BALANCE ERRORS IN NUMERICAL SOLUTIONS OF SHALLOW WATER EQUATIONS

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Abstract: An analysis of the conservative properties of shallow water equations is presented, focused on the consistency of their numerical solution with the conservation laws of mass and momentum. Two different conservative forms are considered, solved by an implicit box scheme. Theoretical analysis supported with numerical experiments is carried out for a rectangular channel and arbitrarily assumed flow conditions. The improper conservative form of the dynamic equation is shown not to guarantee a correct solution with respect to the conservation of momentum. Consequently, momentum balance errors occur in the numerical solution. These errors occur when artificial diffusion is simultaneously generated by a numerical algorithm.

Keywords: shallow water equations, nonlinear advection equations, numerical errors, conservation laws, mass and momentum balance

1. Introduction

Shallow water equations are widely used for solving unsteady flow problems in open channels, rivers and water reservoirs of small depth. Analytical solutions of these equations are limited to special and simplified cases, due to their nonlinear character. Therefore, shallow water equations in engineering problems are solved by means of appropriate numerical methods.

The system of shallow water equations comprises two partial differential equations derived from the conservation laws of mass and momentum. The solutions of these equations should be consistent with the basic physical laws of conservation. However, the results of their numerical solutions have demonstrated that this condition is not satisfied if the system of equations is written in an improper form, resulting in errors of mass and momentum balance.

Shallow water equations are often used in a variety of forms, conservative or non-conservative, and with varying dependent variables (see [1] for a review of these forms). The use of an adequate conservative form of the differential equations and the numerical algorithm will yield a solution free of balance errors, whereas the solution of the same equations written in the non-conservative form is usually inaccurate with respect to its consistency with the conservation law [2]. The same governing equation written in various conservative forms is also known to generate different solutions if discontinuities such as shock waves or transient flow are present in the solution. Different conservative forms of the shallow water equations are equivalent if and only if their solutions are smooth. These problems were discussed by Cunge [3], Toro [4], LeVeque [5], Lai [6] and others.

However, discontinuity is not the only cause of discrepancies between solutions of equations written in different conservative forms. The study indicates that truncation errors in a Taylor series expansion can also produce balance errors in the solutions of equations written in their improper form. This effect is particularly significant if the leading term of the truncation error is first-order accurate, that is when the numerical solution is dominated by artificial diffusion.

The present paper is focused on the above problem in the context of momentum conservation in numerical solutions of one-dimensional shallow water equations. The differential and integral forms with both sets of dependent variables of these equations are discussed first. An implicit box scheme is considered in order to solve the equations. The scheme's accuracy and the influence of numerical diffusion on the momentum balance error are examined using the modified equation method. Theoretical analysis is supported with numerical tests carried out for a rectangular, frictionless channel where unsteady flow conditions are simulated.

2. Conservative and non-conservative forms of shallow water equations

One-dimensional shallow water equations without a source term are a hyperbolic system and can be written in the following form [7]:

$$\frac{\partial h}{\partial t}+\frac{\partial (uh)}{\partial x}=0, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+g\frac{\partial h}{\partial x}=0, \tag{2}$$

where x is distance, t – time, u – mean flow velocity in the x direction, h – flow depth, and g – acceleration due to gravity.

Equation (1) represents the law of mass conservation, while Equation (2) is a dynamical equation derived from the law of momentum conservation. This system can be also expressed in the following vector form:

$$\frac{\partial \boldsymbol{U}}{\partial t}+\boldsymbol{A}(\boldsymbol{U})\frac{\partial \boldsymbol{U}}{\partial x}=0, \tag{3}$$

where:

$$\boldsymbol{U}=\begin{bmatrix} h \\ u \end{bmatrix}, \quad \boldsymbol{A}=\begin{bmatrix} u & H \\ g & u \end{bmatrix}. \tag{4}$$

Vector equation (3) describes advective transport of vector quantity $\boldsymbol{U}$, where depth, h, and velocity, u, are dependent variables. Both vector equation (2) and Equation (3) are written in the non-conservative form. As numerical solutions of nonlinear equations of this form do not guarantee consistency with the physical conservation laws, the appropriate conservative form of the differential equation is required. In accordance with the assumption of a smooth solution of function $\boldsymbol{U}$, we can derive various conservative forms for different dependent variables. It is possible to obtain the following conservative form from the Equations (1)–(2):

$$\frac{\partial \boldsymbol{U}}{\partial t}+\frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial x}=0, \tag{5}$$

where $\boldsymbol{U}$ is a vector of the same dependent variables as in Equations (1)–(2), while $\boldsymbol{F}$ is a vector of flux:

$$\boldsymbol{F}=\begin{bmatrix} uh \\ 0.5u^2+gh \end{bmatrix}. \tag{6}$$

An alternative conservative form can be derived by re-applying the simple transformation to Equations (1)–(2):

$$\frac{\partial \boldsymbol{V}}{\partial t}+\frac{\partial \boldsymbol{G}(\boldsymbol{V})}{\partial x}=0, \tag{7}$$

where $\boldsymbol{V}$ is a vector of new dependent variables and $\boldsymbol{G}$ is a corresponding vector of flux, respectively given by:

$$\boldsymbol{V}=\begin{bmatrix} h \\ uh \end{bmatrix}, \quad \boldsymbol{G}=\begin{bmatrix} uh \\ u^2h+0.5gh^2 \end{bmatrix}. \tag{8}$$

In this case, the dependent variables are depth, h, and flow rate per unit of width, uh. There is an important difference between the presented conservative form of Equation (5) and that of Equation (7) in terms of the dynamical equation, whereas the equations of conservation of mass are the same in both systems.

System (7) can be also expressed in the following form:

$$\frac{\partial \boldsymbol{V}}{\partial t}+\boldsymbol{K}(\boldsymbol{V})\frac{\partial \boldsymbol{V}}{\partial x}=0, \tag{9}$$

where matrix $\boldsymbol{K}$ is Jacobian of fluxes $\boldsymbol{G}$:

$$\boldsymbol{K}=\frac{\partial \boldsymbol{G}}{\partial \boldsymbol{V}}=\begin{bmatrix} 0 & 1 \\ c^2-u^2 & 2u \end{bmatrix}, \tag{10}$$

where the $c=\sqrt{gh}$ parameter is celerity.

Equation (9) describes an advective transport of the vector quantity $\boldsymbol{V}$ in the same way as Equation (3) with a vector $\boldsymbol{U}$.

Equations (3) and (9) demonstrate that the vectors of dependent variables $\boldsymbol{U}$ and $\boldsymbol{V}$ are mutually connected by a transformation matrix $\boldsymbol{N}$ as follows:

$$\frac{\partial \boldsymbol{U}}{\partial t}=\boldsymbol{N}^{-1}\frac{\partial \boldsymbol{V}}{\partial t}, \quad \frac{\partial \boldsymbol{U}}{\partial x}=\boldsymbol{N}^{-1}\frac{\partial \boldsymbol{V}}{\partial x}. \tag{11}$$

These transformations also lead to the relations between matrixes $\boldsymbol{A}$ and $\boldsymbol{K}$:

$$\boldsymbol{NA}=\boldsymbol{KN}, \quad \boldsymbol{AN}^{-1}=\boldsymbol{N}^{-1}\boldsymbol{K}. \tag{12}$$

3. Integral form of the conservation equations

The basic physical laws of conservation of mass and momentum can be expressed in differential or integral form. The integral form guarantees the conservation of these basic physical laws in a natural way. Therefore, when analysing the conservative properties, it is convenient to write the governing equation in its integral form.

Let us consider a one-dimensional conservation equation written in the integral form over the solution domain $0 \le x \le L$ and $0 \le t \le T$:

$$\int_0^T \int_0^L \left[\frac{\partial \boldsymbol{V}}{\partial t} + \frac{\partial \boldsymbol{G}}{\partial x} \right] dx\, dt = 0. \tag{13}$$

Integration of Equation (13) with respect to x between $x = 0$ and $x = L$ results in:

$$\frac{d}{dt} \int_0^L \boldsymbol{V}(x,t) dx = \boldsymbol{G}(\boldsymbol{V})|_0 - \boldsymbol{G}(\boldsymbol{V})|_L. \tag{14}$$

Equation (14) states that the time rate of change of the total quantity of $\boldsymbol{V}$ within the control length, L, of a channel is due to net fluxes, $\boldsymbol{G}$, through the endpoints $x = 0$ and $x = L$. If the total quantity is not conserved, Equation (14) contains a source term causing generation or loss of the corresponding quantity.

Similarly, we can integrate the shallow water equations written in the differential form of (5) and (7) over a channel reach of length L. For the equation of conservation of mass, the form of the integral relation is the same in both systems:

$$\frac{\partial}{\partial t} \int_0^L h\, dx = (uh)_0 - (uh)_L, \tag{15}$$

whereas integration of dynamic equations results in different relations:

$$\frac{\partial}{\partial t} \int_0^L uh\, dx = (u^2 h)_0 - (u^2 h)_L + (0.5gh^2)_0 - (0.5gh^2)_L, \tag{16}$$

for the system in the form of (7) and

$$\frac{\partial}{\partial t} \int_0^L u\, dx = (0.5u^2)_0 - (0.5u^2)_L + (gh)_0 - (gh)_L, \tag{17}$$

for the system in the form of (5).

Both dynamic equations exhibit conservative properties. Integral Equation (16) indicates that the time rate of change of momentum uh over a channel reach of length L in time T is equal to the net flux of momentum $u(uh)$ respectively introduced by the upstream end, $x = 0$, and the downstream end, $x = L$, and the pressure force of $0.5gh^2$ (assuming hydrostatic distribution), whereas Equation (17) does not express the change of momentum, uh, but the change of flow velocity, u. A comparison of integral relations (16) and (17) indicates that the dynamic equation, depending on the conservative form, can describe transport of various quantities. As a consequence, the alternative forms can represent various conservation laws. In system (7) the dynamic equations represent the conservation law of momentum, whereas in system (5) the conservation law described by Equation (17) does not make sense from the physical point of view [4]. The form of shallow water equations is irrelevant if their solutions are smooth. However, in the presence of discontinuities, the conservative forms (5) and (7) yield different solutions.

The influence of the conservative form of shallow water equations on the accuracy of the solution is also observable when the algorithm introduces numerical errors in the form of artificial diffusion. A description of this effect for an implicit box scheme is presented in detail below.

4. Accuracy analysis of a box scheme for advection equations

One-dimensional shallow water equations written in their conservative or non-conservative forms can be solved over a domain $0 \le x \le L$ and $t \ge 0$ by means of the finite difference method using an implicit box scheme [8].

A linear advection equation describing transport of a scalar quantity U at the constant speed A is considered first:

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = 0. \tag{18}$$

An approximation of time and spatial derivatives in Equation (18) by means a box scheme produces the following algebraic equation:

$$\psi\frac{U_j^{n+1}-U_j^n}{\Delta t} + (1-\psi)\frac{U_{j+1}^{n+1}-U_{j+1}^n}{\Delta t} + A\left[(1-\theta)\frac{\left(U_{j+1}^n - U_j^n\right)}{\Delta x} + \theta\frac{\left(U_{j+1}^{n+1}-U_j^{n+1}\right)}{\Delta x}\right] = 0, \tag{19}$$

where j and n are consecutive numbers of cross-section and time level, respectively, ψ and θ are weighing parameters, while Δt and Δx are the time and spatial step, respectively.

The modified equation method can be used to demonstrate the accuracy of numerical algorithm (19) [9]. If we take into account the terms up to the third order in the Taylor series expansion, Equation (18) modified by a box scheme has the following form:

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = \nu_n\frac{\partial^2 U}{\partial x^2} + \varepsilon_n\frac{\partial^3 U}{\partial x^3} + \ldots, \tag{20}$$

where

$$\nu_n = \left(\theta - \frac{1}{2}\right)A^2\Delta t + \left(\frac{1}{2} - \psi\right)A\Delta x, \tag{21}$$

$$\varepsilon_n = \left(\frac{1}{3} - \frac{\theta}{2}\right)A^3\Delta t^2 + (\psi + \theta - 1)\frac{A^2\Delta x\Delta t}{2} + \left(\frac{1}{6} - \frac{\psi}{2}\right)A\Delta x^2. \tag{22}$$

The even-order derivative of modified Equation (20) is associated with artificial dissipation which attenuating the wave amplitude. The odd-order term is associated with artificial dispersion, which leads to significant oscillations if the numerical solution is not dominated by artificial dissipation [9, 10]. Using the modified equation enables determination of the numerical algorithm's properties. It follows from Equations (21) and (22) that numerical diffusion can be avoided if the weighing parameters are $\theta = 0.5$ and $\psi = 0.5$. Moreover, if the Courant number,

$$\mathrm{Cr} = A\frac{\Delta t}{\Delta x}, \tag{23}$$

is equivalent to $\mathrm{Cr} = 1$, the dispersion term disappears as well. As a consequence, for values of $\theta = 0.5$, $\psi = 0.5$ and $\mathrm{Cr} = 1$, the modified equation will be consistent with the differential equation ($\nu_n = 0$ and $\varepsilon_n = 0$) and the box scheme will not introduce numerical errors into the solution.

The above strategy is also applicable to shallow water equations written in the conservative form (9). In this case, the equation is assumed to represent the advection transport of the scalar quantity V at a constant speed K. In accordance with this assumption, the following linear equation is obtained:

$$\frac{\partial V}{\partial t} + K\frac{\partial V}{\partial x} = 0, \tag{24}$$

which can be solved using the box scheme. Application of accuracy analysis to the linear equation results in values of parameters υ_n and ε_n similar to those of the modified Equation (20). As a consequence, the modified equation derived for Equation (24) assumes the following form:

$$\frac{\partial V}{\partial t} + K\frac{\partial V}{\partial x} = \nu_n \frac{\partial^2 V}{\partial x^2} + \varepsilon_n \frac{\partial^3 V}{\partial x^3} + \ldots \tag{25}$$

In Equation (25), parameters υ_n and ε_n are determined by relations (21) and (22), but speed A is replaced with speed K.

5. Balance errors generated by numerical diffusion in shallow water equations

It is assumed that the above relations and statements applied to linear scalar equations are also applicable to nonlinear vector equations. This assumption implies that modified Equations (20) and (25) correspond to the nonlinear equations and that the transported scalar quantity is replaced with vector quantity $\boldsymbol{U}$ or $\boldsymbol{V}$. If terms of orders higher than the second order are neglected for the sake of simplicity, the modified equations can be written in the following forms:

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{A}(\boldsymbol{U})\frac{\partial \boldsymbol{U}}{\partial x} = \frac{\partial}{\partial x}\left[\nu_n \frac{\partial \boldsymbol{U}}{\partial x}\right] + \ldots \tag{26}$$

$$\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{K}(\boldsymbol{V})\frac{\partial \boldsymbol{V}}{\partial x} = \frac{\partial}{\partial x}\left[\nu_n \frac{\partial \boldsymbol{V}}{\partial x}\right] + \ldots \tag{27}$$

It is apparent from the nonlinear character of modified Equations (26) and (27) that numerical diffusivity, ν_n, is also dependent on vector function $\boldsymbol{U}$ or $\boldsymbol{V}$ and is given by the following relation in the case of Equation (26):

$$\nu_n = \left(\theta - \frac{1}{2}\right)\boldsymbol{A}^2(\boldsymbol{U})\Delta t + \left(\frac{1}{2} - \psi\right)\boldsymbol{A}(\boldsymbol{U})\Delta x. \tag{28}$$

Substituting this numerical diffusivity into Equations (26) and (27) produces the following relations:

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{A}(\boldsymbol{U})\frac{\partial \boldsymbol{U}}{\partial x} = \frac{\partial}{\partial x}\left[\left(\alpha \boldsymbol{A}^2(\boldsymbol{U}) + \beta \boldsymbol{A}(\boldsymbol{U})\right)\frac{\partial \boldsymbol{U}}{\partial x}\right], \tag{29}$$

$$\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{K}(\boldsymbol{V})\frac{\partial \boldsymbol{V}}{\partial x} = \frac{\partial}{\partial x}\left[\left(\alpha \boldsymbol{K}^2(\boldsymbol{V}) + \beta \boldsymbol{K}(\boldsymbol{V})\right)\frac{\partial \boldsymbol{V}}{\partial x}\right], \tag{30}$$

where

$$\alpha = \left(\theta - \frac{1}{2}\right)\Delta t, \quad \beta = \left(\frac{1}{2} - \psi\right)\Delta x. \tag{31}$$

The obtained equations are similar to governing Equations (3) and (9) and describe the transport of quantities $\boldsymbol{U}$ and $\boldsymbol{V}$, since it is advection-diffusion transport.

However, it should be emphasized that the diffusion process results from numerical errors.

It is also possible to write modified Equations (29) and (30) in their conservative forms:

$$\frac{\partial \boldsymbol{U}}{\partial t}+\frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial x}=\frac{\partial}{\partial x}\left[\left(\alpha \boldsymbol{A}(\boldsymbol{U})+\beta\right)\frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial x}\right], \tag{32}$$

$$\frac{\partial \boldsymbol{V}}{\partial t}+\frac{\partial \boldsymbol{G}(\boldsymbol{V})}{\partial x}=\frac{\partial}{\partial x}\left[\left(\alpha \boldsymbol{K}(\boldsymbol{V})+\beta\right)\frac{\partial \boldsymbol{G}(\boldsymbol{V})}{\partial x}\right]. \tag{33}$$

Modified Equations (32) and (33) are associated with governing Equations (5) and (7), which are also written in their conservative forms:

$$\frac{\partial \boldsymbol{U}}{\partial t}+\frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial x}=0, \tag{34}$$

$$\frac{\partial \boldsymbol{V}}{\partial t}+\frac{\partial \boldsymbol{G}(\boldsymbol{V})}{\partial x}=0. \tag{35}$$

Equation (34) can be derived from Equation (35). However, the same equivalence does not exist in the case of modified Equations (32) and (33). Equation (32) can not be directly transformed into Equation (33) due to the appearance of additional terms with second order derivatives. Modified Equation (29) will be reconsidered in order to demonstrate this feature. Substituting relation (11) into this equation yields:

$$\boldsymbol{N}^{-1}\frac{\partial \boldsymbol{V}}{\partial t}+\boldsymbol{A}\boldsymbol{N}^{-1}\frac{\partial \boldsymbol{V}}{\partial x}=\frac{\partial}{\partial x}\left[\alpha \boldsymbol{A}^2\boldsymbol{N}^{-1}\frac{\partial \boldsymbol{V}}{\partial x}\right]+\frac{\partial}{\partial x}\left[\beta \boldsymbol{A}\boldsymbol{N}^{-1}\frac{\partial \boldsymbol{V}}{\partial x}\right], \tag{36}$$

and the use of relations (12) and $\boldsymbol{N}\boldsymbol{N}^{-1}=\boldsymbol{I}$ leads to the following equation:

$$\frac{\partial \boldsymbol{V}}{\partial t}+\boldsymbol{K}\frac{\partial \boldsymbol{V}}{\partial x}=\frac{\partial}{\partial x}\left[(\alpha \boldsymbol{K}^2+\beta \boldsymbol{K})\frac{\partial \boldsymbol{V}}{\partial x}\right]+\boldsymbol{N}(\alpha \boldsymbol{K}+\beta)\boldsymbol{K}\frac{\partial \boldsymbol{N}}{\partial x}\frac{\partial \boldsymbol{V}}{\partial x}. \tag{37}$$

Equation (37) is modified Equation (29) with vector function $\boldsymbol{U}$ replaced with vector $\boldsymbol{V}$. Notably, transformation of Equation (29) produces an equation different than Equation (30) by an additional term of the following form:

$$\boldsymbol{S}_n=\boldsymbol{N}(\alpha \boldsymbol{K}+\beta)\boldsymbol{K}\frac{\partial \boldsymbol{N}}{\partial x}\frac{\partial \boldsymbol{V}}{\partial x}. \tag{38}$$

The $\boldsymbol{S}_n$ source term results from artificial diffusion in the solution of Equation (34). In order to study the influence of this term on the solution's accuracy, modified Equation (37) will be integrated over the solution domain $[0;L]\times[0;T]$:

$$\int_0^L (\boldsymbol{V}|_T-\boldsymbol{V}|_0)\,dx=\int_0^T\left(\left[\boldsymbol{G}-(\alpha \boldsymbol{K}+\beta)\frac{\partial \boldsymbol{G}}{\partial x}\right]_0-\left[\boldsymbol{G}-(\alpha \boldsymbol{K}+\beta)\frac{\partial \boldsymbol{G}}{\partial x}\right]_L\right)dt+\delta M. \tag{39}$$

Using relation (31), the additional source term δM assumes the following form:

$$\delta M=\int_0^T\!\!\int_0^L \boldsymbol{S}_n\,dx\,dt=\int_0^T\!\!\int_0^L \boldsymbol{N}\left[\left(\theta-\frac{1}{2}\right)\Delta t\boldsymbol{K}+\left(\frac{1}{2}-\psi\right)\Delta x\right]\boldsymbol{K}\frac{\partial \boldsymbol{N}}{\partial x}\frac{\partial \boldsymbol{V}}{\partial x}dx\,dt. \tag{40}$$

It follows from Equation (39) that the change of quantity $\boldsymbol{V}$ over a channel reach of length L in time T is due to the net advection – diffusion flux $\boldsymbol{G}$ through endpoints

$x=0$ and $x=L$ and the additional term δM determined by relation (40). This term is responsible for the different numerical solution and leads to balance errors despite the governing system (34) being written in the conservative form. It is expected from integral Equation (17) that the conservation discrepancy occurs only in the dynamic equation. Thus, taking into account the momentum transport, the occurrence of additional term δM produces momentum balance error in the system (34). Therefore, the balance equation for the dynamic equation of this system assumes the following form:

$$M-\big(\Phi_M-\Phi_d\big)=\delta M, \tag{41}$$

where:

$$M=\int_0^L\big[(uh)_T-(uh)_0\big]\,dx, \tag{42}$$

$$\Phi_M=\int_0^T\big[(u^2h)_0-(u^2h)_L+(0.5gh^2)_0-(0.5gh^2)_L\big]\,dt, \tag{43}$$

$$\Phi_d=\int_0^T\Big[(\alpha\boldsymbol{K}+\beta)\frac{\partial}{\partial x}\Big((u^2h)_0-(u^2h)_L+(0.5gh^2)_0-(0.5gh^2)_L\Big)\Big]\,dt, \tag{44}$$

$$\delta M=\int_0^T\int_0^L f\left[\left(\theta-\frac{1}{2}\right)\Delta t,\left(\frac{1}{2}-\psi\right)\Delta x,\frac{\partial uh}{\partial x}\right]dx\,dt. \tag{45}$$

It follows from Equation (40) or Equation (45) that the momentum balance error in the nonlinear dynamic equation of system (34) is determined by the numerical diffusion generated by the box scheme. Detailed analysis of Equation (40) leads to the following main conclusions:

- for $\theta=0.5$ and $\psi=0.5$ numerical diffusion is avoided and, consequently, the momentum balance error disappears ($\delta M=0$),
- for $\psi=0$ and $\theta=1$ the maximum numerical diffusion is generated and the balance error reaches the maximum value ($\delta M\rightarrow\max$),
- for $\theta=0.5$ and $\psi<0.5$ the balance error depends mainly on the spatial step, Δx ($\delta M=f(\Delta x)$), whereas it has a constant value for varying time step, Δt,
- for $\psi=0.5$ and $\theta>0.5$ the situation is opposite – the balance error is a function of time step only $\delta M=f(\Delta t)$ and has constant values for varying spatial step, Δx,
- the momentum balance error, δM, also depends on the spatial derivative of function $\boldsymbol{V}$, thus being greater for rapidly propagated waves than for slowly propagated waves.

6. Numerical tests

In order to illustrate above analysis, numerical tests were carried out for a rectangular, frictionless channel of length $L'=500$ m. Equations (34) and (35) were solved subject to the following initial and boundary conditions:

- at time $t=0$ water depth is uniform over channel $h_S(x)=1$ m and the body of water remains at rest, $u_S(x)=0$ m/s,

- a wave of the following form is introduced at the upstream end, $x=0$:

$$u_0(t)=\begin{cases}0 & t=0,\\ \dfrac{t}{t_{\max}}u_{\max} & 0<t\le t_{\max},\\ u_{\max} & t>t_{\max},\end{cases} \tag{46}$$

where $u_{\max}$ is the peak velocity of the inflow and $t_{\max}$ is the time corresponding to the peak velocity,

- constant water depth is imposed at the downstream end $x=L'$, $h_L(t)=1\,\text{m}$.

The computations were carried out for $u_{\max}=1\,\text{m/s}$, $t_{\max}=40\,\text{s}$, $\Delta x=5\,\text{m}$ and $\Delta t=5\,\text{s}$ and for different values of weighing parameters θ and ψ. Mass and momentum balance were computed for the numerical solution of shallow water equations. The conservation of momentum was investigated with relative balance error, which can be written as follows:

$$\Delta M=\frac{M-\Phi_M}{M}100\%, \tag{47}$$

with M and Φ_M determined by formulas (42) and (43), respectively. This error is expressed in an analogical form for the conservation of mass:

$$\Delta m=\frac{m-\Phi_m}{m}100\%, \tag{48}$$

where

$$m=\int_0^L\big[(h)_T-(h)_0\big]dx, \tag{49}$$

$$\Phi_m=\int_0^L\big[(uh)_0-(uh)_L\big]dt. \tag{50}$$

All computations were carried out for time $T=100\,\text{s}$ for a control cross-section of $x=L=200\,\text{m}$. The integrals were computed by numerical integration.

Computations of the conservation of mass for corresponding systems indicated that the balance error, Δm, was less than 0.01% independently of the θ and ψ parameters. Thus, the conservation law of mass was sufficiency satisfied. The results obtained for the conservation of momentum are shown in Figure 1.

The momentum balance error is notably absent from the solution of system (35). However, in the solution of system (34), the errors are significant and cause excessive accumulation of momentum. For $\psi=0$ and $\theta=1$, when the numerical solution is influenced by the maximum artificial diffusion, the balance error, ΔM, also reaches its maximum value about -3%. This error is not observed ($\Delta M=0.01\%$) if numerical diffusion is minimized by selecting appropriate values of weighing parameters ($\psi=0.5$ and $\theta=0.5$).

The relationship between balance error, ΔM, and grid spacing, Δx, on the one hand and time step, Δt, on the other in the solution of system (34) is presented in Figure 2.

It follows from Figure 2a that, for $\theta=0.5$ and $\psi=0.3$, balance error, ΔM, increases with increasing spatial step, Δx. For a small value of Δx, the error is near zero and for $\Delta x=40\,\text{m}$ its value achieves a maximum. However, for $\psi=0.5$ and

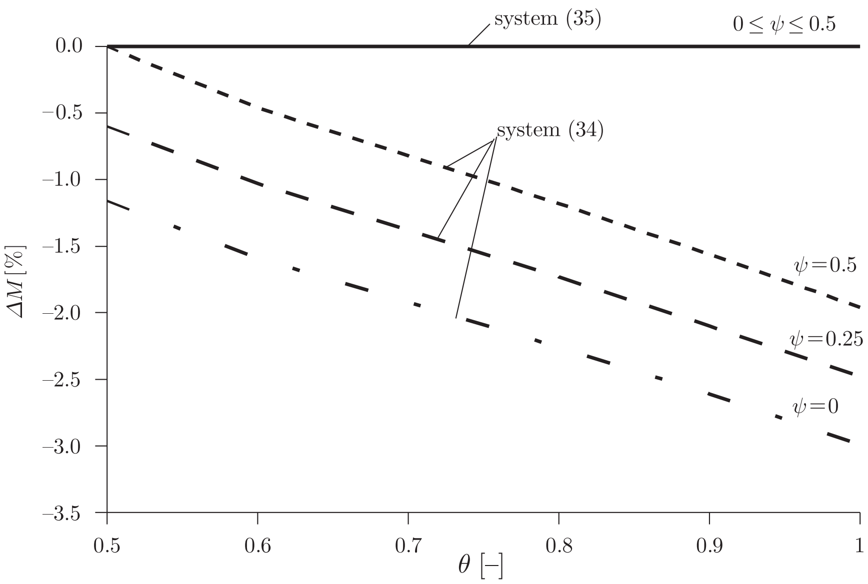

Figure 1. Momentum balance error, ΔM, in the solution of system (34) and (35); $\Delta x = 5$m and $\Delta t = 5$s

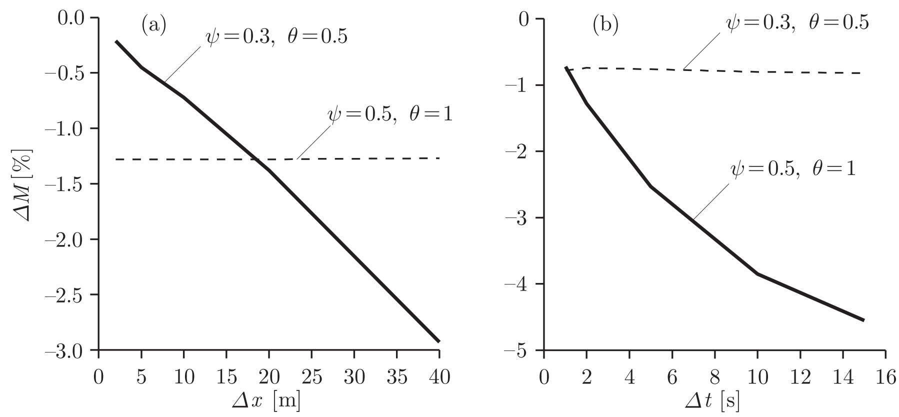

Figure 2. Momentum balance error, ΔM, in the solution of shallow water Equation (34) for various values of: (a) spatial step, Δx ($\Delta t = 2$s); (b) time step, Δt ($\Delta x = 10$m); $t_{\max} = 40$s, $u_{\max} = 1$m/s

$\theta = 1$, balance error assumes constant values throughout the range of Δx. The error is presented in Figure 2b as a function of time step, Δt. Here, we encounter the opposite situation. For $\theta = 0.5$ and $\psi = 0.3$ balance error, ΔM, has a constant value, while for $\psi = 0.5$ and $\theta = 1$ the value of momentum error increases with Δt.

A comparison of computed profiles of water depth, h, of systems (34) and (35) is presented in Figure 3 for selected values of weighing parameters. For $\theta = 0.65$ and $\psi = 0.5$ (Figure 3a) artificial diffusion is minimized and so the shallow water equations in their conservative form yield similar solutions with respect to the shape of waves. No similar agreement can be observed for $\theta = 1$ and $\psi = 0$ (Figure 3b). This significant difference between outflows is due the numerical solution being influenced by large artificial diffusion, which produces a momentum balance error of $\Delta M = -5\%$.

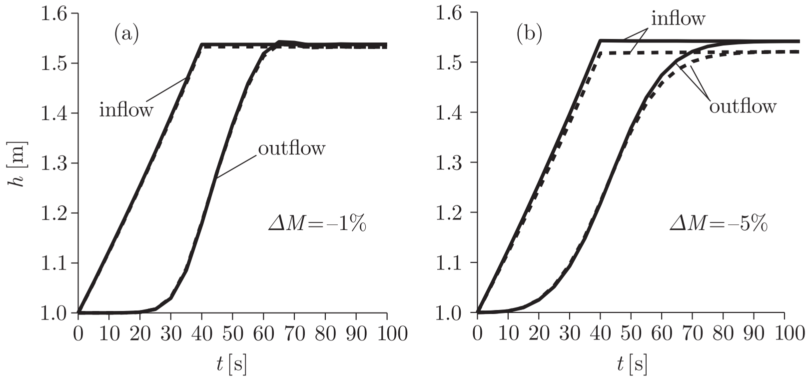

Figure 3. Comparison of numerical solutions of systems (34) and (35) for selected values of weighing parameters: (a) $\theta = 0.65$ and $\psi = 0.5$ – minimal numerical diffusion; (b) $\theta = 1$ and $\psi = 0$ – maximal numerical diffusion; $t_{\max} = 40$s, $u_{\max} = 1.5$m/s, $\Delta x = 5$m and $\Delta t = 5$s

The influence of the spatial derivative on balance error, ΔM, was considered in another test. For this purpose, equations in the form of (34) were solved with various wave shapes described by relation (46). All waves were assumed to have the same maximum velocity of $u_{\max} = 1$m/s, but different values of $t_{\max}$ (ranging from 25s to 400s). Thus, the waves differed from each other in their rates of propagation and, consequently, the corresponding values of the spatial derivative were obtained.

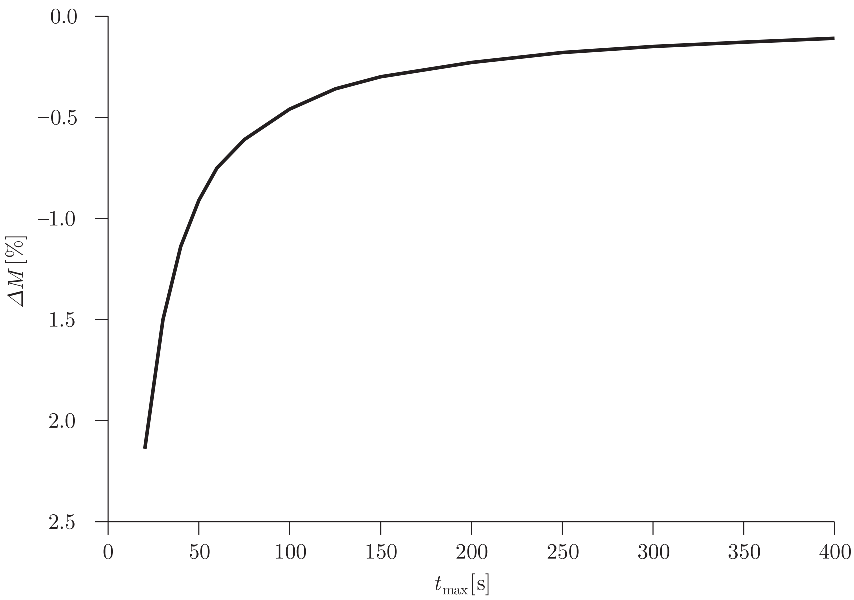

Figure 4. Momentum balance error, ΔM, in the solution of shallow water Equation (34) for various $t_{\max}$ ($\Delta x = 10$m, $\Delta t = 5$s, $\psi = 0.45$, $\theta = 0.65$).

As indicated in Figure 4, momentum balance error, ΔM, becomes significant depending on $t_{\max}$. For rapidly propagated waves with a short $t_{\max} = 25$s, balance error is about -2.2%. This is due to numerical diffusion, manifested by excessive smoothing of the outflow wave. Slowly propagated waves with longer $t_{\max} = 400$s are only slightly deformed and momentum balance error is small ($\Delta M = -0,1\%$). In this case, numerical diffusion is negligible.

7. Summary and conclusions

A study of the conservative properties of shallow water equations was conducted. Theoretical and numerical analyses were carried out for systems of equations written in two conservative forms with two different sets of dependent variables. The conservative properties differed in the two systems. In the system written in the conservative form with dependent variables u and h, momentum was not conserved when the solution was influenced by numerical errors. As a consequence, balance errors were observed in the form of excessive accumulation of momentum.

Using the modified equation approach, it was possible to explain the mechanism of introducing balance errors into numerical solutions. This error was demonstrated to result from a combination of numerical errors and an inadequate form of the dynamic equation. Balance errors are directly generated by an additional source term arising from artificial diffusion generated by the numerical algorithm. Therefore, momentum balance error depends on numerical parameters such as time step, Δt, and spatial step, Δx, and weighing parameters θ and ψ. It also depends on the spatial derivative of functions u and h. As a result, the errors achieve significant values when rapidly propagated waves are modelled.

Solutions of the governing equation written in different conservative forms are equivalent provided that the numerical algorithm does not introduce numerical errors. However, this condition is practically difficult to satisfy for nonlinear equations. As numerical diffusion is often introduced into the solution in order to control non-physical oscillations associated with numerical dispersion. Therefore, the improper conservative form of differential equations leads to balance errors in numerical solutions.

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