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# Lower bound on the weakly connected domination number of a tree

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**Abstract**

We prove that the weakly connected domination number of every tree $T$ on $n \geq 3$ vertices and with $n_1$ end vertices satisfies the inequality $\gamma_w(T) \geq (n(T) + 1 - n_1(T))/2$ and we characterize the extremal trees.

## 1 Introduction

Let $G = (V, E)$ be a connected undirected graph. The *neighbourhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set of all vertices adjacent to $v$. For a set $X \subseteq V(G)$, the *open neighbourhood* $N_G(X)$ is defined to be $\bigcup_{v \in X} N_G(v)$ and the *closed neighbourhood* $N_G[X] = N_G(X) \cup X$. A subset $D$ of $V(G)$ is *dominating* in $G$ if every vertex of $V(G) - D$ has at least one neighbour in $D$. Let $\gamma(G)$ be the minimum cardinality among all dominating sets in $G$. The degree of a vertex $v$ is $d_G(v) = |N_G(v)|$. Further, $D \subset V(G)$ is a *connected dominating set* in $G$ if $D$ is dominating and the subgraph $G[D]$ induced by $D$ in $G$, is connected.

A dominating set $D \subset V(G)$ is a *weakly connected dominating set* in $G$ if the subgraph $G[D]_w = (N_G[D], E_w)$ weakly induced by $D$, is connected, where $E_w$ is the set of all edges having at least one vertex in $D$. Dunbar et al. [1] define the *weakly connected domination number* $\gamma_w(G)$ of a graph $G$ to be the minimum cardinality among all weakly connected dominating sets in $G$.

The *distance* $d_G(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of the shortest $(u - v)$-path in $G$. For unexplained terms and symbols see [3].

Here we consider trees on at least three vertices. If $T$ is a tree, let $n = n(T)$ be the order of $T$ and let $n_1 = n_1(T)$ denote the number of end-vertices of $T$. The set of all end-vertices in $T$ is denoted by $\Omega(T)$. A vertex $v$ is called a *support* if it is adjacent to an end-vertex.

Let $D$ be a weakly connected dominating set of a tree $T$. We say that $D$ has the property $\mathcal{F}$ if $D$ contains no end-vertex of $T$. It is obvious that in every tree on at least three vertices exists a minimum weakly connected dominating set having property $\mathcal{F}$.

We say that vertices $x, y \in D$ are *adjacent in* $D$ if there exists an edge $e = xy$ or if there exists an $(x-y)$-path whose vertices different from $x$ and $y$ do not belong to $D$.

It is known [4] that $\gamma(T) \geq (n(T) + 2 - n_1(T))/3$ for a tree $T$ on $n \geq 3$ vertices. Here we prove a similar inequality for the weakly connected domination number, i.e. we show, that $\gamma_w(T) \geq (n(T) + 1 - n_1(T))/2$ and we characterize the family of all trees $T$ for which $\gamma_w(T) = (n(T) + 1 - n_1(T))/2$.

## 2 Results

We begin with the following result.

**Theorem 1** *If $T$ is a tree of order at least three, then $\gamma_w(T) \geq (n(T)+1-n_1(T))/2$.*

**Proof.** We use induction on $n$, the order of a tree. The result is trivial for each tree of order 3. Let $T$ be a tree of order $n > 3$ and assume that $\gamma_w(T^{'}) \geq (n(T^{'}) + 1 - n_1(T^{'}))/2$ for each tree $T^{'}$ with $3 \leq n(T^{'}) \leq n-1$. Let $D$ be a minimum weakly connected dominating set having property $\mathcal{F}$ in $T$, let $P = (v_0, v_1, \ldots, v_l)$ be a longest path in $T$ and let $T^{'} = T - \{v_0\}$ be the subtree of $T$. Without loss of generality we may assume, that $P$ is chosen in such a way that $d_T(v_1)$ is as large as possible. We consider two cases: $d_T(v_1) > 2$ and $d_T(v_1) = 2$.

*Case 1.* If $d_T(v_1) > 2$, then $D$ is a minimum weakly connected dominating set of $T^{'} = T - \{v_0\}$ and thus $\gamma_w(T^{'}) = \gamma_w(T)$. By induction, $\gamma_w(T^{'}) \geq (n(T^{'})+1-n_1(T^{'}))/2$ and therefore $\gamma_w(T) \geq (n(T)+1-n_1(T))/2$ as $n_1(T^{'}) = n_1(T) - 1$ and $n(T^{'}) = n(T) - 1$.

*Case 2.* If $d_T(v_1) = 2$, then we consider two subcases: $\gamma_w(T^{'}) < \gamma_w(T)$ and $\gamma_w(T^{'}) = \gamma_w(T)$.

*Subcase 2.1.* If $\gamma_w(T^{'}) < \gamma_w(T)$, then it is easy to observe that $\gamma_w(T^{'}) = \gamma_w(T) - 1$. By induction, $\gamma_w(T^{'}) \geq (n(T^{'}) + 1 - n_1(T^{'}))/2$ and consequently, $\gamma_w(T) \geq (n(T) + 1 - n_1(T))/2$ as $n_1(T^{'}) = n_1(T)$ and $n(T^{'}) = n(T) - 1$.

*Subcase 2.2.* If $\gamma_w(T^{'}) = \gamma_w(T)$, then $v_2 \notin D$ (otherwise $D - \{v_1\}$ would be a weakly connected dominating set of $T^{'}$ and equality $\gamma_w(T^{'}) = \gamma_w(T)$ would not hold) and $l \geq 4$. It is easy to observe that $D^{''} = D - \{v_1\}$ is a minimum weakly connected dominating set of $T^{''} = T - \{v_0, v_1\}$. We consider separately $d_T(v_2) > 2$ and $d_T(v_2) = 2$.

If $d_T(v_2) > 2$, then $n_1(T^{''}) = n_1(T) - 1, n(T^{''}) = n(T) - 2$ and $\gamma_w(T^{''}) = \gamma_w(T) - 1$. Then, by induction, $\gamma_w(T^{''}) \geq (n(T^{''})+1-n_1(T^{''}))/2$ and therefore $\gamma_w(T) \geq (n(T) - n_1(T) + 1)/2$.

If $d_T(v_2) = 2$, then, by induction, $\gamma_w(T^{''}) \geq (n(T^{''}) + 1 - n_1(T^{''}))/2$ and thus $\gamma_w(T) \geq (n(T) - n_1(T) + 1)/2$ as $n_1(T^{''}) = n_1(T), n(T^{'}) = n(T) - 2$ and $\gamma_w(T^{''}) = \gamma_w(T) - 1$. ■

Now we consider a family $\mathcal{R}$ which will provide the extremal trees in Theorem 1.

If $T_1$ and $T_2$ are two trees, then let $K(T_1, T_2)$ denote a tree obtained from $T_1$ and $T_2$ by identifying one end-vertex of $T_1$ with one end-vertex of $T_2$.

We say that a tree $T$ belongs to the family $\mathcal{R}$ if and only if there exists a sequence of stars $S_1, \ldots, S_p$ such that every star has at least three vertices and $T = K(K(\ldots K(K(S_1, S_2), S_3) \ldots), S_p)$ (see Fig. 1).

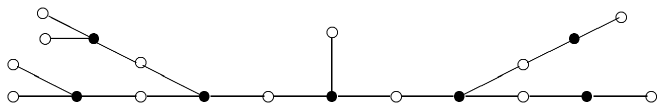

Figure 1: An example of a tree from the family $\mathcal{R}$

The following two observations are obvious from definition of the family $\mathcal{R}$.

**Observation 1** *If $T$ is a tree belonging to the family $\mathcal{R}$, then the distance between any two end-vertices of $T$ is an even number.*

**Observation 2** *If $T$ belonging to the family $\mathcal{R}$ is a tree obtained from the stars $S_1, \ldots, S_p$ and if vertices $c_1, \ldots, c_p$ are centers of the stars, then the set $D = \{c_1 \ldots, c_p\}$ is a unique minimum weakly connected dominating set of $T$. Additionally, every non-end vertex belonging to $V(T) - D$, has degree two.*

**Proof.** Reader may use induction on $p$ to show that $D = \{c_1 \ldots, c_p\}$ is a unique minimum weakly connected dominating set of $T$ belonging to the family $\mathcal{R}$. ■



Now we prove the following theorem.

**Theorem 2** *For every tree $T$ on $n \geq 3$ vertices is $\gamma_w(T) = (n(T) - n_1(T) + 1)/2$ if and only if $T$ belongs to the family $\mathcal{R}$.*

**Proof.** Let $T$ be a tree belonging to the family $\mathcal{R}$ and let $D$ be a minimum weakly connected dominating set having property $\mathcal{F}$ in $T$. From Observation 1 and Observation 2 it follows that for every two vertices $u, v \in D$, the distance $d_T(u, v)$ is an even number and the distance between any two adjacent in $D$ vertices is exactly two. Any two vertices adjacent in $D$ are separated by exactly one vertex of degree two belonging to $V(T) - D$. Thus we have $n(T) = |D| + |V(T) - D| = |D| + n_1(T) + (|D| - 1)$, and hence $n(T) = n_1(T) + 2\gamma_w(T) - 1$, which finally gives $\gamma_w(T) = (n(T) - n_1(T) + 1)/2$.

Now we show that if $\gamma_w(T) = (n(T) - n_1(T) + 1)/2$, then $T$ belongs to the family $\mathcal{R}$. It suffices to show that $\gamma_w(T) > (n(T) - n_1(T) + 1)/2$ if $T$ does not belong to $\mathcal{R}$.

Let $T$ be a tree not belonging to the family $\mathcal{R}$. It is obvious, that $T$ is not a star, because every star belongs to $\mathcal{R}$. Thus $n \geq 4$ and for $n = 4$ $T$ is a path on four vertices and it is easy to check that $\gamma_w(T) > (n(T) - n_1(T) + 1)/2$. Assume that $T$ has at least five vertices and assume that $\gamma_w(T') > (n(T') - n_1(T') + 1)/2$ for each tree $T'$ not belonging to the family $\mathcal{R}$, where $n(T') \leq n(T) - 1$. Let $P = (v_0, v_1, v_2, \ldots, v_l)$ be a longest path in $T$ and let $D$ be a minimum weakly connected dominating set satisfying property $\mathcal{F}$ in $T$. Without loss of generality we may assume, that $P$ is chosen in such a way that $d_T(v_1)$ is as large as possible. We consider two cases:

$d_T(v_1) > 2$ and $d_T(v_1) = 2$.

*Case 1.* If $d_T(v_1) > 2$, then $T^{'} = T - \{v_0\}$ does not belong to $\mathcal{R}$ and $D$ is a minimum weakly connected dominating set of $T^{'}$. Thus we have $\gamma_w(T^{'}) = \gamma_w(T)$. By induction, $\gamma_w(T^{'}) > (n(T^{'}) + 1 - n_1(T^{'}))/2$ and therefore $\gamma_w(T) > (n(T) + 1 - n_1(T))/2$ as $n_1(T^{'}) = n_1(T) - 1, n(T^{'}) = n(T) - 1$.

*Case 2.* If $d_T(v_1) = 2$, then we consider $T^{''} = T - \{v_0, v_1\}$. It is easy to observe that $D - \{v_1\}$ is a minimum weakly connected dominating set of $T^{''}$. Thus we have $\gamma_w(T^{'}) = \gamma_w(T) - 1$.

If $d_T(v_2) = 2$, then $T^{''} \notin \mathcal{R}$ (in the other case $T$ would belong to the family $\mathcal{R}$). By induction $\gamma_w(T^{''}) > (n(T^{''}) - n_1(T^{''}) + 1)/2$ and hence $\gamma_w(T) > (n(T) - n_1(T) + 1)/2$ as $n_1(T^{''}) = n_1(T)$ and $n(T^{''}) = n(T) - 2$.

If $d_T(v_2) > 2$, then we have $n_1(T^{''}) = n_1(T) - 1, n(T^{''}) = n(T) - 2$ and $\gamma_w(T^{'}) = \gamma_w(T) - 1$. If $T \notin \mathcal{R}$, then $\gamma_w(T) - 1 > (n(T) - 2 - n_1(T) + 1 + 1)/2$, so $\gamma_w(T) > (n(T) - n_1(T) + 2)/2 > (n(T) - n_1(T) + 1)/2$.

If $T^{''} \in \mathcal{R}$, then $\gamma_w(T^{''}) = (n(T^{''}) - n_1(T^{''}) + 1)/2$. Since $T^{''} \in \mathcal{R}$, we have $d_T(v_2) > 2$ (in the other case $T$ would belong to $\mathcal{R}$). Thus $\gamma_w(T) - 1 = (n(T) - 2 - n_1(T) + 1 + 1)/2$, so $\gamma_w(T) = (n(T) - n_1(T) + 2)/2 > (n(T) - n_1(T) + 1)/2$. ■

Since every connected dominating set is a weakly connected dominating set and every weakly connected dominating set is a dominating set, we have the following inequality chain for every tree $T$: $\gamma(T) \leq \gamma_w(T) \leq \gamma_c(T)$. It is known [5] that $\gamma_c(T) = n(T) - n_1(T)$, so we also have an upper bound for $\gamma_w(T)$ in terms of $n(T)$ and $n_1(T)$. We can characterize all trees $T$ for which the equality $\gamma_w(T) = n(T) - n_1(T)$ holds.

**Theorem 3** *For a tree $T$ on at least three vertices $\gamma_w(T) = n(T) - n_1(T)$ if and only if every non-end vertex of $T$ is a support.*

**Proof.** Let $T$ be a tree of order $n \geq 3$ and let $\gamma_w(T) = n(T) - n_1(T)$. Let $D$ be a minimum weakly connected dominating set with property $\mathcal{F}$ in $T$. Since $D$ has property $\mathcal{F}$ and $\gamma_w(T) = n(T) - n_1(T)$, every non-end vertex of $T$ belongs to $D$. Suppose that in $T$ there exists a non-end vertex $u$ which is not a support. Then $D - \{u\}$ is a weakly connected dominating set of $T$, a contradiction.

Now assume that $T$ is a tree in which every non-end-vertex is a support and let $D$ be a minimum weakly connected dominating set with property $\mathcal{F}$ in $T$. Every support belongs to $D$ and no end vertex belongs to $D$, so $|D| = n(T) - n_1(T) = \gamma_w(T)$. ■

The bound $n - n_1$ is also an upper bound for the domination number $\gamma$. Favaron [2] proved that $\gamma(T) \leq (n(T) + n_1(T))/3$ for every tree $T$ on at least 3 vertices. For trees with a great number of end vertices, i.e. for every tree $T$ with $n_1(T) > (n(T))/2$ we have $(n(T) + n_1(T))/3 > n(T) - n_1(T)$, so the bound $n(T) - n_1(T)$ is better than $(n(T) + n_1(T))/3$.

## 3 Concluding remarks

From [5] and the above results it follows that $(n(T)+1-n_1(T))/2 \leq \gamma_w(T) \leq n(T)-n_1(T)$ for every tree $T$ on at least three vertices. The example of the caterpillar given in Fig. 2 showes that the difference between $\gamma_w(T)$ and $(n(T)+1-n_1(T))/2$ can be arbitrarily large. It is no problem to observe that $\gamma_w(T_l) - (n(T_l)+1-n_1(T_l))/2 = l/6$ for any integer $l \equiv 0 \pmod 3$.

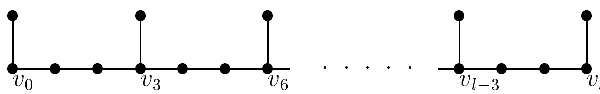


Figure 2: Caterpillar $T_l$

## References


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