

# MODELING OF THE FINITE AMPLITUDE PLANE WAVE PROPAGATION IN NON-DISSIPATIVE MEDIUM

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*The paper presents results of theoretical analysis of the finite amplitude plane wave propagation problem. The case of harmonic plane wave propagation in non-dissipative medium was considered. The mathematical model and some results of numerical investigations are presented. The mathematical model was built on the basis of one-dimensional continuity equation, equation of motion in differential form and state equation. The finite difference method was applied to solve the problem numerically. The pressure changes and harmonic pressure amplitude changes were analysed. The results of computer calculations were compared with solution of the Burgers equation.*

## INTRODUCTION

The mathematical model of finite amplitude wave propagation problem is described basing on continuity, motion and state equations. The system of these equations describes pressure, velocity and density changes in medium. They are usually converted to one nonlinear partial differential equation. The nonlinear equation of acoustics is one of them. It allows to analyse pressure changes along sound beam. However in practice equations which have easier form are used. For example, the KZK equation is very often used during theoretical studies. These equations have not analytical solutions till now and consequently it is necessary to solve them numerically. Theoretical analysis of the plane wave propagation is possible using the Westervelt equation and the Burgers equation [1, 2]. Existence of the Burgers equation solution together with knowledge of theoretical formulas of harmonic pressure amplitudes enable to verify correctness and to discuss accuracy of calculations.

The aim of the paper was to present mathematical and numerical models of the finite amplitude wave propagation problem in non-dissipative medium which are built on the basis of continuum mechanics equations. The convergence and accuracy of obtained discrete equations are discussed. Numerical calculations were carried out for different values of physical and numerical parameters.

## 1. MATHEMATICAL MODEL

To build the mathematical model of finite amplitude plane wave propagation problem the one-dimensional continuity equation and equation of motion in differential form are considered:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0 \quad (1)$$

$$\rho \frac{\partial v}{\partial t} + \rho v \frac{\partial v}{\partial x} + \frac{\partial p}{\partial x} = 0 \quad (2)$$

where  $p$  denotes pressure,  $\rho$  – density of the medium and  $v$  – velocity. To close the system state equation is added

$$\rho = \rho_0 + \frac{1}{c_0^2} p' - \frac{\varepsilon - 1}{c_0^4 \rho_0} p'^2 \quad (3)$$

where  $p' = p - p_0$  is acoustic pressure,  $\rho_0$  – medium density at rest,  $c_0$  – speed of sound,  $\varepsilon$  – nonlinear coefficient. Substituting equation (3) to equations (1) and (2) we obtain system of two first order partial differential equations:

$$\frac{\partial p'}{\partial t} = -v \frac{\partial p'}{\partial x} - \frac{\rho_0 + \frac{1}{c_0^2} p' - \frac{\varepsilon - 1}{c_0^4 \rho_0} p'^2}{\frac{1}{c_0^2} - \frac{\varepsilon - 1}{c_0^4 \rho_0} 2p'} \frac{\partial v}{\partial x} \quad (4)$$

$$\frac{\partial v}{\partial t} = -v \frac{\partial v}{\partial x} - \frac{1}{\rho_0 + \frac{1}{c_0^2} p' - \frac{\varepsilon - 1}{c_0^4 \rho_0} p'^2} \frac{\partial p'}{\partial x}$$

Assuming that harmonic plane wave propagates in water, the boundary condition can be written in following form:

$$\begin{aligned} p'(x = 0, t) &= p_0 \sin \omega t \\ v(x = 0, t) &= v_0 \sin \omega t \end{aligned} \quad (5)$$

where  $\omega = 2\pi f$ ,  $f$  - fundamental wave frequency. Moreover it is assumed that pressure  $p'$  and velocity  $v$  are periodic functions of the time coordinate.

The Burgers equation is often used to analyse finite amplitude plane wave propagation problem. If the acoustics Reynolds number is big ( $Re_a \gg 1$ ), the dissipative term can be omitted and then the Burgers equation is written in following form:

$$\frac{\partial p'}{\partial x} - \frac{\varepsilon}{\rho_0 c_0^3} p' \frac{\partial p'}{\partial \tau} = 0 \quad (6)$$

where  $\tau = t - x/c_0$ .

The waveform change is equivalent with spectrum change. The harmonic analysis is very often used to investigate wave distortion. The harmonic pressure amplitudes are defined in following form:

$$p_n(x) = \frac{2x_N}{nx} J_n \left( \frac{nx}{x_N} \right) \quad (7)$$

where  $x_N = \frac{\rho_0 c_0^3}{\varepsilon \omega p_0}$  and  $J_n$  is the Bessel function of the first kind of order  $n$ .

## 2. NUMERICAL SOLUTION

Let us define new normalized functions  $P = \frac{p'}{p_0}$  and  $V = \frac{\rho_0 c_0}{p_0} v$ . Substituting them into (4) we obtain system of two partial differential equations:

$$\begin{aligned} \frac{\partial P}{\partial t} &= -\frac{p_0}{c_0 \rho_0} V \frac{\partial P}{\partial x} - \frac{1 + \frac{p_0}{c_0^2 \rho_0} P - \frac{\varepsilon - 1}{c_0^4 \rho_0^2} p_0^2 P^2}{\frac{1}{c_0} - \frac{\varepsilon - 1}{c_0^3 \rho_0} 2 p_0 P} \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial t} &= -\frac{p_0}{c_0 \rho_0} V \frac{\partial V}{\partial x} - \frac{c_0}{1 + \frac{p_0}{c_0^2 \rho_0} P - \frac{\varepsilon - 1}{c_0^4 \rho_0^2} p_0^2 P^2} \frac{\partial P}{\partial x} \end{aligned} \quad (8)$$

To solve this system of equations the Lax method is used. Let we define nodal points as follows  $t_m = m\Delta t$ ,  $x_i = i\Delta x$ . After approximation of derivatives equations (8) may be written in following way:

$$\begin{aligned} \delta_t P_i^m &= -\frac{p_0}{c_0 \rho_0} V_i^m \delta_x P_i^m - \frac{A(P_i^m)}{B(P_i^m)} \delta_x V_i^m \\ \delta_t V_i^m &= -\frac{p_0}{c_0 \rho_0} V_i^m \delta_x V_i^m - \frac{c_0}{A(P_i^m)} \delta_x P_i^m \end{aligned} \quad (9)$$

where:  $P_i^m = P(x_i, t_m)$ ,  $V_i^m = V(x_i, t_m)$ ,

$\delta_t$  - difference operator of differential operator  $\partial / \partial t$

$\delta_x$  - difference operator of differential operator  $\partial / \partial x$

$$A(P_i^m) = 1 + \frac{p_0}{c_0^2 \rho_0} P_i^m - \frac{\varepsilon - 1}{c_0^4 \rho_0^2} p_0^2 (P_i^m)^2$$

$$B(P_i^m) = \frac{1}{c_0} - \frac{\varepsilon - 1}{c_0^3 \rho_0} 2 p_0 P_i^m$$



### 3. NUMERICAL INVESTIGATIONS

Solving the system of difference equations (9) we obtain pressure and velocity at the mesh points  $(x_i, t_m)$ . During finite amplitude wave propagation we observe shape changes step by step. Figure 1 presents normalized pressure as a function of time obtained for different distances. Numerical calculations were carried out assuming that harmonic wave which frequency  $f=1$  MHz and amplitude  $p_0=150$  kPa is propagated in water where medium density at rest  $\rho_0=1000$  kg/m<sup>3</sup>, speed of sound  $c_0=1500$  m/s, nonlinear coefficient  $\epsilon=3.5$ . Curve number 1 presents shape of primary wave curves 2 and 3 show pressure at distance  $x=0.3$  m and  $x=0.6$  m respectively. Similar results we obtain for velocity. Figure 2 presents four first harmonic pressure amplitudes along  $x$  axis calculated for the same values of physical and numerical parameters.

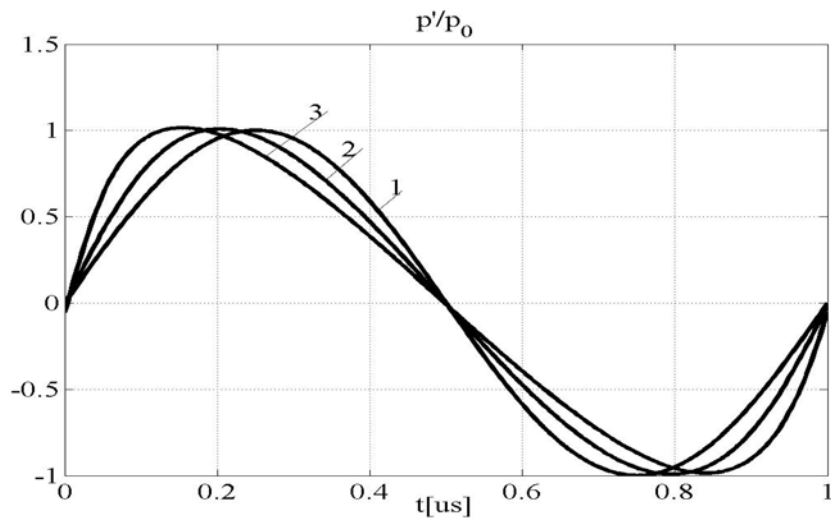


Fig.1 Normalized pressure as a function of time: 1-  $x=0$ , 2 -  $x=0.3$ m, 3 -  $x=0.6$  m

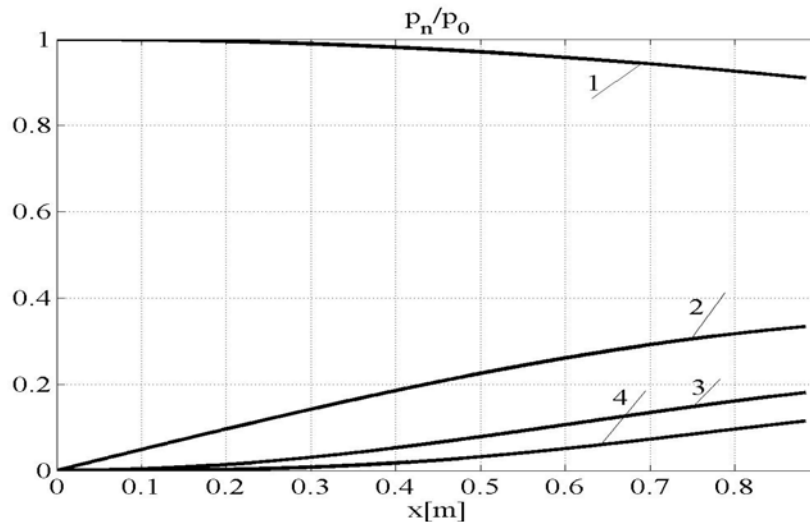


Fig.2 Normalized harmonic pressure amplitudes along  $x$  axis

The correct choice of the net spacing values is very important during numerical calculations. To analyse this problem in detail, the results of numerical calculations of pressure and their harmonic amplitudes were compared with solution of the Burgers equation.

The results of numerical calculations presented in Fig. 1 were carried out for  $\Delta t=0.0039 \mu\text{s}$ . Similar results of pressure changes obtained for  $\Delta t=0.0156 \mu\text{s}$  shows next figure. Note that in this example the time net spacing  $\Delta t$  and consequently distance one  $\Delta x$  were not small enough and for higher distances we observe numerical errors.

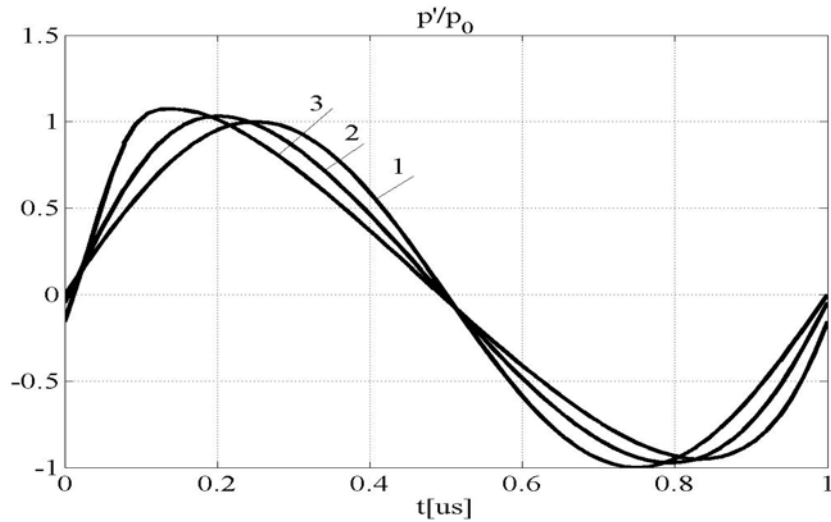


Fig.3 Normalized pressure as a function of time: 1-  $x=0$ , 2 -  $x=0.3\text{m}$ , 3-  $x=0.6 \text{ m}$

The correct choice of numerical parameters is not only one problem during numerical calculations. It is important to remember that proposed numerical approximation of the derivatives requires continuity of solution. Solution of our problem becomes non-continuous for distances higher then  $x_N$ . For using till now physical parameters, this distance is equal  $x_N=1 \text{ m}$ .

Figure 4 shows normalized pressure calculated for two different distances and two different values of time steps. Left figure presents results obtained for distance  $x=0.4 \text{ m}$ , right figure presents similar results obtained for distance  $x=0.8 \text{ m}$  respectively. Curve number 1 (dashed line) shows waveform obtained for  $\Delta t=0.0039 \mu\text{s}$ . Curve number 2 (solid line) was obtained for  $\Delta t=0.0156 \mu\text{s}$ .

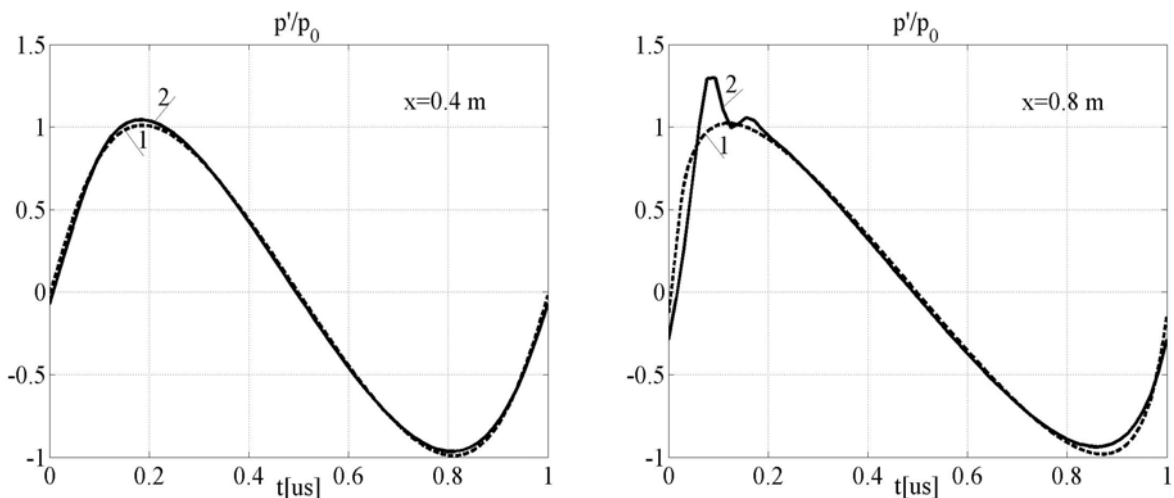


Fig.4 Normalized pressure as a function of time: 1 -  $\Delta t=0.0039 \mu\text{s}$ , 2 -  $\Delta t=0.0156 \mu\text{s}$

Figure 5 shows four first harmonic pressure amplitudes along  $x$  axis obtained for different net spacing  $\Delta t$ . Left figure presents first harmonic pressure amplitude calculated for  $\Delta t=0.0156 \mu\text{s}$  (solid line) and  $\Delta t=0.0039 \mu\text{s}$  (dashed line). Similar results obtained for second, third and fourth harmonic component are shown in right figure.

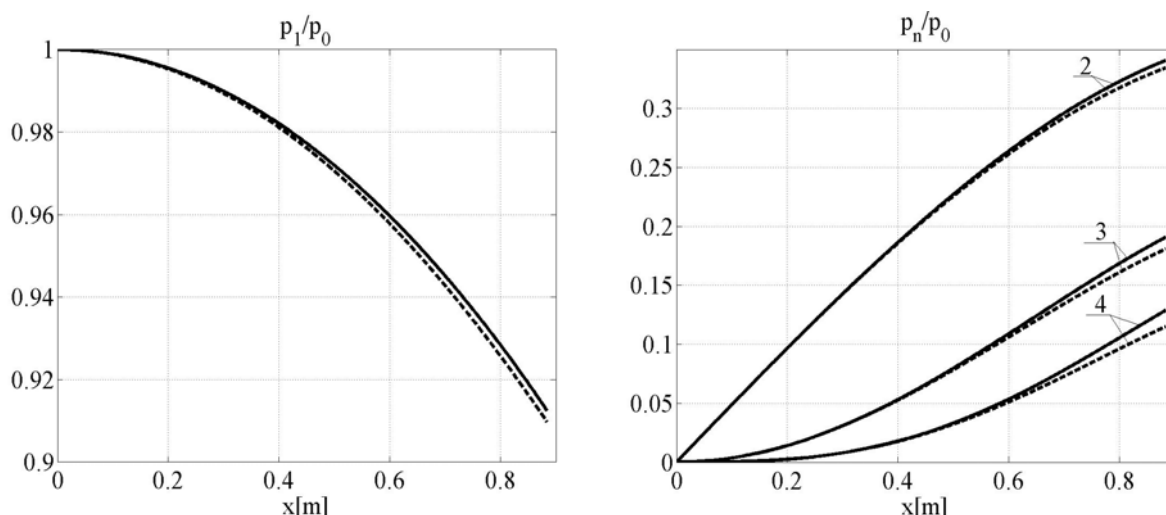


Fig.5 From first to fourth harmonic pressure amplitudes along  $x$  axis for different net spacing  $\Delta t$

To analyse correctness of the results obtained numerically, the results of calculations were compared with theoretical solutions. The absolute error  $E_n(x) = |p_n(x) - \tilde{p}_n(x)|$  and relative error  $R_n(x) = |p_n(x) - \tilde{p}_n(x)| / |p_n(x)|$  were analysed. Functions  $p_n$  and  $\tilde{p}_n$  denote harmonic pressure amplitude calculated using formula (7) and numerically respectively. Numerical calculations were carried out for different values of time net spacing. The calculations were done for distances to  $x=0.9$  m.

Tables 1 and 2 collect the maximum absolute errors and relative error of first five harmonic pressure amplitudes for fixed steps  $\Delta t$  respectively.

Table 1. Maximum absolute error of harmonic pressure amplitudes

	$E_1$	$E_2$	$E_3$	$E_4$	$E_5$
$\Delta t=0.0156 \mu\text{s}$	$2.95 \cdot 10^{-3}$	$6.63 \cdot 10^{-3}$	$1.05 \cdot 10^{-2}$	$1.44 \cdot 10^{-2}$	$1.85 \cdot 10^{-2}$
$\Delta t=0.0078 \mu\text{s}$	$7.25 \cdot 10^{-4}$	$1.54 \cdot 10^{-3}$	$2.31 \cdot 10^{-3}$	$2.98 \cdot 10^{-3}$	$3.59 \cdot 10^{-3}$
$\Delta t=0.0039 \mu\text{s}$	$1.82 \cdot 10^{-4}$	$3.83 \cdot 10^{-4}$	$5.66 \cdot 10^{-4}$	$7.20 \cdot 10^{-4}$	$8.49 \cdot 10^{-4}$

Table 2. Maximum relative error of harmonic pressure amplitudes

	$R_1$	$R_2$	$R_3$	$R_4$	$R_5$
$\Delta t=0.0156 \mu\text{s}$	$3.22 \cdot 10^{-3}$	$1.99 \cdot 10^{-2}$	$5.82 \cdot 10^{-2}$	$1.26 \cdot 10^{-1}$	$2.34 \cdot 10^{-1}$
$\Delta t=0.0078 \mu\text{s}$	$7.91 \cdot 10^{-4}$	$4.62 \cdot 10^{-3}$	$1.28 \cdot 10^{-2}$	$2.61 \cdot 10^{-2}$	$4.53 \cdot 10^{-2}$
$\Delta t=0.0039 \mu\text{s}$	$1.99 \cdot 10^{-4}$	$1.15 \cdot 10^{-3}$	$3.14 \cdot 10^{-3}$	$6.30 \cdot 10^{-3}$	$1.07 \cdot 10^{-2}$

Figure 6 presents relative errors of first harmonic pressure amplitude obtained for different values of time step  $\Delta t$ . Next figure shows the relative error for fixed harmonic pressure amplitudes calculated with step  $\Delta t=0.0039 \mu\text{s}$ . In this example the maximum relative error of waveform at distance  $x=0.8$  m is equal 0.033.

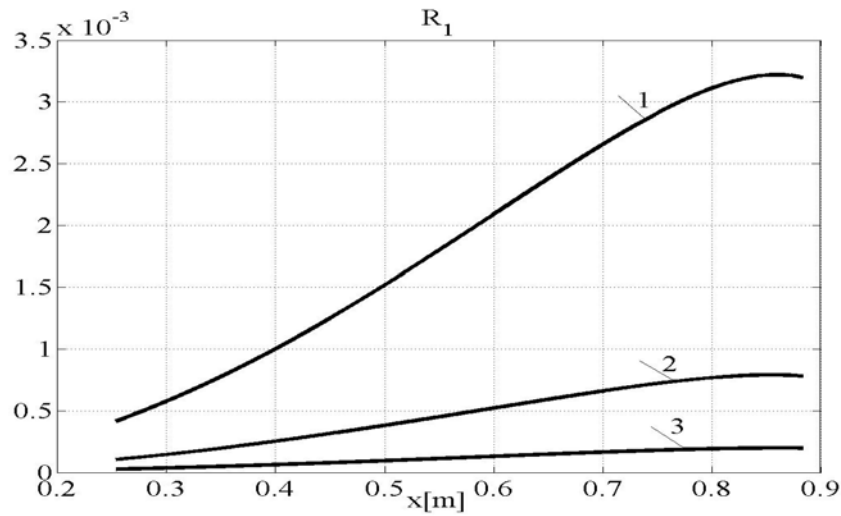


Fig.6 Relative error of first harmonic pressure amplitude:  
 1 -  $\Delta t=0.0156 \mu s$ , 2 -  $\Delta t=0.0078 \mu s$ , 3 -  $\Delta t=0.0039 \mu s$

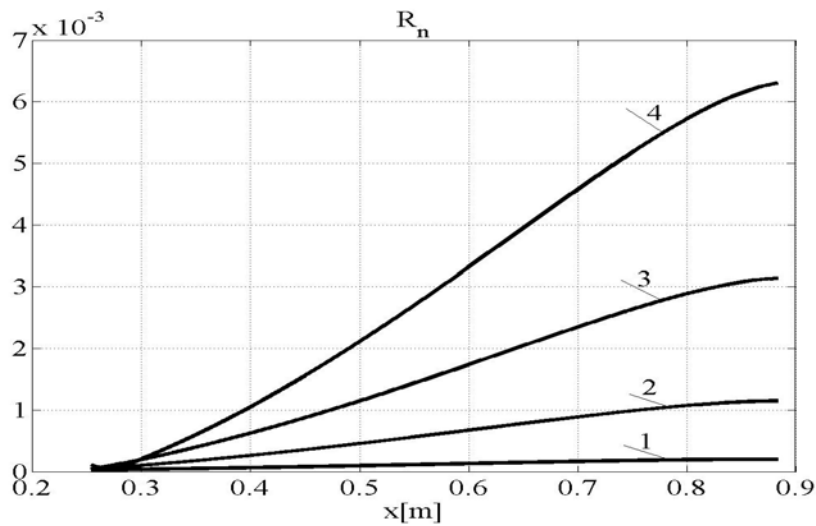


Fig.7 Relative error for different harmonic pressure amplitudes

To complete the discussion about accuracy, solution of equations (9) and solution of the Burgers equation obtained numerically were compared. The relative error of harmonic pressure amplitudes at distance  $x=0.8$  m obtained using presented in this paper numerical model when  $\Delta t=0.0039 \mu s$  are follows:

$$R_1=1.92 \cdot 10^{-4}, \quad R_2=1.07 \cdot 10^{-3}, \quad R_3=2.88 \cdot 10^{-3}, \quad R_4=5.71 \cdot 10^{-3}.$$

Solving numerically the Burgers equation we obtain:

$$R_1=3.01 \cdot 10^{-3}, \quad R_2=2.09 \cdot 10^{-2}, \quad R_3=1.19 \cdot 10^{-2}, \quad R_4=1.94 \cdot 10^{-2}.$$

The finite difference method with the same values of physical and numerical parameters then earlier was used to solve equation (6). These results show that convergence of calculations is faster for proposed model than for numerical solution of the Burgers equation.

Presented till now results of computer calculations were carried out with different values of numerical parameters for the same values of primary wave frequency and pressure, i.e. for  $f=1\text{MHz}$  and  $p_0=150\text{ kPa}$  respectively. Numerical investigations were done not only for different step sizes but also for different values of physical parameters. Figure 8 shows harmonic pressure amplitudes obtained for wave which frequency  $f=1\text{ MHz}$  and pressure  $p_0=300\text{ kPa}$ . The results of computations with pressure  $p_0=150\text{ kPa}$  and frequency  $f=0.5\text{ MHz}$  presents right figure. Note that now calculations were carried out for different distances than earlier. In these examples the discontinuity appears at distance  $x_N=0.5\text{ m}$  and  $x_N=2\text{ m}$  respectively and it is the reason why investigated distances are different.

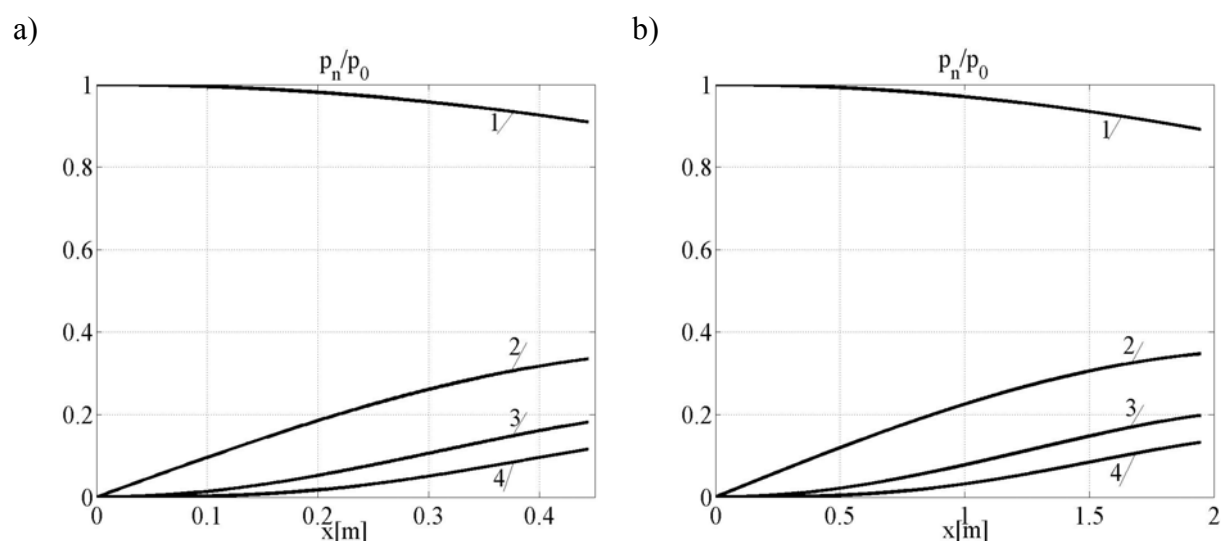


Fig.8 Normalized harmonic pressure amplitudes along  $x$  axis:  
a)  $f=1\text{ MHz}$ ,  $p_0=300\text{ kPa}$ , b)  $f=0.5\text{ MHz}$ ,  $p_0=150\text{ kPa}$

#### 4. CONCLUSIONS

The finite amplitude plane wave propagation problem was considered. The mathematical model, which was worked out on the basis of mechanics equations and some results of numerical investigations have been presented. The numerical calculations were carried out using own computer program that was worked out on the basis of obtained mathematical model. The results of numerical calculations confirm that proposed numerical model is convergent. Moreover this method allows to obtain the same accuracy using higher step sizes than solving numerically the Burgers equation. However it is necessary to remember that it can be used only for continuous solutions of the problem. The presented in this paper mathematical and numerical models can be used to analyse the wave propagation in non-dissipative medium or for small distances when dissipation effects are not very big. When they cannot be omitted this model must be extended. However in this situation we obtain different type of partial differential equations and it is necessary to build more complicated difference equations.

#### REFERENCES

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