN-FRAMES AND CLASSICAL GAUGE FREEDOM

The 4-momentum $k^a = k^a(\boldsymbol{k}) = (|\boldsymbol{k}|, \boldsymbol{k})$ of a massless particle can be written in spinor notation [12] as $k^a = \pi^A(\boldsymbol{k})\bar{\pi}^{A'}(\boldsymbol{k})$, where $\pi^A(\boldsymbol{k})$ is a spinor field defined by k^a up to a phase factor. For any $\pi^A(\boldsymbol{k})$ there exists another spinor $\omega^A(\boldsymbol{k})$ satisfying the spin-frame condition $\omega_A(\boldsymbol{k})\pi^A(\boldsymbol{k}) = 1$. Given $\pi^A(\boldsymbol{k})$ we cannot find a unique $\omega^A(\boldsymbol{k})$, since for any function $\phi(\boldsymbol{k})$ the new field

$$\tilde{\omega}^A(\boldsymbol{k}) = \omega^A(\boldsymbol{k}) + \phi(\boldsymbol{k})\pi^A(\boldsymbol{k}) \tag{1}$$

also satisfies $\tilde{\omega}_A(\boldsymbol{k})\pi^A(\boldsymbol{k}) = 1$. This leads to the equivalence relation: $\tilde{\omega}^A(\boldsymbol{k}) \sim \omega^A(\boldsymbol{k})$ iff $\tilde{\omega}^A(\boldsymbol{k}) - \omega^A(\boldsymbol{k})$ is proportional to $\pi^A(\boldsymbol{k})$.

Free classical electromagnetic fields are related to $\pi^A(\boldsymbol{k})$ by

$$F_{ab}(x) \;=\; \partial_a A_b(x) - \partial_b A_a(x) = -\int d\Gamma(\boldsymbol{k})\pi_A(\boldsymbol{k})\pi_B(\boldsymbol{k})\varepsilon_{A'B'}\big(\alpha(\boldsymbol{k},-)e^{-ik\cdot x} + \overline{\alpha(\boldsymbol{k},+)}e^{ik\cdot x}\big) + \text{c.c.} \tag{2}$$

where $d\Gamma(\boldsymbol{k})$ is the invariant measure on the light-cone. The 4-potential in a Lorenz [13] gauge can be taken in the form (cf. [14])

$$A_a(x) \;=\; i\int d\Gamma(\boldsymbol{k})e^{-ik\cdot x}\big(\omega_A(\boldsymbol{k})\bar{\pi}_{A'}(\boldsymbol{k})\alpha(\boldsymbol{k},+) + \pi_A(\boldsymbol{k})\bar{\omega}_{A'}(\boldsymbol{k})\alpha(\boldsymbol{k},-)\big) + \text{c.c.} \tag{3}$$

Now, if we replace $\omega^A(\boldsymbol{k})$ by $\tilde{\omega}^A(\boldsymbol{k})$ belonging to the same equivalence class, i.e. satisfying (1), then

$$A_a(x) \;\mapsto\; \tilde{A}_a(x) = A_a(x) - \partial_a\Phi(x) \tag{4}$$

where

$$\Phi(x) \;=\; \int d\Gamma(\boldsymbol{k})\phi(\boldsymbol{k})\big(\alpha(\boldsymbol{k},+)e^{-ik\cdot x} + \overline{\alpha(\boldsymbol{k},-)}e^{ik\cdot x}\big) + \text{c.c.} \tag{5}$$

is a solution of $\Box\Phi(x) = 0$. It follows that the equivalence class of spin-frames corresponds to an equivalence class of Lorenz-gauge potentials.

III. TETRADS ASSOCIATED WITH SPIN-FRAMES

Following [12] we introduce two types of tetrads associated with the spin-frames. The null tetrad employs the 4-momentum as one of its elements:

$$\begin{aligned} k_a(\boldsymbol{k}) &= \pi_A(\boldsymbol{k})\bar{\pi}_{A'}(\boldsymbol{k}) && (6)\\ \omega_a(\boldsymbol{k}) &= \omega_A(\boldsymbol{k})\bar{\omega}_{A'}(\boldsymbol{k}) && (7)\\ m_a(\boldsymbol{k}) &= \omega_A(\boldsymbol{k})\bar{\pi}_{A'}(\boldsymbol{k}) && (8)\\ \bar{m}_a(\boldsymbol{k}) &= \pi_A(\boldsymbol{k})\bar{\omega}_{A'}(\boldsymbol{k}). && (9) \end{aligned}$$

Complex 4-vectors $m_a(\boldsymbol{k})$, $\bar{m}_a(\boldsymbol{k})$ occur in (3) and play the role of circular polarization vectors. The Lorenz condition satisfied by (3) follows from the transversality property $k^a(\boldsymbol{k})m_a(\boldsymbol{k}) = k^a(\boldsymbol{k})\bar{m}_a(\boldsymbol{k}) = 0$. The null tetrad is related to the Minkowski-space metric tensor of signature $(+,-,-,-)$ in the standard way [12]

$$g_{ab} = k_a(\boldsymbol{k})\omega_b(\boldsymbol{k}) + \omega_a(\boldsymbol{k})k_b(\boldsymbol{k}) - m_a(\boldsymbol{k})\bar{m}_b(\boldsymbol{k}) - m_b(\boldsymbol{k})\bar{m}_a(\boldsymbol{k}). \tag{10}$$

This formula is independent of the choice of the representative $\omega_A(\boldsymbol{k})$ of a given equivalence class.

The tetrad defined by

$$\begin{aligned} x_a(\boldsymbol{k}) &= \tfrac{1}{\sqrt{2}}\big(m_a(\boldsymbol{k}) + \bar{m}_a(\boldsymbol{k})\big) && (11)\\ y_a(\boldsymbol{k}) &= \tfrac{i}{\sqrt{2}}\big(m_a(\boldsymbol{k}) - \bar{m}_a(\boldsymbol{k})\big) && (12)\\ z_a(\boldsymbol{k}) &= \tfrac{1}{\sqrt{2}}\big(\omega_a(\boldsymbol{k}) - k_a(\boldsymbol{k})\big) && (13)\\ t_a(\boldsymbol{k}) &= \tfrac{1}{\sqrt{2}}\big(\omega_a(\boldsymbol{k}) + k_a(\boldsymbol{k})\big) && (14) \end{aligned}$$

is, in the terminology of [12], a restricted Minkowski tetrad, satisfying

$$g_{ab} = t_a(\boldsymbol{k})t_b(\boldsymbol{k}) - x_a(\boldsymbol{k})x_b(\boldsymbol{k}) - y_a(\boldsymbol{k})y_b(\boldsymbol{k}) - z_a(\boldsymbol{k})z_b(\boldsymbol{k}) \tag{15}$$

The potential now can be written as

$$A_a(x) = i\int d\Gamma(\boldsymbol{k})\big(x_a(\boldsymbol{k})\alpha(\boldsymbol{k},1) + y_a(\boldsymbol{k})\alpha(\boldsymbol{k},2)\big)e^{-ik\cdot x} + \text{c.c.} \tag{16}$$

The link between the two types of amplitudes

$$\alpha(\boldsymbol{k},\pm) = \tfrac{1}{\sqrt{2}}\big(\alpha(\boldsymbol{k},1) \pm i\,\alpha(\boldsymbol{k},2)\big) \tag{17}$$

is analogous to this between circular and linear polarizations. The link is not accidental.

IV. TRANSFORMATION PROPERTIES OF SPIN-FRAMES

The transformation properties of spin-frames we shall discuss below do not depend on their explicit realization. Let us denote by $\boldsymbol{\Lambda k}$ the spacelike part of the 4-vector $\Lambda_a{}^b k_b(\boldsymbol{k})$, and by $\Lambda_A{}^B$ the unprimed SL(2,C) matrix corresponding to $\Lambda_a{}^b \in$ SO(1,3). The spinor-field transformation

$$\pi_A(\boldsymbol{k}) \mapsto \Lambda\pi_A(\boldsymbol{k}) = \Lambda_A{}^B\pi_B(\boldsymbol{\Lambda^{-1}k}) \tag{18}$$

implies that $k_a(\boldsymbol{k}) = \Lambda\pi_A(\boldsymbol{k})\overline{\Lambda\pi}_{A'}(\boldsymbol{k})$ and, hence,

$$\Lambda\pi_A(\boldsymbol{k}) = e^{-i\Theta(\Lambda,k)}\pi_A(\boldsymbol{k}) \tag{19}$$

The phase factor

$$e^{-i\Theta(\Lambda,k)} = \Lambda^{AB}\omega_A(\boldsymbol{k})\pi_B(\boldsymbol{\Lambda^{-1}k}) \tag{20}$$

is the one occuring in the unitary spin-1/2 zero-mass representation of the (covering space of the) Poincaré group, and does not depend on the choice of the representative $\omega_A(\boldsymbol{k})$. The angle $\Theta(\Lambda,k)$ is known as the Wigner phase.

An analogously defined

$$\Lambda\omega_A(\boldsymbol{k}) = \Lambda_A{}^B\omega_B(\boldsymbol{\Lambda^{-1}k}) \tag{21}$$

satisfies

$$\Lambda\omega_A(\boldsymbol{k})\Lambda\pi^A(\boldsymbol{k}) = 1 \tag{22}$$

and thus

$$\Lambda\omega_A(\boldsymbol{k}) = e^{i\Theta(\Lambda,k)}\big(\omega_A(\boldsymbol{k}) + \phi(\boldsymbol{k})\pi_A(\boldsymbol{k})\big) \tag{23}$$

$$= e^{i\Theta(\Lambda,k)}\tilde{\omega}_A(\boldsymbol{k}) \tag{24}$$

with some $\phi(\boldsymbol{k})$. The new $\tilde{\omega}_A(\boldsymbol{k})$ belongs to the equivalence class of $\omega_A(\boldsymbol{k})$. The gauge transformation $\omega_A(\boldsymbol{k}) \mapsto \tilde{\omega}_A(\boldsymbol{k})$ is in general nontrivial and depends on the explicit form of the spin-frame one works with. This is the reason why 4-potentials are not 4-vector fields: A Lorentz transformation changes Lorenz gauges within the equivalence class of spin-frames. In order to guarantee independence of physical quantities of particular representatives one employs the result described in the next Section.

V. EQUIVALENCE CLASSES OF SPIN FRAMES AND BOGOLIUBOV TRANSFORMATION

Of crucial importance for the formalism we shall develop below is the link between the transformation (1) and a transformation of a Bogoliubov type, i.e. the one mixing creation and annihilation operators in a way that preserves canonical commutation relations.

To begin with, consider four annihilation operators a_j satisfying $[a_j, a_{j'}^\dagger] = \delta_{jj'}1$ and the operator

$$V_a = x_a a_1 + y_a a_2 + z_a a_3 + t_a a_0^\dagger. \tag{25}$$

The change of spin-frame by $\omega_A \mapsto \omega_A + |\phi|e^{-i\theta}\pi_A$ is represented at the tetrad level by

$$\begin{pmatrix} t_a \\ x_a \\ y_a \\ z_a \end{pmatrix} \mapsto \begin{pmatrix} \tilde{t}_a \\ \tilde{x}_a \\ \tilde{y}_a \\ \tilde{z}_a \end{pmatrix} = \begin{pmatrix} 1+|\phi|^2/2 & |\phi|\cos\theta & |\phi|\sin\theta & -|\phi|^2/2 \\ |\phi|\cos\theta & 1 & 0 & -|\phi|\cos\theta \\ |\phi|\sin\theta & 0 & 1 & -|\phi|\sin\theta \\ |\phi|^2/2 & |\phi|\cos\theta & |\phi|\sin\theta & 1-|\phi|^2/2 \end{pmatrix} \begin{pmatrix} t_a \\ x_a \\ y_a \\ z_a \end{pmatrix} \tag{26}$$

The matrix in (26) is a special Lorentz transformation in Minkowski space with metric tensor $\mathrm{diag}(1,-1,-1,-1)$. Let us now introduce four new operators b_j defined implicitly by

$$V_a = t_a a_0^\dagger + x_a a_1 + y_a a_2 + z_a a_3 = \tilde{t}_a b_0^\dagger + \tilde{x}_a b_1 + \tilde{y}_a b_2 + \tilde{z}_a b_3 \tag{27}$$

The explicit form

$$\begin{aligned} b_1 &= a_1 - |\phi|\cos\theta a_3 - |\phi|\cos\theta a_0^\dagger \\ b_2 &= a_2 - |\phi|\sin\theta a_3 - |\phi|\sin\theta a_0^\dagger \\ b_3 &= |\phi|\cos\theta a_1 + |\phi|\sin\theta a_2 + (1-\tfrac{1}{2}|\phi|^2)a_3 - \tfrac{1}{2}|\phi|^2 a_0^\dagger \\ b_0 &= -|\phi|\cos\theta a_1^\dagger - |\phi|\sin\theta a_2^\dagger + \tfrac{1}{2}|\phi|^2 a_3^\dagger + (1+\tfrac{1}{2}|\phi|^2)a_0 \end{aligned}$$

implies that $[b_j, b_{j'}^\dagger] = \delta_{jj'}1$ and, hence, there exists [15] a unitary Bogoliubov-type transformation B satisfying $b_j = B^\dagger a_j B$. In order to explicitly construct B [16] we introduce the representation of the Lie algebra $\mathrm{so}(1,3)$,

$$J_1 = i(a_3^\dagger a_2 - a_2^\dagger a_3), \tag{28}$$

$$J_2 = i(a_1^\dagger a_3 - a_3^\dagger a_1), \tag{29}$$

$$J_3 = i(a_2^\dagger a_1 - a_1^\dagger a_2), \tag{30}$$

$$K_1 = i(a_0^\dagger a_1^\dagger - a_0 a_1), \tag{31}$$

$$K_2 = i(a_0^\dagger a_2^\dagger - a_0 a_2), \tag{32}$$

$$K_3 = i(a_0^\dagger a_3^\dagger - a_0 a_3), \tag{33}$$

and their combinations that form a representation of e(2),

$$\begin{aligned}
L_1 &= K_1 + J_2 = i(a_0^\dagger a_1^\dagger - a_0 a_1 + a_1^\dagger a_3 - a_3^\dagger a_1), && (34)\\
L_2 &= K_2 - J_1 = i(a_0^\dagger a_2^\dagger - a_0 a_2 - a_3^\dagger a_2 + a_2^\dagger a_3), && (35)\\
L_3 &= J_3, && (36)\\
[L_1, L_3] &= -iL_2, && (37)\\
[L_2, L_3] &= iL_1, && (38)\\
[L_1, L_2] &= 0. && (39)
\end{aligned}$$

Then

$$B = B(\phi) = e^{i|\phi|(L_1 \cos\theta + L_2 \sin\theta)}, \tag{40}$$

as can be verified by a straightforward computation. Obviously, $[B(\phi_1), B(\phi_2)] = 0$ for any ϕ_1, ϕ_2, since the map $\phi \mapsto B(\phi)$ is a representation of translations.

VI. CONSTRUCTION OF THE REDUCIBLE REPRESENTATION OF CCR

The construction is analogous to the one introduced in [6]. The modification with respect to [6] is that here we introduce four types of annihilation operators, and not just two corresponding to the polarization degrees of freedom.

One begins with four operators, a_0, a_2, a_2, a_3, satisfying commutation relations typical of an *irreducible* representation of CCR: $[a_j, a_{j'}^\dagger] = \delta_{jj'}1$. Let $|0\rangle$ denote their common vacuum, i.e. $a_j|0\rangle = 0$. Now take the kets $|\boldsymbol{k}\rangle$ normalized with respect to the light-cone delta function

$$\langle \boldsymbol{k}|\boldsymbol{k}'\rangle = \delta_\Gamma(\boldsymbol{k}, \boldsymbol{k}') = (2\pi)^3 2|\boldsymbol{k}|\delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}'). \tag{41}$$

What we call the $N = 1$ (or 1-oscillator) representation of CCR acts in the Hilbert space $\mathcal{H}$ spanned by kets of the form

$$|\boldsymbol{k}, n_0, n_1, n_2, n_3\rangle = |\boldsymbol{k}\rangle \otimes \frac{(a_0^\dagger)^{n_0}(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}(a_3^\dagger)^{n_3}}{\sqrt{n_0!n_1!n_2!n_3!}}|0\rangle.$$

Physically, $\mathcal{H}$ may be regarded as representing the space of states of a single four-dimensional oscillator. The 1-oscillator representation is defined by

$$a(\boldsymbol{k}, j) = |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes a_j. \tag{42}$$

This representation is reducible since the commutator

$$[a(\boldsymbol{k}, j), a(\boldsymbol{k}', j')^\dagger] = \delta_{jj'}\delta_\Gamma(\boldsymbol{k}, \boldsymbol{k}')|\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes 1 \tag{43}$$

involves at the right-hand-side the operator-valued distribution $I(\boldsymbol{k}) = |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes 1$ belonging to the center of the algebra, $[a(\boldsymbol{k}, j), I(\boldsymbol{k}')] = [I(\boldsymbol{k}), a(\boldsymbol{k}', j')^\dagger] = 0$, for all $\boldsymbol{k}$, $\boldsymbol{k}'$, j, j'. The $I(\boldsymbol{k})$ form a resolution of unity

$$\int d\Gamma(\boldsymbol{k}) I(\boldsymbol{k}) = \int d\Gamma(\boldsymbol{k})|\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes 1 = I. \tag{44}$$

Here I is the identity operator in $\mathcal{H}$.

For arbitrary N the representation is constructed as follows. Let $\mathcal{H}$ be the representation space of the $N = 1$ representation. Define

$$\underline{\mathcal{H}} = \underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{N} \tag{45}$$

and let A be an arbitrary operator defined for $N = 1$. Let

$$A^{(n)} = \underbrace{I \otimes \cdots \otimes I}_{n-1} \otimes A \otimes \underbrace{I \otimes \cdots \otimes I}_{N-n}. \tag{46}$$

The N oscillator extension of $a(\boldsymbol{k}, j)$ is defined by

$$\underline{a}(\boldsymbol{k}, j) \;=\; \tfrac{1}{\sqrt{N}}\textstyle\sum_{n=1}^{N} a(\boldsymbol{k}, j)^{(n)} \tag{47}$$

and satisfies the reducible representation

$$[\underline{a}(\boldsymbol{k}, j), \underline{a}(\boldsymbol{k}', j')^{\dagger}] \;=\; \delta_{jj'}\delta_{\Gamma}(\boldsymbol{k}, \boldsymbol{k}')\underline{I}(\boldsymbol{k}) \tag{48}$$

where

$$\underline{I}(\boldsymbol{k}) \;=\; \tfrac{1}{N}\textstyle\sum_{n=1}^{N} I(\boldsymbol{k})^{(n)}. \tag{49}$$

As before we find the resolution of unity

$$\textstyle\int d\Gamma(\boldsymbol{k})\underline{I}(\boldsymbol{k}) \;=\; \underline{I} \tag{50}$$

where $\underline{I}$ is the identity operator in $\underline{\mathcal{H}}$.

VII. THERMODYNAMIC LIMIT AND QUANTUM LAW OF LARGE NUMBERS

The asymptotic properties of the formalism for $N \to \infty$ can be anticipated already at this stage if one recognizes in the formula (49) the *frequency operator* employed in the analysis of quantum laws of large numbers [25, 26, 27, 28]. Indeed, let P_θ be a projector. The frequency operator corresponding to the random variable (proposition) represented by P_θ reads

$$P_{\theta,N} \;=\; \tfrac{1}{N}\textstyle\sum_{n=1}^{N} P_{\theta}^{(n)}, \tag{51}$$

Let $P_\theta|\theta\rangle = |\theta\rangle$ and $|\psi\rangle = \sum_\theta \psi_\theta|\theta\rangle$ be a state. Let $|\underline{\psi}\rangle = |\psi\rangle \otimes \dots \otimes |\psi\rangle$ (N times). Then the following weak law of large numbers holds true

$$\lim_{N\to\infty} \parallel (P_{\theta,N})^m|\underline{\psi}\rangle - |\psi_\theta|^{2m}|\underline{\psi}\rangle \parallel = 0 \tag{52}$$

for $m = 1, 2, 3, \dots$. The weak law states that effectively, for $N \to \infty$, the frequency operators act by multiplication of N-copy states by appropriate probabilities. This is essentially why also in our field theory we will find averages where, in the limit $N \to \infty$, the operators $\underline{I}(\boldsymbol{k})$ will be replaced by their corresponding probabilities $Z(\boldsymbol{k})$ associated with the choice of the vacuum state.

VIII. QUANTIZATION OF THE POTENTIAL

The potential operator at the level of the N-oscillator representation reads

$$\underline{A}_a(x) \;=\; i\textstyle\int d\Gamma(\boldsymbol{k})\big(x_a(\boldsymbol{k})\underline{a}(\boldsymbol{k}, 1) + y_a(\boldsymbol{k})\underline{a}(\boldsymbol{k}, 2) + z_a(\boldsymbol{k})\underline{a}(\boldsymbol{k}, 3) + t_a(\boldsymbol{k})\underline{a}(\boldsymbol{k}, 0)^{\dagger}\big)e^{-ik\cdot x} + \text{h.c.} \tag{53}$$

$$\;=\; i\textstyle\int d\Gamma(\boldsymbol{k})\big(m_a(\boldsymbol{k})\underline{a}(\boldsymbol{k}, +) + \bar{m}_a(\boldsymbol{k})\underline{a}(\boldsymbol{k}, -) + z_a(\boldsymbol{k})\underline{a}(\boldsymbol{k}, 3) + t_a(\boldsymbol{k})\underline{a}(\boldsymbol{k}, 0)^{\dagger}\big)e^{-ik\cdot x} + \text{h.c.} \tag{54}$$

It is better to think of (53) and (54) as operators representing a system quantized at the $N = 1$ level, and then extended to arbitrary N by

$$\underline{A}_a(x) \;=\; \tfrac{1}{\sqrt{N}}\textstyle\sum_{n=1}^{N} A_a(x)^{(n)}. \tag{55}$$

From such a perspective it is easier to understand the structure of generators of the Poincaré group and other observables.

Let us note that the potential is Hermitian and the Hilbert space $\underline{\mathcal{H}}$ involves a positive-definite scalar product. A change (1) of spin-frame can be compensated by an N-oscillator Bogoliubov transformation $\underline{B} = B \otimes \dots \otimes B$, where B is of the type discussed in Section V.

The commutator of fields taken at arbitrary space-time points

$$[\underline{A}_a(x), \underline{A}_b(y)] \;=\; ig_{ab}\underline{D}(x - y) \tag{56}$$

involves the operator analogue of the Jordan-Pauli function

$$\underline{D}(x) = i\int d\Gamma(\boldsymbol{k})\underline{I}(\boldsymbol{k})\big(e^{-ik\cdot x} - e^{ik\cdot x}\big). \tag{57}$$

The correct signature of the metric tensor in (56) comes from the Bogoliubov-type structure of the positive-frequency part of $\underline{A}_a(x)$, i.e. the combination of annihilation and creation operators. If one had replaced $a_0^\dagger$ by a_0 one would have been forced to depart either from positivity of the scalar product or unitarity of evolution. A reducible version of such a (Gupta-Bleuler) formalism is possible [17], but one can show that contradictions with probability interpretation of the theory would necessary occur (for a brief discussion of the problem cf. Section XIX).

IX. JORDAN-PAULI OPERATOR: THE ROOTS OF REGULARIZATION

To understand why the formalism we construct is less singular than the one based on irreducible representations it is instructive to take a closer look at (57). In the first place, formula (57) is typical of all the representations of CCR, reducible or irreducible, the differences boiling down to different explicit forms of the central element $\underline{I}(\boldsymbol{k})$. The standard Pauli-Jordan function corresponds to representations where $\underline{I}(\boldsymbol{k})$ equals the identity. In our representation we can write (cf. Eq. (49))

$$\underline{D}(x) = \underline{D}^{(+)}(x) + \underline{D}^{(-)}(x) \tag{58}$$

$$\underline{D}^{(\pm)}(x) = \pm i\int d\Gamma(\boldsymbol{k})\underline{I}(\boldsymbol{k})e^{\mp ik\cdot x} = \tfrac{1}{N}\textstyle\sum_{n=1}^{N} D^{(\pm)}(x)^{(n)}. \tag{59}$$

The operator whose N-oscillator extensions occur in (59) reads explicitly

$$D^{(\pm)}(x) = \pm i\int d\Gamma(\boldsymbol{k})|\boldsymbol{k}\rangle\langle\boldsymbol{k}|e^{\mp ik\cdot x}\otimes 1 = \pm ie^{\mp i\hat{k}\cdot x}\otimes 1\hat{k}_a = \int d\Gamma(\boldsymbol{k})k_a|\boldsymbol{k}\rangle\langle\boldsymbol{k}|. \tag{60}$$

As we can see, the operators $D^{(\pm)}(x)$ are unitary representations of 4-translations, and their generators are given by $\hat{k}_a$. In particular,

$$D^{(\pm)}(0) = \pm iI, \quad \underline{D}^{(\pm)}(0) = \pm i\underline{I}. \tag{61}$$

Quantization in terms of our reducible representation replaces distributions by unitary operators. This is the main difference with respect to the schemes based on regularizations of distributions [18, 19, 20, 21, 22, 23, 24]. In our approach there are no cut-off functions in Heisenberg-picture operators. They appear effectively at the level of averages and are due to the properties of *states* (e.g. compare the operator (135), involving the frequency operator $\underline{I}(\boldsymbol{k})$ and no cut-off, with the average (171), involving the probability $Z(\boldsymbol{k})$ and the cut-off function $\chi(\boldsymbol{k})$). As one of the consequences, spectra of Hamiltonians occurring in reducibly quantized theories will not depend on cut-offs.

These facts show that the regularization occuring in our approach is of an entirely different origin than the ones we know from quantizations based on irreducible representations.

X. POINCARÉ COVARIANCE OF FREE FIELDS

The Poincaré transformations will be taken in the form

$$\underline{a}(\boldsymbol{k},\pm) \mapsto e^{\pm 2i\Theta(\Lambda,\boldsymbol{k})}e^{ik\cdot y}\underline{a}(\boldsymbol{\Lambda^{-1}k},\pm) = \underline{U}^\dagger_{\Lambda,y}\underline{a}(\boldsymbol{k},\pm)\underline{U}_{\Lambda,y} \tag{62}$$

$$\underline{a}(\boldsymbol{k},3) \mapsto e^{ik\cdot y}\underline{a}(\boldsymbol{\Lambda^{-1}k},3) = \underline{U}^\dagger_{\Lambda,y}\underline{a}(\boldsymbol{k},3)\underline{U}_{\Lambda,y} \tag{63}$$

$$\underline{a}(\boldsymbol{k},0)^\dagger \mapsto e^{ik\cdot y}\underline{a}(\boldsymbol{\Lambda^{-1}k},0)^\dagger = \underline{U}^\dagger_{\Lambda,y}\underline{a}(\boldsymbol{k},0)^\dagger\underline{U}_{\Lambda,y} \tag{64}$$

where $\Theta(\Lambda,\boldsymbol{k})$ is the Wigner phase. Transformation (62) is the unitary spin-1 massless representation of the Poincaré group. Transformations (63), (64) imply that the additional two fields are spin-0 and massless. Similarly to [6] we reduce the construction to the problem of finding $U_{\Lambda,y}$ satisfying

$$\underline{U}_{\Lambda,y} = \underbrace{U_{\Lambda,y}\otimes\dots\otimes U_{\Lambda,y}}_{N}. \tag{65}$$

A. Four-translations

The 4-momentum for $N=1$ reads

$$
\begin{aligned}
P_a &= \int d\Gamma(\boldsymbol{k}) k_a |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes \big(a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3 - a_0^\dagger a_0\big)\\
&= \underbrace{\int d\Gamma(\boldsymbol{k}) k_a |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes \big(a_1^\dagger a_1 + a_2^\dagger a_2\big)}_{P_a^I} + \underbrace{\int d\Gamma(\boldsymbol{k}) k_a J(\boldsymbol{k})}_{P_a^{II}},
\end{aligned}
\tag{66}
$$

with

$$
J(\boldsymbol{k}) = |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes (a_3^\dagger a_3 - a_0^\dagger a_0) \tag{67}
$$

One immediately verifies that

$$
e^{iP\cdot x} a(\boldsymbol{k},\pm) e^{-iP\cdot x} = a(\boldsymbol{k},\pm) e^{-ix\cdot k} \tag{68}
$$

$$
e^{iP\cdot x} a(\boldsymbol{k},3) e^{-iP\cdot x} = a(\boldsymbol{k},3) e^{-ix\cdot k} \tag{69}
$$

$$
e^{iP\cdot x} a(\boldsymbol{k},0)^\dagger e^{-iP\cdot x} = a(\boldsymbol{k},0)^\dagger e^{-ix\cdot k} \tag{70}
$$

implying

$$
\underline{U}_{\boldsymbol{1},y}^\dagger \underline{A}_a(x) \underline{U}_{\boldsymbol{1},y} = \underline{A}_a(x-y). \tag{71}
$$

The operator $J(\boldsymbol{k})$ will later reappear as the generator of rotations in the group E(2) associated with the 4-potential. The part P_a^I is identical to the 4-momentum operator introduced in [6]. The 4-momentum for arbitrary N reads

$$
\underline{P}_a = \sum_{n=1}^N P_a^{(n)} \tag{72}
$$

$$
= \underline{P}_a^I + \int d\Gamma(\boldsymbol{k}) k_a \underline{J}(\boldsymbol{k}) \tag{73}
$$

Only for $N=1$ the expression coincides with the generator found by standard Noether formulas (cf. the discussion of this point in [5]). The form (72) is characteristic of a 4-momentum of N non-interacting particles. These particles (four-dimensional oscillators) have no counterpart in classical electrodynamics. It should be stressed that these are not the oscillators of Heisenberg, Born, and Jordan [29] since there is no relationship between N, which is finite, and the number of different frequencies, which is infinite.

B. Rotations and boosts

To find an analogous representation of

$$
\underline{a}(\boldsymbol{k},\pm) \mapsto e^{\pm 2i\Theta(\Lambda,\boldsymbol{k})} \underline{a}(\boldsymbol{\Lambda^{-1}k},\pm) = \underline{U}_{\Lambda,0}^\dagger \underline{a}(\boldsymbol{k},\pm) \underline{U}_{\Lambda,0} \tag{74}
$$

$$
\underline{a}(\boldsymbol{k},3) \mapsto \underline{a}(\boldsymbol{\Lambda^{-1}k},3) = \underline{U}_{\Lambda,0}^\dagger \underline{a}(\boldsymbol{k},3) \underline{U}_{\Lambda,0} \tag{75}
$$

$$
\underline{a}(\boldsymbol{k},0)^\dagger \mapsto \underline{a}(\boldsymbol{\Lambda^{-1}k},0)^\dagger = \underline{U}_{\Lambda,0}^\dagger \underline{a}(\boldsymbol{k},0)^\dagger \underline{U}_{\Lambda,0} \tag{76}
$$

we take the same definition as in [6], i.e.

$$
U_{\Lambda,0} = \exp\Big(\sum_{s=\pm} 2is \int d\Gamma(\boldsymbol{k}) \Theta(\Lambda,\boldsymbol{k}) |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes a_s^\dagger a_s\Big)\Big(\int d\Gamma(\boldsymbol{p}) |\boldsymbol{p}\rangle\langle\boldsymbol{\Lambda^{-1}p}| \otimes 1\Big). \tag{77}
$$

Taking into account the properties of spin-frames and tetrads one verifies that

$$
\begin{aligned}
\underline{U}_{\Lambda,0}^\dagger \underline{A}_a(x) \underline{U}_{\Lambda,0} &= i\int d\Gamma(\boldsymbol{k}) e^{-ik\cdot\Lambda^{-1}x}\\
&\times \Big(\tilde{m}_a(\boldsymbol{\Lambda k}) e^{2i\Theta(\Lambda,\boldsymbol{\Lambda k})} \underline{a}(\boldsymbol{k},+) + \bar{\tilde{m}}_a(\boldsymbol{\Lambda k}) e^{-2i\Theta(\Lambda,\boldsymbol{\Lambda k})} \underline{a}(\boldsymbol{k},-) + \tilde{z}_a(\boldsymbol{\Lambda k}) \underline{a}(\boldsymbol{k},3) + \tilde{t}_a(\boldsymbol{\Lambda k}) \underline{a}(\boldsymbol{k},0)^\dagger\Big) + \text{h.c.}\\
&= \Lambda_a{}^b \underline{\tilde{A}}_b(\Lambda^{-1}x) = \Lambda_a{}^b \underline{B}(\Lambda)^\dagger \underline{A}_b(\Lambda^{-1}x) \underline{B}(\Lambda)
\end{aligned}
\tag{78}
$$

where $\underline{B}(\Lambda)$ compensates the change of gauge caused by (24). One can costruct $\underline{B}(\Lambda)$ by first finding an appropriate $B(\Lambda)$ of the form analogous to (40), and then defining $\underline{B}(\Lambda) = B(\Lambda)^{\otimes N}$. It is more elegant to assume that $\underline{U}_{\Lambda,0}$ is accompanied by a redefinition of vacuum (see below) $|\underline{O}\rangle \mapsto \underline{B}(\Lambda)^\dagger|\underline{O}\rangle$. Then the transformation of the potential becomes effectively

$$\underline{U}^\dagger_{\Lambda,y}\underline{A}_a(x)\underline{U}_{\Lambda,y} = \Lambda_a{}^b\underline{A}_b\big(\Lambda^{-1}(x-y)\big) \tag{79}$$

i.e. that of a 4-vector field.

A still simpler way is to assume the transformation rule

$$\Lambda_A{}^B\omega_B(\boldsymbol{\Lambda^{-1}k}) \;=\; e^{i\Theta(\Lambda,k)}\omega_A(\boldsymbol{k}) \tag{80}$$

i.e. to replace (24) by the form typical of the entire equivalence class. Then one can put $\underline{B}(\Lambda) = \underline{I}$.

The operators ocuring at right-hand-sides of field commutators transform as translation invariant scalar fields

$$\underline{U}^\dagger_{\Lambda,y}\underline{I}(\boldsymbol{k})\underline{U}_{\Lambda,y} \;=\; \underline{I}(\boldsymbol{\Lambda^{-1}k}) \tag{81}$$

$$\underline{U}^\dagger_{\Lambda,y}\underline{D}(x)\underline{U}_{\Lambda,y} \;=\; \underline{D}(\Lambda^{-1}x). \tag{82}$$

The representation we have introduced is a direct sum of a massless, spin-1 unitary representation (corresponding to the indices 1 and 2) and a massless spin-0 unitary representation (corresponding to the indices 3 and 0) of the Poincaré group. In such a structure the 'longitudinal' and 'timelike' components do not transform as parts of a four-vector, but as two scalar fields. However, there is a lot of freedom here. There exists, for example, an interesting representation which employs the link between Bogoliubov and SO$(1,3)$ transformations, and where all the four components behave as those of a four-vector. A particular example of this link was employed in Sec. V. It is interesting to compare the two representations in the context of field quantization, but this will be done in a separate paper [30].

XI. LORENZ CONDITION AND EUCLIDEAN GROUP

The field tensor $\underline{F}_{ab}(x) = \partial_a\underline{A}_b(x) - \partial_b\underline{A}_a(x)$ consists of two parts corresponding to spin-1 and spin-0 fields (formulas (83) and (84), respectively)

$$\underline{F}_{ab}(x) \;=\; -\int d\Gamma(\boldsymbol{k})\pi_A(\boldsymbol{k})\pi_B(\boldsymbol{k})\varepsilon_{A'B'}\big(\underline{a}(\boldsymbol{k},-)e^{-ik\cdot x} + \underline{a}(\boldsymbol{k},+)^\dagger e^{ik\cdot x}\big) + \text{h.c.} \tag{83}$$

$$-\;\sqrt{2}\int d\Gamma(\boldsymbol{k})^*M_{ab}(\boldsymbol{k})\big(\underline{\Pi}_1(\boldsymbol{k})\cos k\cdot x + \underline{\Pi}_2(\boldsymbol{k})\sin k\cdot x\big) \tag{84}$$

$$^*M_{ab}(\boldsymbol{k}) \;=\; -k_a\omega_b(\boldsymbol{k}) + k_b\omega_a(\boldsymbol{k}) \tag{85}$$

$$\underline{\Pi}_1(\boldsymbol{k}) \;=\; \tfrac{1}{2}\big(\underline{a}(\boldsymbol{k},3) + \underline{a}(\boldsymbol{k},3)^\dagger + \underline{a}(\boldsymbol{k},0) + \underline{a}(\boldsymbol{k},0)^\dagger\big)$$

$$\underline{\Pi}_2(\boldsymbol{k}) \;=\; \tfrac{1}{2i}\big(\underline{a}(\boldsymbol{k},3) - \underline{a}(\boldsymbol{k},3)^\dagger + \underline{a}(\boldsymbol{k},0)^\dagger - \underline{a}(\boldsymbol{k},0)\big)$$

The tensor $^*M_{ab}(\boldsymbol{k})$ is the dual of

$$M_{ab}(\boldsymbol{k}) \;=\; i\pi_{(A}\omega_{B)}\varepsilon_{A'B'} - i\bar{\pi}_{(A'}\bar{\omega}_{B')}\varepsilon_{AB} \tag{86}$$

$$k^{b*}M_{ab}(\boldsymbol{k}) \;=\; -k_a \tag{87}$$

One immediately recognizes in (86) and (87) spinor formulas for a massless angular momentum tensor and the Pauli-Lubanski vector of helicity -1 (cf. [32], Eq. (6.3.2)).

The gauge transformation (1) influences the part (84) in $\underline{F}_{ab}(x)$ according to

$$^*M_{ab}(\boldsymbol{k}) \;\mapsto\; {}^*M_{ab}(\boldsymbol{k}) - k_a(\boldsymbol{k})q_b(\boldsymbol{k}) + k_b(\boldsymbol{k})q_a(\boldsymbol{k}) \tag{88}$$

$$q_a(\boldsymbol{k}) \;=\; \phi(\boldsymbol{k})\bar{m}_a(\boldsymbol{k}) + \bar{\phi}(\boldsymbol{k})m_a(\boldsymbol{k}) \tag{89}$$

that is, in a way typical of angular momentum. The 4-vector $q_a(\boldsymbol{k})$ can be used to reexpress the gauge transformed spin-frame as a twistor [32]

$$\tilde{\pi}_A(\boldsymbol{k}) \;=\; \pi_A(\boldsymbol{k}) \tag{90}$$

$$\tilde{\omega}_A(\boldsymbol{k}) \;=\; \omega_A(\boldsymbol{k}) + q_{AA'}(\boldsymbol{k})\bar{\pi}^{A'}(\boldsymbol{k}). \tag{91}$$

Mutual relations within an equivalence class are thus determined by the twistor equation. Change of origin in the space of coordinates q_a can be compensated by the Bogoliubov transformation $\underline{B}$.

Together with $\underline{J}(\boldsymbol{k})$ occuring in (73) we obtain the algebra $e(2)$

$$\begin{aligned} [\underline{\Pi}_1(\boldsymbol{k}), \underline{J}(\boldsymbol{k}')] &= i\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}')\underline{\Pi}_2(\boldsymbol{k}) && (92)\\ [\underline{J}(\boldsymbol{k}), \underline{\Pi}_2(\boldsymbol{k}')] &= i\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}')\underline{\Pi}_1(\boldsymbol{k}) && (93)\\ [\underline{\Pi}_1(\boldsymbol{k}), \underline{\Pi}_2(\boldsymbol{k}')] &= 0 && (94) \end{aligned}$$

It is interesting that the removal of the scalar fields by the constraint

$$\langle\Psi'|\underline{\Pi}_1(\boldsymbol{k})|\Psi\rangle = \langle\Psi'|\underline{\Pi}_2(\boldsymbol{k})|\Psi\rangle = 0 \tag{95}$$

is analogous to the condition leading to the classical Maxwell field if one starts from induced representations [31]. Indeed, all massless discrete-spin representations are found if one requires that the two translation generators of $e(2)$ annihilate vectors from the representation space. It might be therefore tempting to impose the stronger constraint

$$\underline{\Pi}_1(\boldsymbol{k})|\Psi\rangle = \underline{\Pi}_2(\boldsymbol{k})|\Psi\rangle = 0 \tag{96}$$

also here. To see why this condition would be too strong we write the potential in terms of $e(2)$:

$$\underline{A}_a(x) = -\sqrt{2}\int d\Gamma(\boldsymbol{k})\omega_a(\boldsymbol{k})\big(\underline{\Pi}_2(\boldsymbol{k})\cos kx - \underline{\Pi}_1(\boldsymbol{k})\sin kx\big) - \sqrt{2}\int d\Gamma(\boldsymbol{k})k_a\big(\underline{Q}_1(\boldsymbol{k})\cos kx + \underline{Q}_2(\boldsymbol{k})\sin kx\big) + \dots \tag{97}$$

where

$$\begin{aligned} \underline{Q}_1(\boldsymbol{k}) &= -\tfrac{1}{2i}\big(\underline{a}(\boldsymbol{k},3) - \underline{a}(\boldsymbol{k},3)^\dagger - \underline{a}(\boldsymbol{k},0)^\dagger + \underline{a}(\boldsymbol{k},0)\big)\\ \underline{Q}_2(\boldsymbol{k}) &= -\tfrac{1}{2}\big(\underline{a}(\boldsymbol{k},3) + \underline{a}(\boldsymbol{k},3)^\dagger - \underline{a}(\boldsymbol{k},0)^\dagger - \underline{a}(\boldsymbol{k},0)\big) \end{aligned}$$

and the dots stand for the part involving only the spin-1 fields. The part involving $\underline{\boldsymbol{Q}}(\boldsymbol{k})$ is a gauge term and this is why we do not see it in $\underline{F}_{ab}(x)$. The entire algebra reads

$$\begin{aligned} [\underline{\Pi}_1(\boldsymbol{k}), \underline{J}(\boldsymbol{k}')] &= i\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}')\underline{\Pi}_2(\boldsymbol{k}) && (98)\\ [\underline{J}(\boldsymbol{k}), \underline{\Pi}_2(\boldsymbol{k}')] &= i\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}')\underline{\Pi}_1(\boldsymbol{k}) && (99)\\ [\underline{Q}_1(\boldsymbol{k}), -\underline{J}(\boldsymbol{k}')] &= i\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}')\underline{Q}_2(\boldsymbol{k}) && (100)\\ [-\underline{J}(\boldsymbol{k}), \underline{Q}_2(\boldsymbol{k}')] &= i\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}')\underline{Q}_1(\boldsymbol{k}) && (101)\\ [\underline{Q}_1(\boldsymbol{k}), \underline{\Pi}_1(\boldsymbol{k}')] &= i\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}')\underline{I}(\boldsymbol{k}) && (102)\\ [\underline{Q}_2(\boldsymbol{k}), \underline{\Pi}_2(\boldsymbol{k}')] &= i\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}')\underline{I}(\boldsymbol{k}) && (103)\\ [\underline{\Pi}_1(\boldsymbol{k}), \underline{\Pi}_2(\boldsymbol{k}')] &= 0 && (104)\\ [\underline{Q}_1(\boldsymbol{k}), \underline{Q}_2(\boldsymbol{k}')] &= 0 && (105)\\ [\underline{Q}_1(\boldsymbol{k}), \underline{\Pi}_2(\boldsymbol{k}')] &= 0 && (106)\\ [\underline{Q}_2(\boldsymbol{k}), \underline{\Pi}_1(\boldsymbol{k}')] &= 0 && (107) \end{aligned}$$

This is the Lie algebra of the 2-dimensional Euclidean group in a phase space. The "position operators" $\underline{\boldsymbol{Q}}(\boldsymbol{k})$ shift the "momenta" $\underline{\boldsymbol{\Pi}}(\boldsymbol{k})$ and the constraint (96) must be inconsistent with dynamics. There is no problem with (95).

The 4-divergence of the potential

$$\partial^a \underline{A}_a(x) = \sqrt{2}\int d\Gamma(\boldsymbol{k})\big(\underline{\Pi}_2(\boldsymbol{k})\sin kx + \underline{\Pi}_1(\boldsymbol{k})\cos kx\big)$$

shows that the weak Lorenz condition

$$\langle\Psi'|\partial^a\underline{A}_a(x)|\Psi\rangle = 0 \tag{108}$$

is equivalent to (95).

XII. LORENZ CONDITION AND POINCARÉ COVARIANCE OF STATES

The 1-oscillator Hilbert space $\mathcal{H}$ consists of vectors

$$|\Psi\rangle \;=\; \sum_{n_0,n_1,n_2,n_3=0}^{\infty}\int d\Gamma(\boldsymbol{k})\Psi(\boldsymbol{k},n_0,n_1,n_2,n_3)|\boldsymbol{k},n_0,n_1,n_2,n_3\rangle \tag{109}$$

satisfying

$$\sum_{n_0,n_1,n_2,n_3=0}^{\infty}\int d\Gamma(\boldsymbol{k})|\Psi(\boldsymbol{k},n_0,n_1,n_2,n_3)|^2<\infty \tag{110}$$

Subspaces consisting of vectors of the form

$$|\Psi_{n_0,n_1,n_2,n_3}\rangle \;=\; \int d\Gamma(\boldsymbol{k})\Psi(\boldsymbol{k},n_0,n_1,n_2,n_3)|\boldsymbol{k},n_0,n_1,n_2,n_3\rangle$$

are invariant subspaces of the representation constructed in Section X.

In particular, all the vectors of the form

$$|\Psi\rangle \;=\; \sum_{n_1,n_2=0}^{\infty}\int d\Gamma(\boldsymbol{k})\Psi(\boldsymbol{k},n_0,n_1,n_2,n_3)|\boldsymbol{k},n_0,n_1,n_2,n_3\rangle \tag{111}$$

belong to the Poincaré-invariant subspace satisfying the weak Lorenz condition (95) for $N=1$. Moreover, if we additionally require $n_0=n_3$ then

$$\langle\Psi'|P_a^{II}|\Psi\rangle=0 \tag{112}$$

An extension to arbitrary N is immediate. One concludes that the Lorenz condition (95) can be imposed in a Poincaré invariant way.

XIII. VACUUM, MULTIPHOTON, AND COHERENT STATES

The subspace corresponding to $n_0=n_1=n_2=n_3=0$ defines the vacuum for $N=1$. Any vector of the form

$$|O\rangle=\int d\Gamma(\boldsymbol{k})O(\boldsymbol{k})|\boldsymbol{k},0\rangle \tag{113}$$

plays a role of a 1-oscillator vacuum. For arbitrary N the vacuum state is taken in the form

$$|\underline{O}\rangle=\underbrace{|O\rangle\otimes\dots\otimes|O\rangle}_{N} \tag{114}$$

All vacuum states are annihilated by all annihilation operators. Vacuum states are translation invariant and SL(2,C) covariant:

$$U_{\Lambda,y}|O\rangle \;=\; \int d\Gamma(\boldsymbol{k})O(\boldsymbol{\Lambda^{-1}k})|\boldsymbol{k},0\rangle \tag{115}$$

$$\underline{U}_{\Lambda,y}|\underline{O}\rangle \;=\; U_{\Lambda,y}|O\rangle\otimes\dots\otimes U_{\Lambda,y}|O\rangle \tag{116}$$

Of particular importance is the scalar field representing vacuum probability density $Z(\boldsymbol{k})=|O(\boldsymbol{k})|^2$. Square integrability implies that $Z(\boldsymbol{k})$ decays at infinity; later on, we will also require $Z(\boldsymbol{k})$ going to zero at $\boldsymbol{k}=0$ in order to avoid infrared divergences. The latter would spoil the explicit constructive nature of the present approach. The number $Z=\max_k\{Z(\boldsymbol{k})\}$ is Poincaré invariant and can be interpreted as a renormalization constant.

Multiphoton states are obtained in the usual way by acting on the vacuum $|\underline{O}\rangle$ with creation operators. Coherent states associated with amplitudes $\alpha(\boldsymbol{k},\pm)$ occurring in (2) are defined in terms of the displacement operator [6]

$$\underline{\mathcal{D}}(\alpha) \;=\; \exp\big(\underline{a}(\alpha)^\dagger-\underline{a}(\alpha)\big) \tag{117}$$

$$\underline{a}(\alpha) \;=\; \sum_{s=\pm}\int d\Gamma(\boldsymbol{k})\overline{\alpha(\boldsymbol{k},s)}\underline{a}(\boldsymbol{k},s) \tag{118}$$

A coherent state is constructed from vacuum by $|\underline{O}(\alpha)\rangle=\underline{\mathcal{D}}(\alpha)|\underline{O}\rangle$. Coherent-state averages are related to classical fields by

$$\langle\underline{O}(\alpha)|\underline{A}_a(x)|\underline{O}(\alpha)\rangle \;=\; i\int d\Gamma(\boldsymbol{k})Z(\boldsymbol{k})\big(m_a(\boldsymbol{k})\alpha(\boldsymbol{k},+)+\bar{m}_a(\boldsymbol{k})\alpha(\boldsymbol{k},-)\big)e^{-ik\cdot x}+\text{c.c.} \tag{119}$$

Let us note that the averages involve the amplitudes $Z(\boldsymbol{k})\alpha(\boldsymbol{k},\pm)$ and not just $\alpha(\boldsymbol{k},\pm)$.

XIV. FIELDS PRODUCED BY A CLASSICAL CURRENT

The interaction Hamiltonian in the interaction picture is assumed in the usual form

$$H(t) = \int d^3x\, J^a(t,\boldsymbol{x})\underline{A}_a(t,\boldsymbol{x}) \tag{120}$$

where $J^a(t,\boldsymbol{x})$ is a classical conserved current. The interaction picture evolution operator satisfies

$$i\tfrac{d}{dt}U(t,t_0) = H(t)U(t,t_0), \quad U(t_0,t_0) = \underline{I} \tag{121}$$

Recalling that $\underline{A}_a(t,\boldsymbol{x})$ depends on time via the free Hamiltonian $H_0 = \underline{P}_0$ (72) we can split the Heisenberg-picture time evolution into parts involving separately the interaction picture $U(t,t_0)$ and the free evolution, i.e.

$$\underline{A}^H_a(x) = U(t,t_0)^\dagger \underline{A}_a(x)U(t,t_0) \tag{122}$$

To obtain the latter we made the usual assumption that there exists a time t_0 at which the field is free. This restriction is eased later on by moving t_0 to $\pm\infty$. The following two splittings of (120) are important

$$\begin{aligned}
H(t) &= H_1(t) + H_1(t)^\dagger = H_2(t) + H_2(t)^\dagger \\
H_1(t) &= i\int d^3x\, J^a(x)\int d\Gamma(\boldsymbol{k})\big(x_a(\boldsymbol{k})\underline{a}(\boldsymbol{k},1) + y_a(\boldsymbol{k})\underline{a}(\boldsymbol{k},2) + z_a(\boldsymbol{k})\underline{a}(\boldsymbol{k},3) + t_a(\boldsymbol{k})\underline{a}(\boldsymbol{k},0)^\dagger\big)e^{-ik\cdot x} && (123)\\
H_2(t) &= i\int d^3x\, J^a(x)\int d\Gamma(\boldsymbol{k})\big(x_a(\boldsymbol{k})\underline{a}(\boldsymbol{k},1) + y_a(\boldsymbol{k})\underline{a}(\boldsymbol{k},2) + z_a(\boldsymbol{k})\underline{a}(\boldsymbol{k},3) - t_a(\boldsymbol{k})\underline{a}(\boldsymbol{k},0)e^{2ik\cdot x}\big)e^{-ik\cdot x} && (124)
\end{aligned}$$

since the commutators

$$\begin{aligned}
[H_i(t_1), H_i(t_2)] &= 0 && (125)\\
[H_i(t_1)^\dagger, H_i(t_2)^\dagger] &= 0 && (126)\\
\big[H_i(t_1), [H_j(t_2), H_j(t_3)^\dagger]\big] &= 0 && (127)\\
\big[H_i(t_1)^\dagger, [H_j(t_2), H_j(t_3)^\dagger]\big] &= 0 && (128)
\end{aligned}$$

hold for all $i,j = 1,2$ and arbitrary times. The commutators

$$[H_1(x_0), H_1(y_0)^\dagger] = i\int d^3x d^3y\, J_a(x)\underline{D}^{(+)}(x-y)J^a(y) \tag{129}$$

$$\begin{aligned}
[H_2(x_0), H_2(y_0)^\dagger] &= i\int d^3x d^3y\, J_a(x)\underline{D}^{(+)}(x-y)J^a(y) \\
&\quad + 2\int d^3x d^3y\, J^a(x)J^b(y)\int d\Gamma(\boldsymbol{k})\underline{I}(\boldsymbol{k})t_a(\boldsymbol{k})t_b(\boldsymbol{k})\cos k\cdot(x-y), && (130)
\end{aligned}$$

are in the center of CCR. Employing continuous Baker-Hausdorff formulas [33, 34]

$$\begin{aligned}
T\exp\Big(\int_{t_0}^t d\tau\big(A(\tau)+B(\tau)\big)\Big) &= \exp\Big(\int_{t_0}^t d\tau(A(\tau)+B(\tau))\Big)\exp\Big(\tfrac12\int_{t_0}^t d\tau_1\int_{t_0}^t d\tau_2\big(\theta(\tau_1-\tau_2)-\theta(\tau_2-\tau_1)\big)[A(\tau_1),B(\tau_2)]\Big)\\
&= \exp\Big(\int_{t_0}^t d\tau A(\tau)\Big)\exp\Big(\int_{t_0}^t d\tau B(\tau)\Big)\exp\Big(\int_{t_0}^t d\tau_1\int_{t_0}^t d\tau_2\theta(\tau_1-\tau_2)[B(\tau_1),A(\tau_2)]\Big) && (131)
\end{aligned}$$

where $A(\tau)$, $B(\tau)$ satisfy relations analogous to (125)–(128), we find that

$$U(t,t_0) = \exp\Big(-i\int_{t_0}^t d^4x J^a(x)\underline{A}_a(x)\Big)\exp\Big(-\tfrac{i}{2}\int_{t_0}^t\int_{t_0}^t d^4x_1 d^4x_2 J_a(x_1)\underline{D}_{\rm adv}(x_1-x_2)J^a(x_2)\Big), \tag{132}$$

where $\underline{D}_{\rm adv}(x) = -\theta(-x_0)\underline{D}(x)$. Formula (131) will be later used to compute the photon statistics.

Employing (132) we find the explicit form of the Heisenberg-picture evolution

$$\underline{A}^H_a(x) = \underline{A}_a(x) + \int_{t_0}^t d^4y\underline{D}(x-y)J_a(y) \tag{133}$$

Our field $\underline{A}^H_a(x)$ is free at $t = t_0$. In the next Section we show that the weak Lorenz condition holds for (133) only in the limit $t_0 = \pm\infty$.

XV. LORENZ CONDITION AND RETARDED/ADVANCED SOLUTIONS

The four divergence of (133) takes the form

$$\partial^a \underline{A}_a^H(x) \;=\; \text{free part} + \int d^3x' \underline{D}(t-t_0, \boldsymbol{x}-\boldsymbol{x}')J_0(t_0,\boldsymbol{x}')$$

Taking an arbitrary coherent-state average

$$\langle \underline{O}(\alpha)|\partial^a \underline{A}_a^H(x)|\underline{O}(\alpha)\rangle = \int d^3x' \langle \underline{O}|\underline{D}(t-t_0, \boldsymbol{x}-\boldsymbol{x}')|\underline{O}\rangle J_0(t_0,\boldsymbol{x}')$$

and requiring the Lorenz gauge for all conserved currents, we obtain a condition on the vacuum-state probability density $Z(\boldsymbol{k}) = |O(\boldsymbol{k})|^2$

$$\langle \underline{O}|\underline{D}(t-t_0,\boldsymbol{x})|\underline{O}\rangle = i\int d\Gamma(\boldsymbol{k})Z(\boldsymbol{k})\big(e^{-ik\cdot x}e^{ik_0t_0} - e^{-ik_0t_0}e^{ik\cdot x}\big) = 0$$

This cannot hold in general if t_0 is finite. However, for $t_0 \to \pm\infty$ the condition becomes equivalent to

$$\lim_{t_0\to\pm\infty} \int d^3k f(\boldsymbol{k})e^{i|\vec{k}|t_0} = 0 \tag{134}$$

where $f(\boldsymbol{k}) = Z(\boldsymbol{k})e^{ik\cdot x}/|\boldsymbol{k}|$. The latter condition requires only that $Z(\boldsymbol{k})/|\boldsymbol{k}|$ satisfies assumptions of the Riemann-Lebesgue lemma.

We thus restrict the analysis to the two cases of either retarded or advanced solutions. The formulas are

$$\underline{A}_a^{\rm ret}(x) \;=\; \underline{A}_a(x) + \int d^4y \underline{D}_{\rm ret}(x-y)J_a(y) \tag{135}$$
$$\underline{A}_a^{\rm adv}(x) \;=\; \underline{A}_a(x) + \int d^4y \underline{D}_{\rm adv}(x-y)J_a(y) \tag{136}$$
$$\underline{D}_{\rm ret}(x) \;=\; \theta(x_0)\underline{D}(x) \tag{137}$$
$$\underline{D}_{\rm adv}(x) \;=\; -\theta(-x_0)\underline{D}(x) \tag{138}$$
$$\underline{D}(x) \;=\; \underline{D}_{\rm ret}(x) - \underline{D}_{\rm adv}(x) \tag{139}$$

Since $\Box\underline{D}(x) = 0$ we find

$$\Box\underline{D}_{\rm ret}(x-y) = \Box\underline{D}_{\rm adv}(x-y) \stackrel{\rm def}{=} \underline{\delta}(x-y) \tag{140}$$

One has to bear in mind that $\underline{\delta}(x-y)$ is *defined* by (140) and that the resulting *operator* is not equivalent to the Dirac delta.

The advanced and retarded potentials satisfy

$$\Box\underline{A}_a^{\rm ret/adv}(x) = \int d^4y \underline{\delta}(x-y)J_a(y) \stackrel{\rm def}{=} \underline{J}_a(x) \tag{141}$$

The weak Lorenz condition implies that the average current

$$\langle J_a(x)\rangle = \langle \underline{O}(\alpha)|\underline{J}_a(x)|\underline{O}(\alpha)\rangle = \langle \underline{O}|\underline{J}_a(x)|\underline{O}\rangle \tag{142}$$

is the conserved physical current that produces the classical electromagnetic field

$$\langle A_a^{\rm ret/adv}(x)\rangle \;=\; \langle \underline{O}(\alpha)|\underline{A}_a^{\rm ret/adv}(x)|\underline{O}(\alpha)\rangle. \tag{143}$$

The modification of the current depends only on the choice of the vacuum state because the displacement operator commutes with $\underline{I}(\boldsymbol{k})$.

To close this Section let us mention that an operator analogue of the Feynman propagator

$$\underline{D}_F(x) \;=\; \theta(x_0)\underline{D}^{(+)}(x) - \theta(-x_0)\underline{D}^{(-)}(x) \tag{144}$$
$$\;=\; \underline{D}_{\rm adv}(x) + \underline{D}^{(+)}(x) = \underline{D}_{\rm ret}(x) - \underline{D}^{(-)}(x) \tag{145}$$

would occur in *perturbative* formulas in exactly the same places as in the standard formalism. The reason is that the algebraic structure of Feynman diagrams is unchanged by the change of representation of CCR. Since $\Box\underline{D}^{(\pm)}(x) = 0$, the Feynman potential

$$\underline{A}_a^F(x) \;=\; \underline{A}_a(x) + \int d^4y \underline{D}_F(x-y)J_a(y) \tag{146}$$

satisfies the same equation as the retarded and advanved fields, but is non-Hermitian for real currents.

XVI. DYNAMICS OF RETARDED AND ADVANCED SOLUTIONS BETWEEN TWO FINITE TIMES

We have solved the Heisenberg equations with free-field "initial" conditions at $t_0 = \pm\infty$ and arrived at retarded and advanced solutions of Maxwell's equations. We have not yet shown what kind of dynamics will map retarded or advanced solutions at a *finite* time t_1 into retarded or advanced solutions at another finite time t. This would be the true solution of the Heisenberg-picture evolution since at a finite initial time the field cannot be free, unless the charge of the current is zero.

One can immediately write down appropriate formulas on the basis of the retarded and advanced solutions

$$\underline{A}_a^{\mathrm{ret/adv}}(t,\boldsymbol{x}) = \lim_{t_0\to-\infty/+\infty} U(t,t_0)^\dagger U_0(t,t_1)^\dagger U(t_1,t_0)\underline{A}_a^{\mathrm{ret/adv}}(t_1,\boldsymbol{x})U(t_1,t_0)^\dagger U_0(t,t_1)U(t,t_0) \tag{147}$$

$$= W_{-/+}(t,t_1)^\dagger \underline{A}_a^{\mathrm{ret/adv}}(t_1,\boldsymbol{x})W_{-/+}(t,t_1) \tag{148}$$

Let us recall that $U_0(t,t_1) = \exp\big(-iH_0(t-t_1)\big)$, where $H_0 = \underline{P}_0$ is the free Hamiltonian defined by the reducible representation of CCR, and $U(t,t_0)$ is the interaction-picture evolution operator. Some care is needed in the definitions of $W_\pm(t,t_1)$ if the limits $\lim_{t_0\to\pm\infty} U(t_1,t_0)^\dagger U_0(t,t_1)U(t,t_0)$ involve divergent phase factors. This is the standard problem and has nothing to do with the divergences of quantum field theory. Keeping this subtlety in mind we arrive at

$$W_\pm(t,t_1) = \exp\Big(i\textstyle\int_{\pm\infty}^{t_1} d^4x\big(J^a(x_0,\boldsymbol{x}) - J^a(x_0+t-t_1,\boldsymbol{x})\big)\underline{A}_a(x_0,\boldsymbol{x})\Big)\exp\Big(-iH_0(t-t_1)\Big) \tag{149}$$

It is clear that for a static charge density the evolution is free. The ranges of integration are finite also in case the currents are static for $t < t_-$ and $t > t_+$ with some $t_\pm$. It should be stressed that this type of "switching on and off" of the current is perfectly consistent with charge conservation.

The corresponding Hamiltonian $H_\pm(t)$ satisfying

$$i\partial_t W_\pm(t,t_1) = W_\pm(t,t_1)H_\pm(t) \tag{150}$$

reads

$$H_\pm(t) = H_0 + \textstyle\int_{\pm\infty}^{t} d^4x\underline{A}_a(x)\frac{\partial}{\partial x_0}J^a(x). \tag{151}$$

In the next section we show the explicit form of $H_\pm(t)$ for a pointlike charge. As we shall see the Hamiltonian has a clear physical interpretation.

XVII. EXPLICIT FORM OF THE NEW HAMILTONIAN FOR A POINTLIKE CHARGE

A pointlike charge q localized on an infinitely long world-line $z^a(t) = \big(t,\boldsymbol{z}(t)\big)$ leads to the conserved current [35, 36]

$$J^a(t,\boldsymbol{x}) = q\big(1,\boldsymbol{v}(t)\big)\delta^{(3)}\big(\boldsymbol{x}-\boldsymbol{z}(t)\big) \tag{152}$$

where $\boldsymbol{v}(t) = d\boldsymbol{z}(t)/dt$. Let us assume that the world-line represents a charge which is at rest for times $t < t_-$ and $t > t_+$. The assumption implies also that $\boldsymbol{v}(t_\pm) = 0$ if we assume that $t \mapsto \boldsymbol{z}(t)$ is twice differentiable. Under these assumptions we find for $t_- \le t \le t_+$ that

$$H_\pm = H_0 - q\underline{\boldsymbol{A}}\big(z(t)\big)\cdot\boldsymbol{v}(t) - q\textstyle\int_{z(t_\pm)}^{z(t)} d\boldsymbol{z}\cdot\underline{\boldsymbol{E}}. \tag{153}$$

$$H_- = H_0 \quad \text{for} \quad t \le t_- \tag{154}$$

$$H_+ = H_0 \quad \text{for} \quad t \ge t_+ \tag{155}$$

The line integral in the third term of (153) is along the part of the charge world-line where the charge velocity is nonzero. The electric field operator takes the usual form $\underline{\boldsymbol{E}} = -\partial_0\underline{\boldsymbol{A}} - \boldsymbol{\nabla}\underline{A}_0$.

Let us note that the terms explicitly involving $\underline{A}_0$ have cancelled out. It is clear from the construction that the electric field occuring in $H_\pm$ is free. Therefore, the Hamiltonian does not contain self-energy terms but, instead, takes into account the work performed by the particle against the electric field.

XVIII. PHOTON STATISTICS

The operator $U(t,\pm\infty)$ (as well as $U_0(t,\pm\infty)$) in general does not exist due to the problem with divergent phase factor. Fortunately we do not really need $U(t,t_0)$ itself, but only its action on operators X

$$\boldsymbol{U}(t,t_0)(X) = U(t,t_0)^\dagger X\, U(t,t_0) \tag{156}$$

Similarly, in order to compute the S-matrix we concentrate on the limiting operator map $\boldsymbol{S} = \boldsymbol{U}(+\infty,-\infty)$.

Eqs. (133), (135), (136) imply that at one hand

$$\underline{A}_a^{\rm ret}(x) \;=\; \lim_{\tau_0\to-\infty}\lim_{\tau\to+\infty} U(\tau,\tau_0)^\dagger \underline{A}_a^{\rm adv}(x) U(\tau,\tau_0) = \boldsymbol{S}\big(\underline{A}_a^{\rm adv}(x)\big) \tag{157}$$

and on the other

$$\underline{A}_a(x) + \int d^4y \underline{D}(x-y) J_a(y) \;=\; \boldsymbol{S}\big(\underline{A}_a(x)\big). \tag{158}$$

Finally, as shown in [6], the S-matrix $\boldsymbol{S}$ gives the action of the displacement operator on the field operators.

More interesting is the question of photon statistics in fields produced by classical currents, especially if the currents are pointlike or stationary. Accelerated pointlike charges lead, in the standard formalism, to infrared catastrophe. In a naive approach, all transition probabilities are zero, which contradicts unitarity of the S-matrix. In manifestly covariant approaches, such as the Gupta-Bleuler formulation, static charge distributions lead to infinite vacuum-to-vacuum probability, which again makes no sense. A mathematically correct treatment is possible [39], but requires a change to an abstract representation. This goes against the philosophy of explicit construction, adhered to in the present paper.

In our approach infrared divergence can be avoided by requiring that the vacuum probability density $Z(\boldsymbol{k})$ tends to zero at $\boldsymbol{k}=0$. Let us take the operator $H_2(t)$ given by (124) and split it into the parts $H_2^{(30)}(t)$ and $H_2^{(12)}(t)$ involving, respectively, the fields of spin-0 (i.e. $\underline{a}(\boldsymbol{k},3)$ and $\underline{a}(\boldsymbol{k},0)$) and spin-1 (i.e. $\underline{a}(\boldsymbol{k},1)$ and $\underline{a}(\boldsymbol{k},2)$). Let us recall that the N-oscillator Hilbert space is spanned by N-fold tensor products of vectors of the form $|\boldsymbol{k},n_1,n_2,n_3,n_0\rangle$. We shall refer to such 1-oscillator states as containing n_1+n_2 transverse excitations and n_3+n_0 longitudinal ones. Any state belonging to the N-oscillator Hilbert space $\underline{\mathcal{H}}$ and containing n transverse excitations, where n is is the sum of the transverse excitations of all the N oscillators, is regarded as a state involving n transverse photons. Similarly we define a general state involving n' longitudinal photons. In particular, the vector

$$H_2^{(12)}(t_1)^\dagger \dots H_2^{(12)}(t_n)^\dagger |\underline{O}\rangle \tag{159}$$

belongs to the subspace of n-transverse-photon states. The state

$$H_2^{(30)}(t_1)^\dagger \dots H_2^{(30)}(t_{n'})^\dagger |\underline{O}\rangle \tag{160}$$

involves n' longitudinal photons.

Denote by $\mathcal{P}_{nn'}$ the projector on the subspace of $\underline{\mathcal{H}}$ that contains states with n transverse and n' longitudinal photons. The probability of finding n transverse and n' longitudinal photons in the state produced from vacuum by a classical current is thus

$$p_{nn'}(t,t_0) \;=\; \langle \underline{O}|U(t,t_0)^\dagger \mathcal{P}_{nn'} U(t,t_0)|\underline{O}\rangle. \tag{161}$$

Employing (131) we find for $p_{nn'} = p_{nn'}(\infty,-\infty)$

$$\begin{aligned}
p_{nn'} &= \frac{1}{n!n'!}\langle \underline{O}|F_{12}^n e^{-F_{12}} F_{30}^{n'} e^{-F_{30}}|\underline{O}\rangle = \frac{1}{n!n'!}\frac{d^n}{d\mu^n}\frac{d^{n'}}{d\nu^{n'}} C(\mu,\nu)\Big|_{\mu=\nu=-1}\\
C(\mu,\nu) &= \langle \underline{O}|e^{\mu F_{12}} e^{\nu F_{30}}|\underline{O}\rangle\\
F_{12} &= \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' [H_2^{(12)}(\tau), H_2^{(12)}(\tau')^\dagger]\\
F_{30} &= \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' [H_2^{(30)}(\tau), H_2^{(30)}(\tau')^\dagger]
\end{aligned}$$

Assuming that detectors react only to spin-1 photons we obtain photon statistics

$$p_n = \sum_{n'=0}^{\infty} p_{nn'} = \frac{1}{n!}\langle \underline{O}|F_{12}^n e^{-F_{12}}|\underline{O}\rangle \tag{162}$$

Alternatively, in order to describe quantum optics of spin-1 observables directly, without any reference to the spin-0 fields, we can consider states defined via reduced density matrices with the spin-0 parts traced out. Probability p_n is an example of an average computed in terms of such a reduced density matrix.

A. Fourier description

To find an explicit formula we first consider a current whose 4-dimensional Fourier transform is a well behaved function $\tilde{J}_a(k) = \int d^4x J_a(x) e^{ik\cdot x}$. Then

$$F_{12} = \int d\Gamma(\boldsymbol{k})\underline{I}(\boldsymbol{k})\Big(|\tilde{J}^a(\boldsymbol{k})m_a(\boldsymbol{k})|^2 + |\tilde{J}^a(\boldsymbol{k})\bar{m}_a(\boldsymbol{k})|^2\Big)$$

$$F_{30} = \int d\Gamma(\boldsymbol{k})\underline{I}(\boldsymbol{k})|\tilde{J}^a(\boldsymbol{k})\omega_a(\boldsymbol{k})|^2$$

where $\tilde{J}^a(\boldsymbol{k})$ is a restriction of $\tilde{J}_a(k)$ to the light-cone. Due to the continuity equation $\tilde{J}^a(\boldsymbol{k})k_a(\boldsymbol{k}) = 0$ the spin-1 expression F_{12} is independent of the choice of $\omega_A(\boldsymbol{k})$.

Employing the relation between $\underline{I}(\boldsymbol{k})$ and $I(\boldsymbol{k}) = |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes 1$ we can write the generating function as

$$C(\mu,\nu) = \left(\int d\Gamma(\boldsymbol{k})Z(\boldsymbol{k})\exp\Big(\frac{\mu}{N}\big(|\tilde{J}^a(\boldsymbol{k})m_a(\boldsymbol{k})|^2 + |\tilde{J}^a(\boldsymbol{k})\bar{m}_a(\boldsymbol{k})|^2\big)\Big)\exp\Big(\frac{\nu}{N}|\tilde{J}^a(\boldsymbol{k})\omega_a(\boldsymbol{k})|^2\Big)\right)^N \tag{163}$$

Of particular interest is the thermodynamic limit $N \to \infty$ for the spin-1 part. The corresponding generating function becomes

$$\lim_{N\to\infty} C(\mu,0) = \exp\Big(\mu\int d\Gamma(\boldsymbol{k})Z(\boldsymbol{k})\big(|\tilde{J}^a(\boldsymbol{k})m_a(\boldsymbol{k})|^2 + |\tilde{J}^a(\boldsymbol{k})\bar{m}_a(\boldsymbol{k})|^2\big)\Big) \tag{164}$$

and

$$p_n = \tfrac{1}{n!}\Big(\int d\Gamma(\boldsymbol{k})Z(\boldsymbol{k})\big(|\tilde{J}^a(\boldsymbol{k})m_a(\boldsymbol{k})|^2 + |\tilde{J}^a(\boldsymbol{k})\bar{m}_a(\boldsymbol{k})|^2\big)\Big)^n \exp\Big(-\int d\Gamma(\boldsymbol{k})Z(\boldsymbol{k})\big(|\tilde{J}^a(\boldsymbol{k})m_a(\boldsymbol{k})|^2 + |\tilde{J}^a(\boldsymbol{k})\bar{m}_a(\boldsymbol{k})|^2\big)\Big)$$

This is basically the well known Poisson distribution, with one modification: The standard infrared-divergent result is found if one puts $Z(\boldsymbol{k}) = 1$. However, we know that $\int d\Gamma(\boldsymbol{k})Z(\boldsymbol{k}) = 1$ and thus $Z(\boldsymbol{k}) \neq 1$. We have remarked earlier that the maximum value of $Z(\boldsymbol{k})$ is a positive Poincaré invariant, denoted by Z. Introducing the new function $\chi(\boldsymbol{k}) = Z(\boldsymbol{k})/Z$, and absorbing $Z^{1/2}$ into a renormalized current $\tilde{J}^a_{\rm ren}(\boldsymbol{k}) = Z^{1/2}\tilde{J}^a(\boldsymbol{k})$ we find

$$p_n = \tfrac{1}{n!}\Big(\int d\Gamma(\boldsymbol{k})\chi(\boldsymbol{k})\big(|\tilde{J}^a_{\rm ren}m_a(\boldsymbol{k})|^2 + |\tilde{J}^a_{\rm ren}\bar{m}_a(\boldsymbol{k})|^2\big)\Big)^n \exp\Big(-\int d\Gamma(\boldsymbol{k})\chi(\boldsymbol{k})\big(|\tilde{J}^a_{\rm ren}m_a(\boldsymbol{k})|^2 + |\tilde{J}^a_{\rm ren}\bar{m}_a(\boldsymbol{k})|^2\big)\Big)$$

Now this is indeed the standard *regularized* expression. The latter provides us with a new information about the vacuum wave function $O(\boldsymbol{k})$: It has to vanish at the origin $\boldsymbol{k} = \boldsymbol{0}$ if one wants the cut-off function $\chi(\boldsymbol{k})$ to regularize the infrared divergence. The origin belongs to the boundary of the light cone. Vanishing at the origin is a Poincaré invariant boundary condition.

We regard this result as very important, as it handles in a natural manner two elements that are imposed in an ad hoc manner in the standard formalism. First of all, we do not need to justify the infrared cut-off by hand-waving arguments on unobservability of "soft photons". Our formalism introduces the cutting-off function automatically. Secondly, we know what is the origin of a renormalization constant: This is simply the Poincaré invariant associated with the vacuum wave function.

B. Pointlike static charge

In this case it makes no sense to switch to the Fourier domain, since the position space calculation is more straightforward. Assume the current is $J_a(x) = (q\delta^{(3)}(\boldsymbol{x}), \boldsymbol{0})$. The generating function becomes

$$C(\mu,\nu) = \Big(\int d\Gamma(\boldsymbol{k})Z(\boldsymbol{k})e^{(2q^2/|\boldsymbol{k}|^2)\big(\frac{\mu}{N}x_0(\boldsymbol{k})^2 + \frac{\mu}{N}y_0(\boldsymbol{k})^2 + \frac{\nu}{N}z_0(\boldsymbol{k})^2 + \frac{\nu}{N}t_0(\boldsymbol{k})^2\big)}\Big)^N \tag{165}$$

In the thermodynamic limit

$$\lim_{N\to\infty} C(\mu,\nu) = \exp\Big(\int d\Gamma(\boldsymbol{k})Z(\boldsymbol{k})(2q^2/|\boldsymbol{k}|^2)\big(\mu x_0(\boldsymbol{k})^2 + \mu y_0(\boldsymbol{k})^2 + \nu z_0(\boldsymbol{k})^2 + \nu t_0(\boldsymbol{k})^2\big)\Big) \tag{166}$$

Identical results are obtained if instead of $U(\infty,-\infty)$ one works with $U(t,\pm\infty)$ for a finite t.

Let us remark that an analogous calculation performed in a reducible version of Gupta-Bleuler formalism [17] leads to vacuum-to-vacuum "probabilities" that are greater than 1. The reason is that for currents whose only nonzero component is $J_0(x)$ the Fourier-space version of continuity equation does not read $\tilde{J}_a(k)k^a=0$, but $\tilde{J}_0(k_0)\delta(k_0)=0$, and one cannot claim that $\tilde{J}_a(k)$ is spacelike. In our formalism the timelike component of the current comes with the correct sign.

C. Rényi statistics for finite N

Generating functions can be written in a unified way for any N in terms of Kolmogorov-Nagumo averages of the form used in Rényi statistics. Let us recall that Rényi's alpha entropies were obtained in [37] as Kolmogorov-Nagumo averages

$$\langle I\rangle_\phi = \phi^{-1}\big(\sum_j p_j\phi(I_j)\big) \tag{167}$$

of the random variable $I_j=\ln(1/p_j)$, and $\phi(x)=e^{(1-\alpha)x}$. For $\alpha=1$ one obtains the standard Boltzmann-Shannon entropy. In [38] it was shown that thermodynamics that employs Rényi type averaging can be used to derive certain equilibrium distributions occuring in linguistics and protein folding. Various arguments based on thermodynamics suggest that $\alpha\neq 1$ statistics may be typical of finite systems. Photon statistics for finite-N representations of CCR supports this intuition.

Indeed, in the thermodynamic limit we found a generating function of the form $C(\mu,0)=e^{\langle j(\mu)\rangle}$ with

$$\langle j(\mu)\rangle = \mu\textstyle\int d\Gamma(\boldsymbol{k})Z(\boldsymbol{k})\big(|\tilde{J}^a(\boldsymbol{k})m_a(\boldsymbol{k})|^2+|\tilde{J}^a(\boldsymbol{k})\bar{m}_a(\boldsymbol{k})|^2\big)$$

being a linear ($\alpha=1$) average of $\mu\big(|\tilde{J}^a(\boldsymbol{k})m_a(\boldsymbol{k})|^2+|\tilde{J}^a(\boldsymbol{k})\bar{m}_a(\boldsymbol{k})|^2\big)$, with probability density $Z(\boldsymbol{k})$. For finite N we find $C(\mu,0)=e^{\langle j(\mu)\rangle_\phi}$ where $\phi(x)=e^{(1-\alpha)x}$, $\alpha=1-1/N$, and we average the same random variable with the same probability distribution. Obviously, the limits $N\to\infty$ and $\alpha\to 1$ are equivalent. It follows that the field theories based on $N<\infty$ or $N=\infty$ representations are related to one another in a way that is analogous to the relation between systems described by $0<\alpha<1$ and $\alpha=1$ entropies. These, on the other hand, are known to apply to fractal and non-fractal geometries, respectively. A natural intuition thus relates the $N<\infty$ case to some "space-time foam", and $N=\infty$ to continuum space-time.

XIX. CLASSICAL FIELDS PRODUCED BY CLASSICAL SOURCES: A QUANTUM WAY

Our previous analysis shows that, having a classical current $J_a(x)$, we obtain a result that agrees with standard calculations if one (a) absorbs $Z^{1/2}$ in the current by means of $J_a^{\rm ren}(x)=Z^{1/2}J_a(x)$ (bare charge renormalization $q\mapsto q^{\rm ren}=Z^{1/2}q$), and (b) compares the result with a regularized formula, which is anyway the one we have to compare with experiment. Z is not a constant but rather an invariant of the Poincaré group that characterizes a given vacuum. We have also obtained a cut-off function $\chi(\boldsymbol{k})=Z(\boldsymbol{k})/Z$. At this stage we do not have much information as to the exact form of $\chi(\boldsymbol{k})$ and can only say that it vanishes for large $\boldsymbol{k}$ and $\boldsymbol{k}=0$, and that $\chi(\boldsymbol{k})/|\boldsymbol{k}|$ fulfills the assumptions of the Riemann-Lebesgue lemma.

Now let us take an arbitrary classical amplitude $\alpha(\boldsymbol{k},\pm)$ corresponding to left- and right-handed Fourier modes of a classical electromagnetic field. We define the *quantum optics regime* by the support of those classical amplitudes that satisfy

$$\alpha(\boldsymbol{k},\pm) \;=\; \alpha(\boldsymbol{k},\pm)\chi(\boldsymbol{k}). \tag{168}$$

The latter formula is meaningful provided $\chi(\boldsymbol{k})=1$ if $\boldsymbol{k}$ belongs to quantum optics regime. Classical fields belonging to quantum optics regime do not contain wavelenghts that are either too large or too small. Let $|\underline{O}(\alpha)\rangle$ be a coherent state with α in quantum optics regime. Let us take the coherent state average of the retarded solution of Heisenberg equation of motion (135) and express it in terms of the renormalized current $J_a^{\rm ren}(x)$. Taking into account that

$$\mathcal{A}_a(x) \;=\; \langle\underline{O}(\alpha)|\underline{A}_a(x)|\underline{O}(\alpha)\rangle = Zi\textstyle\int d\Gamma(\boldsymbol{k})\big(m_a(\boldsymbol{k})\alpha(\boldsymbol{k},+)+\bar{m}_a(\boldsymbol{k})\alpha(\boldsymbol{k},-)\big)e^{-ik\cdot x}+\text{c.c.}, \tag{169}$$

by (119) and the assumption that (168) is fulfilled, the formula

$$\langle \underline{O}(\alpha)|\underline{D}(x-y)|\underline{O}(\alpha)\rangle = Zi\int d\Gamma(\boldsymbol{k})\chi(\boldsymbol{k})\big(e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)}\big), \tag{170}$$

and dividing the entire solution by $Z^{1/2}$, we find that the classical field

$$\langle A_a^{\rm ret}(x)\rangle = Z^{-1/2}\mathcal{A}_a(x) + \int d^4y D_{\rm ret}(x-y)J_a^{\rm phys}(y) \tag{171}$$

$$\partial^a\langle A_a^{\rm ret}(x)\rangle = \partial^a\mathcal{A}_a(x) = 0 \tag{172}$$

$$\partial^a J_a^{\rm phys}(x) = 0 \tag{173}$$

exhibits the textbook relation between the in-field, renormalized current, and renormalization constant.

Here $D_{\rm ret}(x-y)$ is the ordinary retarded Green function and $J_a^{\rm phys}(y)$ is the efective current obtained after charge renormalization and inclusion of $\chi(\boldsymbol{k})$ in its Fourier transform (the convolution of $\underline{D}_{\rm ret}(x-y)$ and $J_a(y)$ in (135) allows to shift the regularization from the Green function to the current, and vice versa). All these objects have occured in our calculation automatically.

Finally, let us have a closer look at the effective current. For simplicity take a static pointlike charge. Employing the relations

$$Z^{1/2}J_0^{\rm phys}(x_0,\boldsymbol{x}) = \int d^4y\langle \underline{O}|\underline{\delta}(x-y)|\underline{O}\rangle J_0(y) = q\langle \underline{O}|\partial_0\underline{D}(0,\boldsymbol{x})|\underline{O}\rangle = Zq\tfrac{1}{2}\int \tfrac{d^3k}{(2\pi)^3}\chi(\boldsymbol{k})\big(e^{i\vec{k}\cdot\vec{x}} + e^{-i\vec{k}\cdot\vec{x}}\big)$$

we find that the effective total charge

$$Q = \int d^3x J_0^{\rm phys}(x_0,\boldsymbol{x}) = Z^{1/2}q\chi(\boldsymbol{0}) = q^{\rm ren}\chi(\boldsymbol{0})$$

vanishes since we require $\chi(\boldsymbol{0}) = 0$.

In order to check the physical meaning of this condition let us consider the simple case of spherically symmetric

$$\chi_{k_1,k_2}(\boldsymbol{k}) = \begin{cases} 1 & \text{for } k_1 \le |\boldsymbol{k}| \le k_2 \\ 0 & \text{otherwise} \end{cases} \tag{174}$$

Then $Q = q^{\rm ren}$ for $k_1 = 0$ and $Q = 0$ for $k_1 > 0$. One can check that the effective charge density $J_0^{\rm phys}(x_0,\boldsymbol{x}) = \rho_{k_2}(\boldsymbol{x}) - \rho_{k_1}(\boldsymbol{x})$, where $\int d^3x\rho_{k_2}(\boldsymbol{x}) = \int d^3x\rho_{k_1}(\boldsymbol{x}) = q^{\rm ren}$ for all $k_2, k_1 > 0$, but simultaneously the pointlike limit $\lim_{k_1\to 0}\rho_{k_1}(\boldsymbol{x}) = 0$ holds. It turns out that the charge density consists of a difference of two densities: One, which is the sharper and more localized the greater k_2, and the other which is the flatter and less localized the smaller k_1. One of them corresponds to localization of the charge $q^{\rm ren}$ in a sinc-like very sharp-peaked density, and the other describes the charge $-q^{\rm ren}$ distributed almost uniformly in a volume which becomes infinite if $k_1 = 0$. Plots of the densities for $k_1 > 0$ and $k_1 = 0$ become indistinguishable even for relatively large k_1 and small k_2, so we leave this exercise to the readers. In practice, one cannot locally distinguish between $k_1 = 0$ and $k_1 \approx 0$, but globally the two cases are inequivalent.

XX. SUMMARY AND CONCLUSIONS

We have discussed a new quantization scheme based on reducible representations of CCR. The principal goal of this research program is to arrive at a mathematically consistent formalism for quantum fields, that should be described by a *theory* and not a *set of working rules*, as Dirac summarized the current status of field quantization [3]. In our approach fields are represented by operators and not operator-valued distributions. The field is a finite quantum system, and the measure of its size is the parameter N. For this reason two sources of infinities are absent in the formalism from the very outset: We can multiply field operators at the same points in space-time and all the tensor products one encounters are finite. The latter condition means that we deal only with factors of type I, in von Neumann's terminology.

We have carefully analyzed Poincaré covariance of the theory. There were two aspects we had to understand to make sure that the new quantization is not inconsistent with special relativity. First of all, we constructed a unitary representation of the Poincaré group whose carrier space is an ordinary Hilbert space involving no indefinite metric. Four-potential operator is self-adjoint, the dynamics is unitary, but commutation relations for fields are nevertheless manifestly Poincaré covariant. The formalism is a promising alterantive to the Gupta-Bleuler quantization, where the price payed for manifest covariance is either in non-positivity of the Hilbert-space metric, or in non-Hermicity of the potential.

Secondly, we had to understand in what way a Lorentz transformation influences the gauge freedom. The latter has led to the observation that a change of gauge due to Lorentz transformations can be always compensated by a Bogoliubov unitary transformation of the vacuum. An inclusion of the Bogoliubov transformation turns the 4-potential into a 4-vector field.

One element that remained arbitrary is what kind of a tetrad one has to associate with a 4-potential. Once one makes a choice then the remaining freedom can be controlled by Bogoliubov transformations, whose explicit form has been given. In our formalism the dynamics is unitary, in the ordinary meaning of this word. We have no problems with negative or greater than 1 "probabilities" that occur in the Gupta-Bleuler formalism. The correct probability interpretation is guaranteed by the Schwartz inequality. Further, if one looks at the radiation fields then our formalism produces the standard regularized formula, which does not depend on a choice of gauge.

Finally, its seems that we have produced the first example of Heisenberg dynamics where the retarded or advanced fields unitarily evolve from, say, $t = 0$ to another finite t. Our construction allowed to systematically derive the form of Hamiltonian that is responsible for such an evolution, and the result turned out to differ from the usual minimal-coupling expression: There is no scalar-potential part and a new term occurs. The term describes the work performed by the charge moving in electric field. This result may have implications for quantum optics where the usual treatments of spontaneous emission or resonance fluorescence are based on initial conditions at $t = 0$ and not $t = -\infty$ (cf. [40, 41], the exception is [42]).

Summing up, we think we have proposed at least a nontrivial answer to the problem posed by Dirac in his last two papers. It looks like the formalism, supplemented by its fermionic analogue introduced in [7], is ready for calculations in full quantum electrodynamics. Some preliminary results on loop diagrams have been already obtained and will be reported in a future paper.

Acknowledgments

We are indebted to Professor David Finkelstein and Ms. Klaudia Wrzask for discussions. This work was performed as a part of the bilateral Flemish-Polish project "Soliton techniques applied to equations of quantum field theory". The research of M.C. was also supported by the Polish Ministry of Scentific Research and Information Technology under the (solicited) grant No. PBZ-Min-008/P03/2003.

[1] I. Białynicki-Birula, Phys. Rev. **155**, 1414 (1967); Phys. Rev. D **2**, 2877 (1970).
[2] R. P. Feynman, Phys. Rev. **76**, 769 (1949).
[3] P. A. M. Dirac, Eur. J. Phys. **5**, 65 (1984).
[4] P. A. M. Dirac, Int. J. Theor. Phys. **23**, 677 (1984).
[5] M. Czachor, J. Phys. A: Math. Gen. **33**, 8081 (2000).
[6] M. Czachor, Phys. Lett. A **313**, 380 (2003)
[7] M. Czachor, J. Nonlin. Math. Phys. **11**, 78 (2004); hep-th/0304144.
[8] M. Wilczewski and M. Czachor, quant-ph/0507093.
[9] M. Czachor and M. Wilczewski, quant-ph/0500117.
[10] M. Brune, F. Schmidt-Kaler, A. Maali, J. Dreyer, E. Hagley, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. **76**, 1800 (1996).
[11] D. R. Finkelstein, "General quantization", quant-ph/0601002.
[12] R. Penrose and W. Rindler, *Spinors and Space-Time, vol. 1: Two-Spinor Calculus and Relativistic Fields* (Cambridge University Press, Cambridge, 1984).
[13] As noted in [12], footnote on p. 321, the gauge condition $\partial^a A_a = 0$ was introduced by L. Lorenz and not by H. A. Lorentz, whose name is associated with Lorentz transformations.
[14] A. Ashtekar, *Lectures on Non-Perturbative Canonical Gravity* (World Scientific, Singapore, 1991)
[15] O. Bratelli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, vol. 2 (Springer, Berlin, 1981).
[16] This explict construction has been done in collaboration with K. Wrzask.
[17] M. Czachor, "Non-Hermitian quantization of potentials", unpublished notes (2004).
[18] G. V. Efimov, *Nonlocal Interactions of Quantized Fields* (Nauka, Moscow, 1977), in Russian.
[19] J. W. Moffat, Phys. Rev. D **39**, 3654 (1989); Phys. Rev. D **41**, 1177 (1990).
[20] D. Evens, J. W. Moffat, G. Kleppe, R. P. Woodard, Phys. Rev. D **43**, 499 (1991).
[21] G. Kleppe, R. P. Woodard, Ann. Phys. (NY) **221**, 106 (1993).
[22] M. A. Clayton, Mod. Phys. Lett. A **16**, 1117 (2001).
[23] N. J. Cornish, Mod. Phys. Lett. A **7**, 1895 (1992).
[24] A. Basu, S. D. Joglekar, Eur. Phys. J. C **4**, 1 (2001).
[25] J. B. Hartle, Amer. J. Phys. **36**, 704 (1968).

[26] E. Farhi, J. Goldstone, and S. Gutmann, Ann. Phys. (NY) **192**, 368 (1989).
[27] Y. Aharonov and B. Reznik, Phys. Rev. A **65**, 052116 (2002).
[28] J. Finkelstein, Phys. Rev. A **67**, 026101 (2003).
[29] W. Heisenberg, M. Born, and P. Jordan, Z. Phys. **35**, 557 (1925)
[30] M. Czachor and K. Wrzask, in preparation.
[31] Y. Ohnuki, *Unitary Representations of the Poincaré Group and Relativistic Wave Equations* (World Scientific, Singapore, 1988).
[32] R. Penrose and W. Rindler, *Spinors and Space-Time, vol. 2: Spinor and Twistor Methods in Space-Time Geometry* (Cambridge University Press, Cambridge, 1986).
[33] I. Białynicki-Birula, B. Mielnik, and J. Plebański, Ann. Phys. (NY) **51**, 187 (1969)
[34] I. Białynicki-Birula and Z. Białynicka-Birula, *Quantum Electrodynamics* (Polish Scientific Publishers, Warszawa, 1974).
[35] J. Kijowski, D. Chruściński, Gen. Rel. Grav. **27**, 267 (1995).
[36] D. Chruściński, J. Kijowski, J. Geom. Phys. **27**, 49 (1998).
[37] A. Rényi, MTA III. Oszt. Közl. **10**, 251 (1960); reprinted in *Selected Papers of Alfréd Rényi,* vol 2. (Akadémiai Kiadó, Budapest, 1976), 526.
[38] M. Czachor and J. Naudts, Phys. Lett. A **298**, 369 (2002).
[39] J. Naudts, W. De Roeck, Int. J. Theor. Phys. **43**, 1111 (2004).
[40] H. J. Kimble, L. Mandel, Phys. Rev. A **13**, 2123 (1976).
[41] K. Rzążewski, W. Żakowicz, J. Phys. A: Math. Gen. **9**, 1159 (1976); J. Math. Phys. **21**, 378 (1988); J. Phys. B **25**, L319 (1992).
[42] K. Pachucki, Phys. Rev. A **44**, 5407 (1991).