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Flood Routing by the Non-Linear Muskingum Model: Conservation of Mass and Momentum

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Abstract

In this paper, the conservative properties of the Muskingum equation, commonly applied to solve river flood routing, are analysed. The aim of this analysis is to explain the causes of the mass balance error, which is observed in the numerical solutions of its non-linear form. The linear Muskingum model has been considered as a semi-discrete form of the kinematic wave equation and therefore it was possible to derive its two non-linear forms. Both forms were derived directly from the kinematic wave equation. It appeared, that depending on the assumed conservative form of the Muskingum equation, this model satisfies either the global mass conservation law or the global momentum conservation law. Both laws are satisfied simultaneously by the linear equation only. The mass balance error can be eliminated from the numerical solution on condition that the non-linear Muskingum equation is written in the proper conservative form.

Key words: flood routing, non-linear Muskingum equation, mass and momentum balance, conservative form

1. Introduction

Unsteady open-channel flow equations describe the principles of mass and momentum conservation. These equations can be expressed in differential or integral form and their solution should be consistent with conservation laws. Indeed, the integral form of equation guarantees the conservation of fundamental principles. However, in the case of non-linear differential equations, the results of numerical solutions show that this condition can be not satisfied if the equations are written in improper form. As a consequence, mass and momentum balance errors may be observed. Non-linear differential equations can be used in a variety of forms – conservative or non-conservative, and with different dependent variables. Various aspects of the conservative and non-conservative forms of non-linear transport equations are presented comprehensively by Gresho and Sani (1998). The adequate conservative form of differential equation and numerical algorithm lead to the solution without balance errors, whereas solution of the same equation written in the non-conservative form (or in an inappropriate conservative form) is usually inaccurate. This effect is especially significant if discontinuities, such as shock waves, are present in the solution (Toro 1997, LeVeque 2002, Lai et al 2002).

To solve the flood routing problem, the system of Saint-Venant equations, which comprises continuity and dynamic equations, is usually used. This system can be written in the following conservative form (Cunge et al 1980):

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0, \tag{1}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + gA\frac{\partial h}{\partial x} - gAs + gAS = 0, \tag{2}$$

where:

- t – time,
- x – longitudinal coordinate,
- h – flow depth,
- Q – flow discharge,
- A – cross-sectional area of flow,
- g – gravitational acceleration,
- s – channel bottom slope,
- S – slope of energy line.

The friction slope S is usually expressed using the Manning formula:

$$S = \frac{n^2|Q|Q}{R^{4/3}A^2}, \tag{3}$$

where:

- n – Manning roughness coefficient,
- R – hydraulics radius.

For practical reasons, hydrologists are still interested in using simplified flood routing models, such as the kinematic and diffusive ones derived from Eqs. (1, 2). Both models can be presented in common general formula taking the form of an advection-diffusion transport equation (Chow et al 1988):

$$\frac{\partial Q}{\partial t} + C\,(Q)\,\frac{\partial Q}{\partial x} - \nu\,(Q)\,\frac{\partial^2 Q}{\partial x^2} = 0, \tag{4}$$

where:

- $C(Q)$ – advective velocity,
- $\nu(Q)$ – hydraulic diffusivity.

For a wide, rectangular channel with a slope in the energy line equal to the slope of the channel bottom, the kinematic wave speed and hydraulic diffusivity can be expressed as follows:

$$C = \frac{1}{m\alpha Q^{m-1}} = \frac{1}{m}U, \tag{5}$$

$$\nu = \frac{Q}{2Bs}, \tag{6}$$

with

$$\alpha = \left(\frac{np^{2/3}}{s^{1/2}}\right)^m, \tag{7}$$

where:

- m – kinematic wave ratio (= 3/5 for the Manning law friction),
- p – wetted perimeter,
- U – cross-sectional average flow velocity,
- B – channel width at water surface.

Eq. (4) with $\nu = 0$ becomes a kinematic wave model. The theory of kinematic and diffusive waves is well known and widely disseminated.

Flood routing is very often carried out using the hydrological lumped model. They are derived from the storage equation, which is obtained by integration of the continuity equation (Eq. (1)) with respect to x:

$$\frac{dV}{dt} = Q_{j-1} - Q_j, \tag{8}$$

where:

- V – storage of the channel reach of length Δx,
- Q_{j-1} – inflow,
- Q_j – outflow,
- j – index of cross-section.

An additional formula relating storage, inflow and outflow (Chow et al 1988):

$$V = K\left[XQ_{j-1} + (1-X)\,Q_j\right], \tag{9}$$

introduced into Eq. (8) leads to the Muskingum model:

$$X\frac{dQ_{j-1}}{dt} + (1-X)\frac{dQ_j}{dt} = \frac{1}{K}\left(Q_{j-1} - Q_j\right), \tag{10}$$

where X and K are empirical constants to be found by trial and error for a given reach (Miller and Cunge 1975). Eq. (10) is obtained with the assumption that both afore-mentioned parameters are constant. For $X = 0$ Eq. (10) becomes the linear reservoir model.

The Muskingum model is commonly used in hydrological applications for its simplicity. Although this model has been known for tens years, its properties are not exactly recognized. For example, there are some difficulties involved in interpreting both the K and X parameters from a physical point of view. Consequently, it seems that Eq. (9) does not represent any physical rule. It is probably for this reason that some undertaken attempts to improve the Muskingum model have appeared unsuccessful. Since the linear model represents a significant simplification of the real flood propagation process, some authors have proposed its refinement by introducing variable parameters. Such an approach was applied, for instance, by Ponce and Yevjevich (1978), Ponce and Chaganti (1994) and Tang et al (1999a, 1999b). Unfortunately, when the variable parameters $K = K(Q)$ and $X = X(Q)$ were taken into account, considerable mass balance errors occurred. Similar results were obtained when a non-linear equation relating storage, inflow and outflow was used (Tung (1984) and Mohan (1997)). In contrast, the computations carried out for constant parameters show that the Muskingum model perfectly satisfies the mass balance. This fact suggests that the mass balance errors are related to the form (conservative or non-conservative) in which the Muskingum non-linear equation is written. Therefore, the attempts to improve the Muskingum model by introducing non-linearity, without an analysis of its conservation properties, result in failure.

2. Kinematic Wave Model vs Muskingum Equation

The kinematic character of the Muskingum model, as well as the numerical nature of the wave attenuation process in its solution was discovered by Cunge (1969). While approximating the linear kinematic wave equation through the difference box scheme and the Muskingum model, employing the implicit trapezoidal rule, Cunge noticed their similarity on condition that:

$$K = \frac{\Delta x}{C}, \tag{11}$$

which means that in this case, K represents time taken by the wave to travel, with kinematic celerity C, from cross-section j–1 to j. The accuracy analysis carried out for the applied approximation of the kinematic wave equation showed that it modifies the advection equation to an advection-diffusion one, similar to Eq. (4):

$$\frac{\partial Q}{\partial t} + C\frac{\partial Q}{\partial x} - \nu_n \frac{\partial^2 Q}{\partial x^2} = 0, \tag{12}$$

where ν_n is the coefficient of numerical diffusion, defined as follows:

$$\nu_n = \left(\frac{1}{2} - X\right) C \cdot \Delta x. \tag{13}$$

Cunge (1969) suggested accepting such a value of the parameter X, as to ensure a numerical diffusivity (Eq. (13)) equal to the hydraulic one given by Eq. (6), i.e. $\nu = \nu_n$. This condition is satisfied for

$$X = \frac{1}{2} - \frac{Q}{2B \cdot s \cdot C \cdot \Delta x}. \tag{14}$$

Consequently, the Muskingum model can reproduce a solution of the linear diffusive wave equation. This version of the Muskingum model is known as the Muskingum-Cunge version (Chow et al 1988). Thus Cunge's approach related the Muskingum model with a diffusive wave one in a particular way. Namely, the proposed formulae for K and X ensure equivalent numerical results from both models. Simply, instead of solving Eq. (4), one can solve Eq. (10) with the values of K and X defined by Cunge (1969). Sometimes this fact is a basis to consider the Muskingum model as an approximation of the diffusive wave. This is true in the sense of action of both models. However, if the term "approximation" is understood in the mathematical sense, the Muskingum equation should be regarded rather as an approximation of the kinematic wave equation. This fact, resulting from the consistency condition, is discussed below.

Taking into account Cunge's experiences, the supposition that the Muskingum model should be regarded as a semi-discrete form of the kinematic wave equation seems to be well founded. In order to prove this statement, let us consider the kinematic wave equation, i.e. Eq. (4) with $\nu = 0$ and $C = \text{const}$. The considered channel reach of length L is divided by N+1 nodes into N space intervals of length Δx. Approximation of the spatial derivative, carried out at point P, located between nodes j–1 and j (Fig. 1), gives:

$$\frac{dQ_p}{dt} + C\frac{Q_j - Q_{j-1}}{\Delta x} = 0, \tag{15}$$

where Q_p represents the discharge at point P.

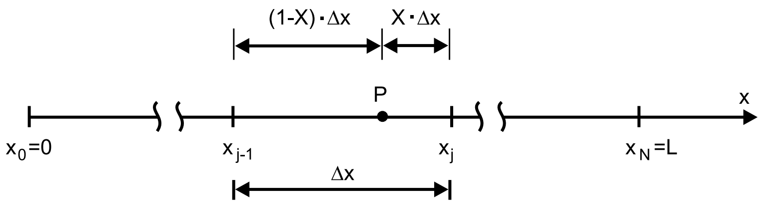

Fig. 1. The discretization along x-axis applied to the kinematic wave equation

Let us assume that Q_P is calculated by linear interpolation between the nodes j–1 and j:

$$Q_p = XQ_{j-1} + (1 - X)\,Q_j, \tag{16}$$

where X is the weighting parameter, which ranges from 0 to 1. This is defined as follows:

$$X = \frac{x_j - x}{x_j - x_{j-1}} \quad \text{for} \quad x_{j-1} \leq x \leq x_j. \tag{17}$$

Substituting Eq. (16) into Eq. (15) and taking into account Eq. (11), one obtains the Muskingum model in the form of Eq. (10). Note that in the proposed approach, the parameter X is interpreted clearly: it has numerical sense. As a weighting parameter, it determines the accuracy of applied spatial approximation of the kinematic wave equation.

Spatial discretization introduces numerical error caused by the truncation of the Taylor series. This error can be estimated directly from Eq. (10) by an analysis of accuracy using the modified equation approach (Fletcher 1991). To this order, the nodal values of Q and dQ/dt in Eq. (10) are replaced by a Taylor series expansion around point P (Fig. 1) including the terms of 2nd order. Finally, one obtains the following modified equation (Szymkiewicz 2002):

$$\frac{\partial Q}{\partial t} + \frac{\Delta x}{K}\frac{\partial Q}{\partial x} - \left(\frac{1}{2} - X\right)\frac{\Delta x^2}{K}\frac{\partial^2 Q}{\partial x^2} = 0. \tag{18}$$

According to the condition of consistency, the modified equation must tend to the governing one if space interval length tends to zero. Note that for $\Delta x \rightarrow 0$ Eq. (18) tends to the kinematic wave equation (Szymkiewicz 2002). This fact proves explicitly that the Muskingum model is an approximation of the kinematic wave.

However, this approximation introduces an error, which is observed as an artificial flood wave's attenuation. This process, controlled by the term representing so-called numerical diffusion, depends on the following coefficient:

$$\nu_n = \left(\frac{1}{2} - X\right)\frac{\Delta x^2}{K}. \tag{19}$$

This expression coincides with Eq. (13) proposed by Cunge (1969). It should be remembered that the numerical diffusion is caused by the spatial approximation only. An additional diffusion can be generated while integrating Eq. (10) over time, by a method of 1st order of accuracy. Usually the implicit trapezoidal rule is applied, which, ensure an accuracy of 2nd order with regard to t, is dissipation free.

In summary, one can say that in fact the numerical solution of the Muskingum model is the numerical solution of the kinematic wave model through a method of lines. In this approach, a solution of the partial differential equation is realized in two stages (Fletcher 1991). Firstly, by spatial discretization, the partial differential equation is reduced to a system of ordinary differential equations in time. Next, this system is integrated using any method for the numerical solution of an initial value problem, for ordinary differential equations.

Since the Muskingum model appeared to be the kinematic wave equation expressed in a particular semi-discrete form, it seems that this model can be derived directly from continuity and simplified momentum equations as in the case of the kinematic wave. This way of derivation, without using the storage equation completed by an additional formula relating storage, inflow and outflow, can be regarded as an alternative one.

Approximation of Eq. (1) with regard to x yields:

$$\frac{dA_p}{dt} + \frac{Q_j - Q_{j-1}}{\Delta x} = 0 \quad (j = 1, 2, ..., N), \tag{20}$$

where A_p, being a cross-sectional area at the point P (Fig. 1), can be expressed as a function of Q_p using the Manning (or Chézy) formula written in the following form:

$$A_p = \alpha (Q_p)^m. \tag{21}$$

Introducing of Eq. (21) into Eq. (20) leads to the following equation:

$$\frac{d\left[\alpha \left(Q_p\right)^m\right]}{dt} = \frac{Q_{j-1} - Q_j}{\Delta x} \quad (j = 1, 2, ..., N). \tag{22}$$

After differentiating with α = const, one obtains:

$$\alpha m (Q_p)^{m-1} \frac{dQ_p}{dt} = \frac{Q_{j-1} - Q_j}{\Delta x}. \tag{23}$$

According to Eq. (5), the kinematic wave celerity at point P is as follows:

$$C_p = \frac{1}{\alpha m (Q_p)^{m-1}}. \tag{24}$$

Assuming C = const, which implies that K = const, and taking into account the relations (11) and (16), Eq. (22) takes the well known form of the Muskingum model (Eq. (10)):

$$X \cdot \frac{dQ_{j-1}}{dt} + (1 - X) \cdot \frac{dQ_j}{dt} = \frac{1}{K}\left(Q_{j-1} - Q_j\right). \tag{25}$$

Let us remember that the kinematic wave model is based on the same governing equation, i.e. on the equation of continuity, as the above derived Eq. (25). Moreover, while deriving both models the same assumptions are applied.

The kinematic origin of the Muskingum equation presented above is confirmed by well known results of the numerical solution. First of all, it is known that the Muskingum equation, integrated in time by the method which does not produce additional numerical diffusion, ensures pure translation of the flood wave for $X = 1/2$. As results from Eq. (19), in this case the numerical diffusion, generated by approximation of the spatial derivative, disappears. For $X = 1/2$, this approximation coincides with the centred difference representing 2nd order of accuracy. Consequently, any attenuation of the flood wave calculated at the downstream end is not observed. Secondly, for some set of the values of K, X, and Δt, the Muskingum equation can produce numerical effects in the form of unphysical oscillations of the hydrograph at the downstream end. Similar results can be obtained while numerically solving the kinematic wave model by the dissipation free method. These effects are connected to the numerical solution of the hyperbolic equation using dispersive methods (Fletcher 1991).

If the kinematic wave equation coincides with the Muskingum one, it is important to explain the meaning of the additional relationship (9) used while deriving the Muskingum model in a standard way, i.e. from the storage equation (8). An interpretation of the formula (9) can be carried out using the previously assumed definition of the weighting parameter X and the steady uniform flow equation.

The aim of introducing the relation (9) was to eliminate storage V from Eq. (8). To obtain the same effect one can calculate the storage approximately, as follows:

$$V \approx \Delta x \cdot A_p. \tag{26}$$

Now, let us introduce the Manning formula written in the form of Eq. (21). This yields:

$$V = \Delta x \cdot \alpha \cdot \left(Q_p\right)^m = \Delta x \cdot \alpha \cdot \left(Q_p\right)^{m-1} \cdot Q_p. \tag{27}$$

According to Eq. (5) or to the continuity equation $Q = A \cdot U$, the mean flow velocity at point P is equal:

$$U_P = \frac{1}{\alpha \left(Q_P\right)^{m-1}}. \tag{28}$$

If we assume a constant flow velocity over the considered channel reach, $U_P =$ const, and we introduce a new constant parameter:

$$K_P = \frac{\Delta x}{U_P}, \tag{29}$$

then with Q_P (the discharge at point P – Fig. 1) expressed using the linear interpolation formula (16), Eq. (27) will take the form:

$$V = K_P \cdot Q_P = K_P \left[X Q_{j-1} + (1 - X) Q_j \right]. \tag{30}$$

Thus, the additional formula relating storage, inflow and outflow is obtained. Note that it was derived using the steady uniform flow equation and by applying a numerical estimation of storage. As a result, this relation should be regarded as having mixed sense: physical and numerical. This feature is reflected by the parameters K_P and X. The first one has physical interpretation, whereas the second – numerical.

Comparing Eq. (29) with Eq. (11), as proposed by Cunge (1969), one may notice a difference. Cunge defines the parameter K as time of the flood wave travelling from the cross-sections j–1 to j with kinematic wave speed C, whereas Eq. (29) defines K as time of a particle of water travelling with the average flow velocity U. One can add that this difference has no essential meaning, since in hydrological practice the proper value of parameter K should be fitted for each case study considered.

3. Mass and Momentum Balance for Non-Conservative and Conservative Forms of the Non-Linear Muskingum Equation

Let us consider a channel reach of length L in which the unsteady flow described by Eqs. (1) and (2) takes place during the time interval T. It is well known that these equations represent mass and momentum conservation principles respectively. To obtain the global conservation laws, these equations are integrated over the domain of solution, i.e. for $0 \le x \le L$ and $0 \le t \le T$. Assuming that:

- water density is constant,
- the time interval T is sufficiently long, and after passing of the flood wave the discharge returns to its initial value,

the mass balance takes the following form:

$$\int_0^T [Q(0,t) - Q(L,t)]\, dt = 0. \tag{31}$$

Introducing:

$$m_0 = \int_0^T Q\,(0,t)\, dt, \tag{32a}$$

$$m_L = \int_0^T Q(L,t)\,dt, \tag{32b}$$

the relative mass balance error is defined as follows:

$$\Delta E_m = \frac{m_L - m_0}{m_0} \cdot 100\%, \tag{33}$$

where:

ΔE_m – the mass balance error expressed in %,
$m_0,\ m_L$ – total volume of inflow and outflow respectively within the time period of T.

Global momentum balance can be carried out similarly. The total variation of momentum over a channel reach of length L, in time interval T, must be equal to the difference of momentum transported at upstream end $x = 0$ and downstream end $x = L$ respectively. If constant water density is assumed, the losses caused by friction, gravitation and pressure are neglected and the time interval T is long enough, this statement is expressed as follows:

$$\int_0^T \left(\frac{Q^2(0,t)}{A(0,t)} - \frac{Q^2(L,t)}{A(L,t)} \right) dt = 0. \tag{34}$$

Introducing the formulae:

$$M_0 = \int_0^T U(0,t)\,Q(0,t)\,dt = \int_0^T \frac{Q^2(0,t)}{A(0,t)} dt, \tag{35a}$$

$$M_L = \int_0^T U(L,t)\,Q(L,t)\,dt = \int_0^T \frac{Q^2(L,t)}{A(L,t)} dt, \tag{35b}$$

representing, divided by water density, the total momentum of inflow and outflow within the time period of T respectively, the relative momentum balance error can be calculated as follows:

$$\Delta E_M = \frac{M_L - M_0}{M_0} \cdot 100\%, \tag{36}$$

where: ΔE_M – the momentum balance error expressed in %.

Eq. (34) and Eqs. (35a, b) hold when both functions $Q(x, t)$ and $A(x, t)$ are known, as in Eqs. (1) and (2). In the case of the Muskingum equation, the momentum balance can be rearranged using the Manning formula expressed by Eq. (21). Therefore Eq. (34) can be rewritten as follows:

$$\int_0^T \left(Q^{2-m}(0,t) - Q^{2-m}(L,t)\right) dt = 0. \tag{37}$$

Note that for the linear kinematic wave model with $U(x, t) = Q/A = \text{const}$, the momentum balance expressed by Eq. (34) takes the form of Eq. (31), which represents the mass balance.

It is interesting to know, which law does the Muskingum equation represent? It appears that this equation can represent both laws depending on its form. This problem is discussed below.

As shown previously, the linear Muskingum equation (10) can be considered as a semi-discrete form of a linear kinematic wave equation. Consequently, it was derived directly from this equation. This way of derivation is very useful, since it allows us to derive easily other forms of the Muskingum equation. To do this, let us reconsider the non-linear kinematic wave equation (Eq. (4) with $\nu = 0$), in which, instead of constant celerity C a variable one $C(Q)$ is taken into account. Then this equation becomes,

$$\frac{\partial Q}{\partial t} + \frac{1}{\alpha \cdot m} \cdot Q^{1-m} \frac{\partial Q}{\partial x} = 0. \tag{38}$$

Semi-discretization of Eq. (38) yields:

$$X \cdot \frac{dQ_{j-1}}{dt} + (1-X) \cdot \frac{dQ_j}{dt} + \frac{\left(XQ_{j-1} + (1-X)\,Q_j\right)^{1-m}}{\Delta x \cdot \alpha \cdot m} \cdot \left(Q_j - Q_{j-1}\right) = 0. \tag{39}$$

The above expression is the non-linear Muskingum equation written in the non-conservative form.

Eq. (38) can be converted into the equivalent one:

$$\frac{\partial Q}{\partial t} + \frac{1}{\alpha \cdot m \cdot (2-m)} \cdot \frac{\partial Q^{2-m}}{\partial x} = 0. \tag{40}$$

Its semi-discretization yields the first conservative form of the non-linear Muskingum equation:

$$X \cdot \frac{dQ_{j-1}}{dt} + (1-X) \cdot \frac{dQ_j}{dt} + \frac{1}{\Delta x \cdot \alpha \cdot m \cdot (2-m)} \cdot \left(Q_j^{2-m} - Q_{j-1}^{2-m}\right) = 0. \tag{41}$$

Eq. (40) is not the only conservative form of the kinematic wave equation. Another one can be derived directly from Eqs. (1) and (21). Combining these equations with the previously accepted assumption that α = const, one obtains:

$$\frac{\partial Q^m}{\partial t} + \frac{1}{\alpha}\frac{\partial Q}{\partial x} = 0. \tag{42}$$

After semi-discretization with respect to x, Eq. (42) yields the second conservative form of the Muskingum model:

$$\frac{d}{dt}\left[X \cdot Q_{j-1} + (1-X)Q_j\right]^m + \frac{1}{\Delta x \cdot \alpha}\left(Q_j - Q_{j-1}\right) = 0. \tag{43}$$

Having three different forms of the non-linear Muskingum equation, one can compare their properties. First of all we should explain which conservative quantity is preserved by them. Since all equations were derived using the continuity and momentum equations for unsteady open channel flow, it is reasonable to expect that both conservation laws should be satisfied. To answer these questions, Eq. (39), Eq. (41) and Eq. (43) must be integrated over the solution domain. In this case the integration over a channel reach is not needed since the Muskingum model is actually the kinematic wave integrated in space – over an interval of length equal to Δx.

Let us integrate the non-linear Muskingum equation, written for one interval – reservoir, in the non-conservative form (Eq. (39) over time interval $\langle 0, T\rangle$:

$$\begin{aligned}&\int_0^T \left(X\frac{dQ_{j-1}}{dt} + (1-X)\,\frac{dQ_j}{dt}\right)dt = \\ &= \frac{1}{\Delta x \cdot \alpha \cdot m}\int_0^T \left[XQ_{j-1} + (1-X)\,Q_j\right]^{1-m}\left(Q_{j.-1} - Q_j\right)dt.\end{aligned} \tag{44}$$

Since the Muskingum model is applied for a cascade of Nreservoirs bounded by cross-section j–1 and j having length of Δx, the outflow from the preceding reservoir is the inflow to the next one. Finally, if the time interval T is long enough, the following integral equation is obtained:

$$\begin{aligned}&\int_0^T \left[(X \cdot Q_0 + (1-X)Q_1)^{1-m}Q_0 - (X \cdot Q_{N-1} + (1-X)\cdot Q_N)^{1-m}\cdot Q_N\right]dt = \\ &= \int_0^T R_M \cdot dt,\end{aligned} \tag{45}$$

in which

$$R_M = \sum_{j=1}^{N-1} \left[\left(X \cdot Q_j + (1 - X)\, Q_{j+1} \right)^{1-m} Q_j - \left(X \cdot Q_j + (1 - X)\, Q_{j+1} \right)^{1-m} Q_{j+1} \right], \quad (46)$$

where:

- N – total number of reservoirs,
- Q_0 – discharge at upstream end,
- Q_N – discharge at downstream end.

The term R_M, which appeared in Eq. (45), results from the internal fluxes between subsequent intervals – reservoirs. They are not equal with one another and, consequently, the total flux at the internal nodes cannot be cancelled.

Eq. (45) shows, that the Muskingum model written in non-conservative form (39) preserves neither the mass conservation nor the momentum conservation law. This fact explains the mass balance error reported by Ponce and Yevjevich (1978) and Tang et al (1999a, b), which was noticed after introducing variable parameters X and K into the linear Muskingum equation. Such an approach cannot be successful because it leads to a non-linear equation written in the non-conservative form. Consequently, an extra term R_M is generated during integration.

In the same way, the global conservation laws for the conservative forms of the non-linear Muskingum equation can be derived. For Eq. (41) one obtains:

$$\int_0^T \left[(Q_0\,(t))^{2-m} - (Q_N\,(t))^{2-m} \right] dt = 0, \quad (47)$$

whereas for Eq. (43) the global conservation law is as follows:

$$\int_0^T (Q_0\,(t) - Q_N\,(t))\, dt = 0. \quad (48)$$

Since the functions $Q_0(t)$ and $Q_N(t)$ denote the hydrographs in the cross-sections $x = 0$ and $x = L$ respectively, Eq. (47) coincides with Eq. (37) being the momentum conservation principle. Therefore Eq. (41) represents the momentum conservation principle, and it cannot preserve mass conservation, whereas Eq. (43) represents the mass conservation principle since Eq. (48) coincides with Eq. (31). In this case, the total volume of the flood wave inflowing by the upstream end ($x = 0$) should be equal to the total volume of water outflowing by the downstream end ($x = L$). Of course, the linear Muskingum model in form of Eq. (25) represents both mass and momentum conservation principles. The conservative properties of the Muskingum model are summarized in Table 1.

Table 1. Conservative properties of the Muskingum equation

Form of equation	Conservation of mass	Conservation of momentum
$\frac{dQ_p}{dt} + \frac{1}{K} \cdot \left(Q_j - Q_{j-1}\right) = 0 \quad (K = \text{const})$ for $j = 1, 2, \ldots, N$	+	+
$\frac{dQ_p}{dt} + \frac{Q_p^{1-m}}{\Delta x \cdot \alpha \cdot m} \cdot \left(Q_j - Q_{j-1}\right) = 0$ for $j = 1, 2, ..., N$	–	–
$\frac{dQ_p}{dt} + \frac{1}{\Delta x \cdot \alpha \cdot m \cdot (2-m)} \cdot \left(Q_j^{2-m} - Q_{j-1}^{2-m}\right) = 0$ for $j = 1, 2, ..., N$	–	+
$\Delta x \cdot \alpha \frac{dQ_p^m}{dt} + \left(Q_j - Q_{j-1}\right) = 0$ for $j = 1, 2, ..., N$	+	–

In all equations displayed in Table 1, the variable Q_P is interpreted as in Eq. (16).

To check how the mass and momentum are preserved by the presented forms of the Muskingum equation, the flow in a rectangular channel of width $B = 50$ m and of length 100 km is considered. Its bottom slope is $s = 0.0005$, whereas the Manning's coefficient is $n = 0.025$ m$^{-1/3}$s. The initial condition corresponds to a uniform steady flow for discharge q_0 with normal depth h_n. At the upstream end the following hydrograph is imposed:

$$q(t) = q_0 + (q_{\max} - q_0)\left(\frac{t}{t_{\max}}\right)^2 \exp\left(1 - \left(\frac{t}{t_{\max}}\right)^2\right), \tag{49}$$

where:

q_0 – baseflow discharge of the inflow,
$q_{\max}$ – peak discharge of the inflow,
$t_{\max}$ – time of the peak flow.

All forms of the Muskingum equation were applied for the same flood wave described by Eq. (49) with $q_0 = 100$ m^3/s, $q_{\max} = 1000$ m^3/s, $t_{\max} = 2.5$ h and $T = 21$ h. Numerical integration was carried out using the trapezoidal implicit method with $N = 10$ and $\Delta x = 5000$ m. The mass and momentum balance errors were calculated for constant and variable weighting parameters K and X.

As could be expected, the linear Muskingum equation (with K = const and X = const), perfectly satisfies both mass and momentum conservation laws. The calculated mass balance errors ΔE_m and ΔE_M were practically always equal to zero, regardless of the assumed values of both parameters. However, the results obtained for the non-linear Muskingum equation, i.e. with $K = K(Q)$ and $X = X(Q)$, depend on the applied conservative or non-conservative form. These are shown in Table 2.

Table 2. Mass and momentum balance errors for the non-linear Muskingum-Cunge model

Equation	Mass balance error ΔE_m [%]	Momentum balance error ΔE_M [%]
Eq. (39)	9.2	20.8
Eq. (41)	–5.8	0.0
Eq. (43)	0.0	8.2

One can notice that the non-linear Muskingum equation in non-conservative form (Eq. (39)) satisfies neither the mass conservation law nor the momentum conservation one. The calculated errors are equal to $\Delta E_m = 9.2\%$ and $\Delta E_M = 20.8\%$ respectively, whereas in the case of Eq. (43) the mass conservation principle is satisfied perfectly. The error ΔE_m is always equal to zero. However, at the same time the momentum balance suffers. For the second conservative form (Eq. (41)) of the Muskingum model one can observe the opposite situation. The solution of Eq. (41) generates a mass balance error while the momentum conservation law is satisfied perfectly. The results given by various forms of the Muskingum equation coincide completely with those given by the kinematic wave model (Gąsiorowski and Szymkiewicz 2007). This fact additionally confirms the kinematic origin of the Muskingum model.

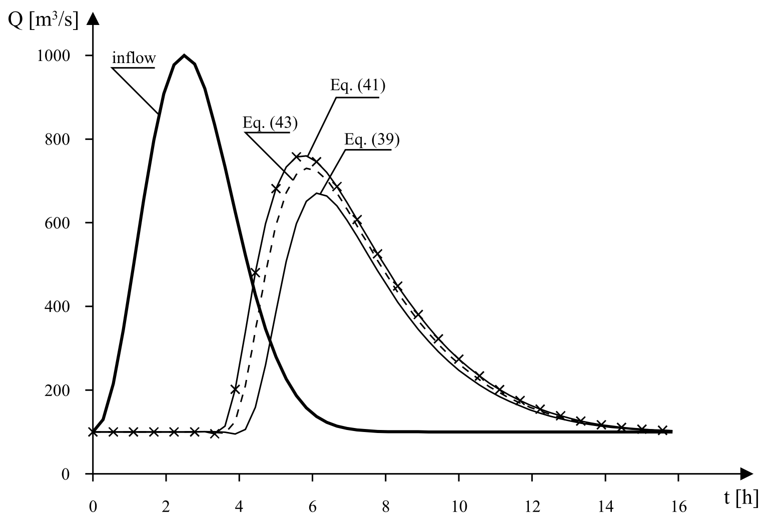

Fig. 2. Numerical solution of the non-linear Muskingum-Cunge model for Δt = 1000 s

The hydrographs computed at the downstream end by all considered forms of the Muskingum equation are shown in Fig. 2. The significant attenuation of the outflow wave is caused by numerical diffusion introduced into the solution according to Eq. (18). In this case, both the numerical diffusion and an inappropriate form of equation result in mass or momentum balance errors. This conclusion has a practical meaning, since artificial diffusion is often introduced into the numerical solution to "smooth" the unphysical oscillations or to damp the wave in order to "simulate" a real flood process, like in the Muskingum-Cunge model. Thus, it seems important to choose proper conservative form of the non-linear equations, which allows us to avoid the balance error of the transported quantity. The presented results suggest that for hydrological applications, the non-linear Muskingum model in the form of Eq. (43) seems to be most suitable, since it satisfies the mass conservation principle.

4. Conclusions

An analysis of mass and momentum conservation carried out for the Muskingum model allows us to find out that the linear Muskingum equation satisfies simultaneously both mass and momentum conservation laws.

The Muskingum equation can be considered as a semi-discrete form of the kinematic wave equation, therefore both models have the same conservative properties. Making use of this fact, various forms of the Muskingum equation can be derived directly from both continuity and steady uniform flow equation. It was also shown that similar assumptions to those applied in deriving of the kinematic wave equation can be found in an additional formula relating storage, inflow and outflow. This formula, applied to derive the Muskingum equation in a standard way, can be interpreted as a result of numerical calculation of storage V and application of the steady uniform flow equation.

The non-linear Muskingum equation can be written in two conservative forms. Each form satisfies one principle only – either the mass or the momentum one. The two principles of conservation are not satisfied simultaneously.

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