



## Note

## A note on the weakly convex and convex domination numbers of a torus

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## ABSTRACT

The distance  $d_G(u, v)$  between two vertices  $u$  and  $v$  in a connected graph  $G$  is the length of the shortest  $(u, v)$  path in  $G$ . A  $(u, v)$  path of length  $d_G(u, v)$  is called a  $(u, v)$ -geodesic. A set  $X \subseteq V$  is called *weakly convex* in  $G$  if for every two vertices  $a, b \in X$ , exists an  $(a, b)$ -geodesic, all of whose vertices belong to  $X$ . A set  $X$  is *convex* in  $G$  if for all  $a, b \in X$  all vertices from every  $(a, b)$ -geodesic belong to  $X$ . The *weakly convex domination number* of a graph  $G$  is the minimum cardinality of a weakly convex dominating set of  $G$ , while the *convex domination number* of a graph  $G$  is the minimum cardinality of a convex dominating set of  $G$ . In this paper we consider weakly convex and convex domination numbers of tori.

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## 1. Definitions

Here we consider simple, undirected and connected graphs  $G = (V, E)$  with  $|V| = n$ . The *open neighbourhood*  $N_G(v)$  of a vertex  $v \in V$  is the set of all vertices adjacent to  $v$  and the *closed neighbourhood*  $N_G[v]$  of a vertex  $v \in V$  is the set  $N_G(v) \cup \{v\}$ . A subset  $D$  of  $V$  is *dominating* if every vertex of  $V - D$  has at least one neighbour in  $D$ . Let  $\gamma(G)$  be the minimum cardinality of a dominating set of  $G$ .

The *distance*  $d_G(u, v)$  between two vertices  $u$  and  $v$  in a connected graph  $G$  is the length of the shortest  $(u, v)$  path in  $G$ . A  $(u, v)$  path of length  $d_G(u, v)$  is called a  $(u, v)$ -geodesic.

A set  $X \subseteq V$  is called *weakly convex* in  $G$  if for every two vertices  $a, b \in X$ , exists an  $(a, b)$ -geodesic, all of whose vertices belong to  $X$ . A set  $X$  is *convex* in  $G$  if for all  $a, b \in X$  all vertices from every  $(a, b)$ -geodesic belong to  $X$ . A set  $X \subseteq V$  is a *weakly convex dominating set* in  $G$  if  $X$  is weakly convex and dominating. Further,  $X$  is a *convex dominating set* if it is convex and dominating. The *weakly convex domination number*  $\gamma_{wcon}(G)$  of a graph  $G$  is the minimum cardinality of a weakly convex dominating set of  $G$ , while the *convex domination number*  $\gamma_{con}(G)$  of a graph  $G$  is the minimum cardinality of a convex dominating set of  $G$ . Convex and weakly convex domination numbers were first introduced by Topp [4].

## 2. Cartesian product

The *Cartesian product* of two graphs  $G_1$  and  $G_2$ , denoted by  $G = G_1 \square G_2$ , is the graph with vertex set  $V(G) = V(G_1) \times V(G_2)$ , where two vertices  $(u_1, u_2), (v_1, v_2)$  are adjacent in  $G_1 \square G_2$  if and only if the following holds:

- (a)  $u_1 = v_1$  and  $u_2 v_2 \in E(G_2)$ ,
- (b)  $u_2 = v_2$  and  $u_1 v_1 \in E(G_1)$ .

For  $v_i \in V(G_2)$ ,  $G_1^i$  denotes the subgraph of  $G_1 \square G_2$  induced by  $V(G_1) \times \{v_i\}$  and we call  $G_1^i$  the  $i$ th copy of  $G_1$  in  $G_1 \square G_2$ . If  $V(G_2) = \{v_1, \dots, v_n\}$ , then  $G_1^i$  and  $G_1^j$  are *neighbouring copies* in  $G_1 \square G_2$  if  $v_i v_j \in E(G_2)$ . The Cartesian product of two cycles  $C_m$  and  $C_n$  is called a *torus*, if  $m \geq 3$  and  $n \geq 3$ . The problem of domination in a torus was considered in [5]. The bondage

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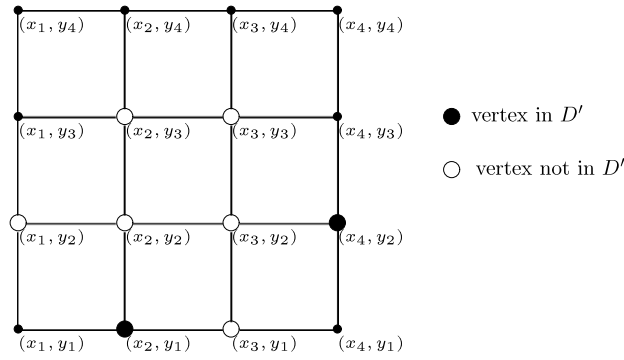


Fig. 1. The case (a).

number and signed domatic number of a torus is studied in [2] and in [3], respectively. Here we consider weakly convex and convex domination numbers of a torus.

In 1963 Vizing conjectured that  $\gamma(G_1 \square G_2) \geq \gamma(G_1)\gamma(G_2)$  for every two graphs  $G_1$  and  $G_2$ . In [1] it was proven that the following Vizing-type inequality for the convex domination number is true.

**Theorem 1 ([1]).** For connected graphs  $G_1$  and  $G_2$ ,

$$\gamma_{\text{con}}(G_1)\gamma_{\text{con}}(G_2) \leq \gamma_{\text{con}}(G_1 \square G_2).$$

It is easy to verify that the convex domination number of a cycle  $C_m$  on  $m \geq 6$  vertices equals  $m$ . Hence, Theorem 1 implies what follows.

**Corollary 2.** Let  $C_m$  and  $C_n$  be cycles on  $m \geq 6$  and  $n \geq 6$  vertices, respectively. Then

$$\gamma_{\text{con}}(C_m \square C_n) = mn.$$

A similar result may be proven for the weakly convex domination number.

**Theorem 3.** Let  $C_m$  and  $C_n$  be the cycles on  $m \geq 7$  and  $n \geq 7$  vertices, respectively. Then

$$\gamma_{\text{wcon}}(C_m \square C_n) = mn.$$

**Proof.** Let  $G = C_m \square C_n$ , where  $V(G) = \{(x_i, y_j) : x_i \in V(C_m), y_j \in V(C_n), 1 \leq i \leq m, 1 \leq j \leq n\}$  and  $m, n \geq 7$ . Suppose  $\gamma_{\text{wcon}}(G) < mn$ . Let  $D$  be a minimum weakly convex dominating set of  $G$ . Since  $|D| < mn$  we assume, without loss of generality, that  $(x_3, y_3) \notin D$ . Then  $(x_3, y_3)$  belongs to the  $((x_3, y_2), (x_3, y_4))$ -geodesic and thus  $(x_3, y_2) \notin D$  or  $(x_3, y_4) \notin D$ . Without loss of generality, let  $(x_3, y_2) \notin D$ . Similarly,  $(x_3, y_3)$  belongs to the  $((x_3, y_1), (x_3, y_4))$ -geodesic and thus we let  $(x_3, y_1) \notin D$ . Further,  $(x_3, y_3)$  belongs to the  $((x_2, y_3), (x_4, y_3))$ -geodesic so, without loss of generality, we let  $(x_2, y_3) \notin D$ .

Each  $((x_2, y_2), (x_3, y_4))$ -geodesic contains  $(x_3, y_3)$  or  $(x_3, y_2)$  or  $(x_2, y_3)$ , so  $(x_2, y_2) \notin D$  or  $(x_3, y_4) \notin D$ .

- (a) If  $(x_2, y_2) \notin D$ , then  $(x_4, y_2) \in D$  dominates  $(x_3, y_2)$ . Moreover, since  $D$  is weakly convex,  $(x_1, y_2) \notin D$ . However this implies that  $(x_2, y_1) \in D$  which is a contradiction, because no  $((x_2, y_1), (x_4, y_2))$ -geodesic is contained in  $D$ . (See Fig. 1, some edges in the figures are omitted to make them more clear.)
- (b) If  $(x_3, y_4) \notin D$ , then  $(x_4, y_3) \in D$  dominates  $(x_3, y_3)$ . Moreover, since  $D$  is weakly convex,  $(x_1, y_3) \notin D$  and  $(x_2, y_4) \notin D$ . However, this implies that  $(x_2, y_2) \in D$  which is a contradiction, because no  $((x_2, y_2), (x_4, y_3))$ -geodesic is contained in  $D$ . (See Fig. 2.)  $\square$

The following propositions give exact values for the convex domination number and the weakly convex domination number for a torus  $G = C_m \square C_n$ , where  $n \in \{3, 4, 5, 6\}$ . We start with a lemma.

**Lemma 4.** Let  $G = C_m \square C_n$ , where  $n \geq 3$  and  $m \geq 5$ . Let  $D$  be a minimum weakly convex dominating set of  $G$  such that  $(x_3, y_j) \notin D$  for  $j \in \{1, 2, 3\}$ . Then, without loss of generality,  $(x_i, y_j) \notin D$  for  $i \in \{2, 3\}, j \in \{1, 2, 3\}$  and  $(x_1, y_2) \in D, (x_4, y_2) \in D$ .

**Proof.** Let  $G$  and  $D$  be defined as above. Since  $D$  is dominating,  $(x_3, y_2)$  has a neighbour in  $D$ . Without loss of generality let  $(x_4, y_2) \in D$ . Since  $m \geq 5$ , each  $((x_2, y_1), (x_4, y_2))$ -geodesic contains  $(x_3, y_1)$  or  $(x_3, y_2)$ , so the weak convexity of  $D$  implies that  $(x_2, y_1) \notin D$ . By similar arguments,  $(x_2, y_3) \notin D$  and  $(x_2, y_2) \notin D$ . Hence,  $(x_1, y_2) \in D$  dominates  $(x_2, y_2)$ . (See Fig. 3.)  $\square$

**Proposition 5.** Let  $G = C_m \square C_3$ , where  $m \geq 3$ . Then

$$\gamma_{\text{wcon}}(G) = \gamma_{\text{con}}(G) = m.$$

**Proof.** Since  $\gamma_{\text{con}}(G) \geq \gamma_{\text{wcon}}(G) \geq \gamma(G) = 3$  and the set  $\{(x_2, y_1), (x_2, y_2), (x_2, y_3)\}$  is a convex dominating set of  $G$ , the result is true for  $m = 3$ . Let  $G = C_m \square C_3$ , where  $m \geq 4$  and denote  $V(G) = \{(x_i, y_j) : x_i \in V(C_m), y_j \in V(C_3), 1 \leq i \leq m\}$ .

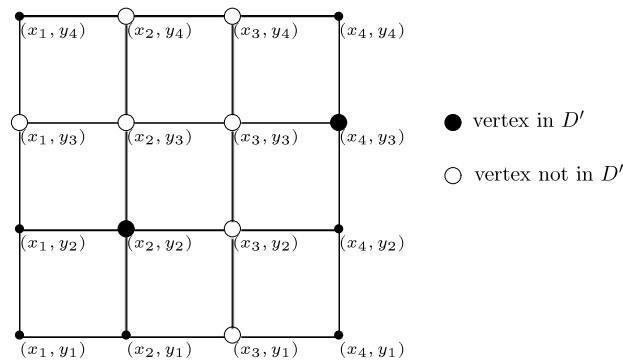


Fig. 2. The case (b).

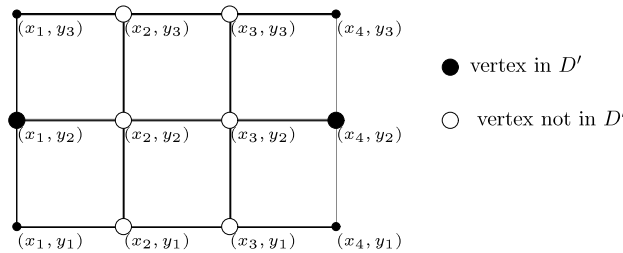


Fig. 3. Illustration for the Lemma 4.

$m, 1 \leq j \leq 3$ . We show that  $D = \{(x_i, y_1) : 1 \leq i \leq m\}$  is a minimum convex and weakly convex dominating set of  $G$ . It is obvious that  $D$  is convex and dominating, so  $\gamma_{wcon}(G) \leq \gamma_{con}(G) \leq m$ . Suppose  $D$  is not a minimum weakly convex dominating set of  $G$ . Then there exists a weakly convex dominating set  $D' \subseteq V(G)$  such that  $|D'| < |D| = m$ . However, it is possible to verify that for  $C_4 \square C_3$  we cannot find a weakly convex dominating set of cardinality at most 3. Hence, in what follows we assume  $m \geq 5$ . Since  $|D'| < m$ , there exists an index  $i$  such that all three vertices of  $V(C_3^i)$  do not belong to  $D'$ . Without loss of generality, let  $i = 3$ . By Lemma 4, we obtain that  $(x_l, y_k) \notin D'$  for  $l \in \{2, 3\}, k \in \{1, 2, 3\}$  and  $(x_1, y_2) \in D', (x_4, y_2) \in D'$  (See Fig. 4.)

Recall that  $\gamma_{wcon}(C_k) = k$  for  $k \geq 7$ . For this reason, since  $D'$  is weakly convex and  $d_G((x_1, y_2), (x_4, y_2)) \leq 3$ , we conclude that  $m \leq 6$ . Hence,  $|D'| < 6$ . Since  $D'$  is dominating,  $(x_1, y_1), (x_1, y_3), (x_4, y_1), (x_4, y_3) \in D'$ . However, this implies that  $|D'| \geq 6$ , a contradiction. Therefore,  $\gamma_{wcon}(G) = \gamma_{con}(G) = m$ .  $\square$

**Proposition 6.** Let  $G = C_m \square C_4$ , where  $m \geq 7$ . Then

$$\gamma_{wcon}(G) = \gamma_{con}(G) = 2m.$$

**Proof.** Let  $G = C_m \square C_4$ , where  $m \geq 7$  and denote  $V(G) = \{(x_i, y_j) : x_i \in V(C_m), y_j \in V(C_4), 1 \leq i \leq m, 1 \leq j \leq 4\}$ . We show that  $D = \{(x_i, y_j) : 1 \leq i \leq m, 1 \leq j \leq 2\}$  is a minimum weakly convex and convex dominating set of  $G$ . It is obvious that  $D$  is convex and dominating, so  $\gamma_{wcon}(G) \leq \gamma_{con}(G) \leq 2m$ . Suppose  $D$  is not a minimum weakly convex dominating set of  $G$ . Then there exists a weakly convex dominating set  $D' \subseteq V(G)$  such that  $|D'| < |D| = 2m$ . Then there exists an index  $i$  such that at least three vertices of  $V(C_4^i)$  do not belong to  $D'$ . Without loss of generality, let  $(x_3, y_1), (x_3, y_2), (x_3, y_3) \notin D'$ . Now, by Lemma 4, we obtain that  $(x_l, y_k) \notin D'$  for  $l \in \{2, 3\}, k \in \{1, 2, 3\}$  and  $(x_1, y_2) \in D', (x_4, y_2) \in D'$ . Now, since  $D'$  is weakly convex and  $d_G((x_1, y_2), (x_4, y_2)) \leq 3$ , we conclude that  $m \leq 6$  which is impossible. Hence,  $\gamma_{wcon}(G) = \gamma_{con}(G) = 2m$ .  $\square$

The straightforward, albeit technical proof of the following observation is omitted.

**Observation 7.** The weakly convex domination number and convex domination numbers of the Cartesian product of  $C_m$  and  $C_4$ , where  $m \in \{4, 5, 6\}$ , are also equal to  $2m$ .

**Proposition 8.** Let  $G = C_m \square C_5$ , where  $m \geq 5$ . Then

$$\gamma_{wcon}(G) = \gamma_{con}(G) = 3m.$$

**Proof.** Let  $G = C_m \square C_5$ , where  $m \geq 5$  and denote  $V(G) = \{(x_i, y_j) : x_i \in V(C_m), y_j \in V(C_5), 1 \leq i \leq m, 1 \leq j \leq 5\}$ . We show that  $D = \{(x_i, y_j) : 1 \leq i \leq m, 1 \leq j \leq 3\}$  is a minimum convex and weakly convex dominating set of  $G$ . It is obvious that  $D$  is convex and dominating, so  $\gamma_{wcon}(G) \leq \gamma_{con}(G) \leq 3m$ . Suppose  $D$  is not a minimum weakly convex dominating set

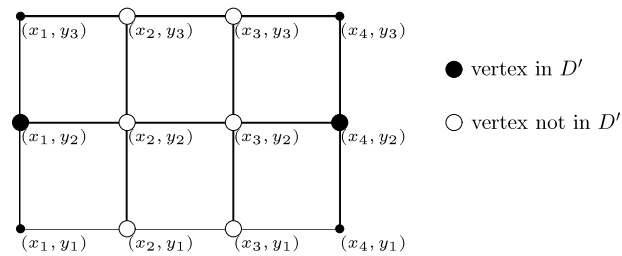


Fig. 4. The  $C_m \square C_3$  (some edges are omitted).

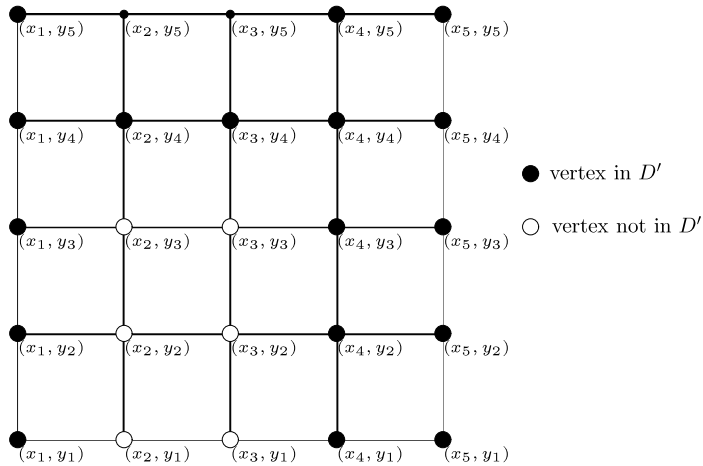


Fig. 5. Case (a) of  $C_m \square C_5$ .

of  $G$ . Then there exists a weakly convex dominating set  $D' \subseteq V(G)$  such that  $|D'| < |D| = 3m$ . Therefore, there exists an index  $i$  such that at least three vertices of  $V(C_5^i)$  do not belong to  $D'$ . Since  $D'$  is weakly convex, those three vertices induce a  $P_3$  in  $G$ . Without loss of generality, let  $(x_3, y_1), (x_3, y_2), (x_3, y_3) \notin D'$ . Now, by Lemma 4, we obtain that  $(x_i, y_k) \notin D'$  for  $l \in \{2, 3\}, k \in \{1, 2, 3\}$  and  $(x_1, y_2) \in D', (x_4, y_2) \in D'$ . Again we conclude that  $m \leq 6$ .

If  $(x_4, y_3) \notin D'$ , then  $(x_3, y_4) \in D'$  and so no  $((x_4, y_2), (x_3, y_4))$ -geodesic is contained in  $D'$ . Thus we conclude that  $(x_4, y_3) \in D'$ . By similar reasoning and symmetry we obtain that  $(x_1, y_3) \in D', (x_4, y_1) \in D'$  and  $(x_1, y_1) \in D'$ . Further, since  $D'$  is weakly convex, we have that  $(x_5, y_k) \in D'$  for  $k \in \{1, 2, 3\}$  and if  $m = 6$  then, additionally,  $(x_6, y_k) \in D'$  for  $k \in \{1, 2, 3\}$ .

- (a) Suppose  $(x_3, y_4) \in D'$ . Then  $(x_4, y_4)$  belongs to the  $((x_3, y_4), (x_4, y_3))$ -geodesic that does not contain vertices from  $V(G) - D'$ , so  $(x_4, y_4) \in D'$ . Similarly, considering the  $((x_3, y_4), (x_1, y_3))$ -geodesic outside of  $D'$  we conclude that  $(x_1, y_4) \in D'$  and  $(x_2, y_4) \in D'$ . This implies that  $(x_5, y_4) \in D', (x_5, y_5) \in D', (x_4, y_5) \in D'$  and  $(x_1, y_5) \in D'$  (see Fig. 5). Then  $|D'| \geq 17$  for  $m = 5$  and  $|D'| \geq 20$  for  $m = 6$ . However this contradicts  $|D'| < 3m$ .
- (b) Thus,  $(x_3, y_4) \notin D'$ . If  $(x_2, y_4) \in D'$ , then each  $((x_2, y_4), (x_4, y_3))$ -geodesic would contain vertices from  $V(G) - D'$ , so  $(x_2, y_4) \notin D'$ . If  $(x_4, y_4) \notin D'$ , then  $(x_3, y_5) \in D'$  dominates  $(x_3, y_4) \notin D'$ . However, each  $((x_3, y_5), (x_4, y_3))$ -geodesic contains vertices from  $V(G) - D'$ . Thus,  $(x_4, y_4) \in D'$ . Similarly we obtain that  $(x_1, y_4) \in D'$ . Since  $D'$  is weakly convex,  $(x_5, y_4) \in D', (x_1, y_5) \in D', (x_4, y_5) \in D'$  and  $(x_5, y_5) \in D'$  (see Fig. 5). Thus,  $|D'| \geq 15$  for  $m = 5$ . If  $m = 6$ , then  $(x_6, y_k) \in D'$  for  $k \in \{1, 2, \dots, 5\}$  and thus  $|D'| \geq 20$ . However both cases contradict the fact that  $|D'| < 3m$  (Fig. 6).  $\square$

Our last result for the Cartesian product of two cycles concerns only weakly convex domination.

**Proposition 9.** Let  $G = C_m \square C_6$ , where  $m \geq 6$ . Then

$$\gamma_{\text{wcon}}(G) = 4m.$$

**Proof.** Let  $G = C_m \square C_6$ , where  $m \geq 6$  and denote  $V(G) = \{(x_i, y_j) : x_i \in V(C_m), y_j \in V(C_6), 1 \leq i \leq m, 1 \leq j \leq 6\}$ . We show that  $D = \{(x_i, y_j) : 1 \leq i \leq m, 1 \leq j \leq 4\}$  is a minimum weakly convex dominating set of  $G$ . It is obvious that  $D$  is weakly convex and dominating, so  $\gamma_{\text{wcon}}(G) \leq 4m$ . Suppose  $D$  is not a minimum weakly convex dominating set of  $G$ . Then there exists a weakly convex dominating set  $D' \subseteq V(G)$  such that  $|D'| < |D| = 4m$  and therefore there exists an index  $i$  such that at least three vertices of  $V(C_6^i)$  do not belong to  $D'$ . Since  $D'$  is weakly convex, those three vertices induce a  $P_3$  in

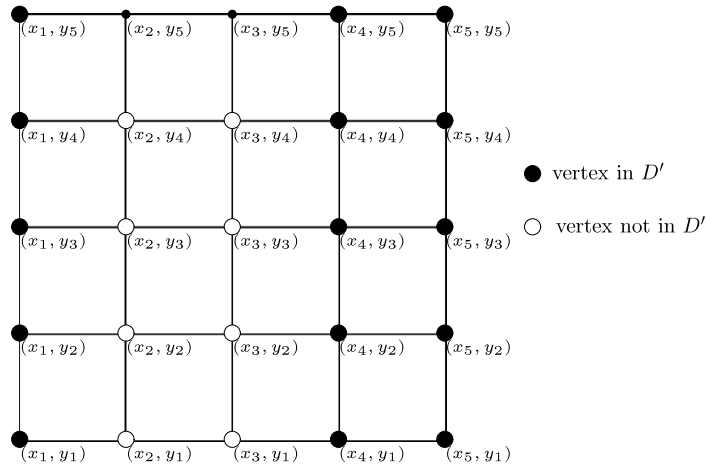


Fig. 6. Case (b) of  $C_m \square C_5$ .

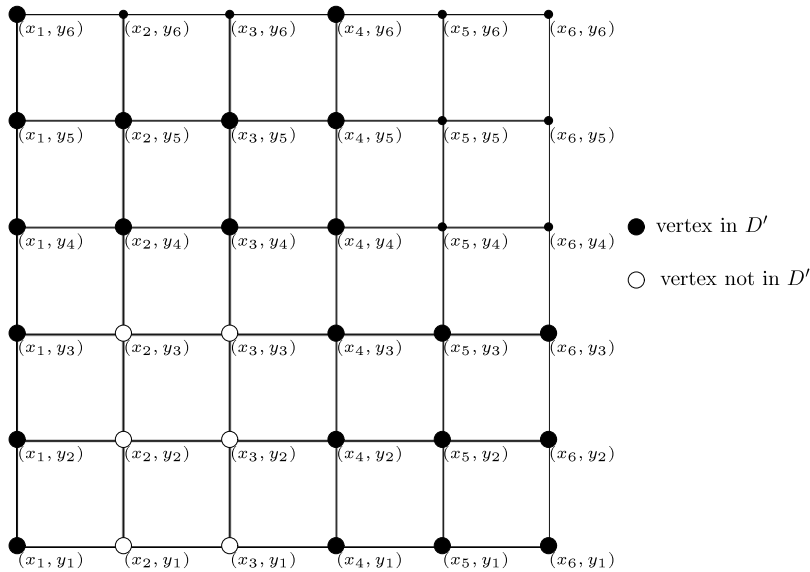


Fig. 7. Case (a): subcase  $(x_3, y_5) \in D'$  of  $C_6 \square C_6$ .

G. Without loss of generality, let  $(x_3, y_1), (x_3, y_2), (x_3, y_3) \notin D'$ . Now, by Lemma 4, we obtain that  $(x_l, y_k) \notin D'$  for  $l \in \{2, 3\}$  and  $k \in \{1, 2, 3\}$  and  $(x_1, y_2) \in D', (x_4, y_2) \in D'$ . Again we conclude that  $m \leq 6$ .

Moreover, by similar reasoning as in the proof of previous proposition, we obtain that  $(x_l, y_k) \in D'$  for  $l \in \{1, 4, 5, 6\}$  and  $k \in \{1, 2, 3\}$ .

(a) If  $(x_3, y_4) \in D'$ , then by considering  $((x_3, y_4), (x_4, y_3))$ -geodesics we conclude that  $(x_4, y_4) \in D'$ . Similarly,  $(x_1, y_4)$  and  $(x_2, y_4)$  belong to a  $((x_3, y_4), (x_1, y_3))$ -geodesic and thus they belong to  $D'$ . If  $(x_3, y_5) \in D'$ , then each  $((x_3, y_5), (x_4, y_1))$ -geodesic in  $D'$  contains  $(x_4, y_6)$ , so  $(x_4, y_6) \in D'$  and further  $(x_4, y_5) \in D'$ . Similarly, each  $((x_3, y_5), (x_1, y_1))$ -geodesic in  $D'$  contains  $(x_1, y_6)$ , so  $(x_1, y_6) \in D'$  and further  $(x_1, y_5) \in D'$  and thus  $(x_2, y_5) \in D'$  (see Fig. 7). Since  $D'$  is weakly convex,  $(x_2, y_6), (x_3, y_6) \in D'$  or  $(x_5, y_6), (x_6, y_6) \in D'$ . In both situations  $|D'| \geq 24 = 4m$ , a contradiction.

Therefore we conclude  $(x_3, y_5) \notin D'$ . Since  $D'$  is weakly convex,  $(x_3, y_6) \notin D'$ . If  $(x_2, y_5) \in D'$ , then each  $((x_2, y_5), (x_4, y_1))$ -geodesic would contain vertices of  $V(G) - D'$  which is impossible. Thus  $(x_2, y_5) \notin D'$  and further  $(x_2, y_6) \notin D'$ . Since  $D'$  is dominating,  $(x_1, y_6) \in D', (x_4, y_6) \in D'$ . Moreover, the weakly convexity of  $D'$  implies that  $(x_l, y_5) \in D'$  for  $l \in \{1, 4, 5, 6\}$  and  $(x_l, y_6) \in D'$  for  $l \in \{5, 6\}$  (see Fig. 8). However then  $|D'| \geq 24$ , a contradiction.

(b) If  $(x_3, y_4) \notin D'$ , then by the similar reasoning as in the proof of Proposition 8, we obtain that  $(x_2, y_4) \notin D'$  and  $(x_l, y_4) \in D'$  for  $l \in \{1, 4, 5, 6\}$ . Now, the rest of the proof is similar to the proof of the Case (a) and (b) of Proposition 8.  $\square$

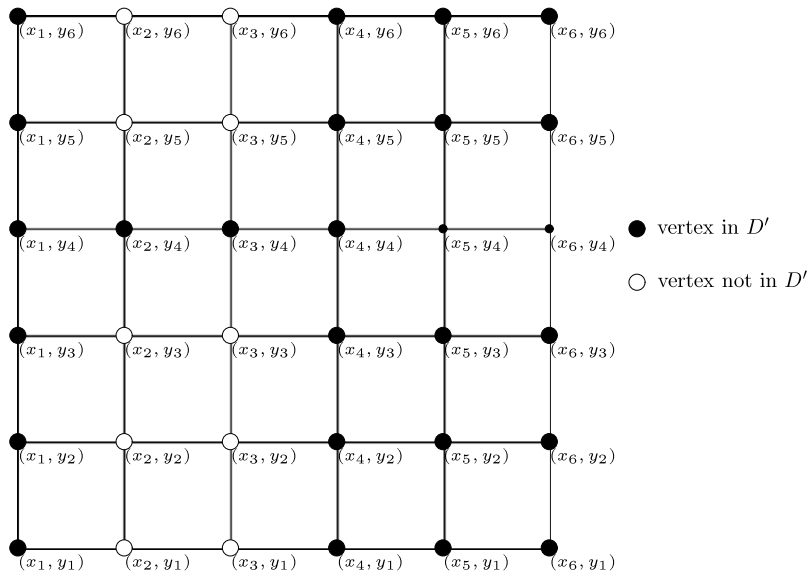


Fig. 8. Case (a): subcase  $(x_3, y_5) \notin D'$  of  $C_6 \square C_6$ .

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