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All graphs with restrained domination number three less than their order

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Abstract

For a graph $G = (V, E)$, a set $S \subseteq V$ is a restrained dominating set if every vertex not in S is adjacent to a vertex in S as well as another vertex in $V - S$. The restrained domination number of G, denoted by $\gamma_r(G)$, is the smallest cardinality of a restrained dominating set of G. In this paper we find all graphs G satisfying $\gamma_r(G) = n - 3$, where n is the order of G.

1 Introduction

All graphs considered in this paper are finite, undirected, without loops and multiple edges. Let $G = (V, E)$ be a graph with the vertex set V and the edge set E. Then we use the convention $V = V(G)$ and $E = E(G)$. The number of vertices of G is called the *order* of G and is denoted by $n(G)$. When there is no confusion we can use the abbreviation $n(G) = n$. Let C_n and P_n denote the cycle and the path of order n, respectively. The *open neighborhood* of a vertex $v \in V$ in G is denoted $N_G(v) = N(v)$ and defined by $N(v) = \{u \in V : vu \in E\}$ and the *closed neighborhood* $N[v]$ of v is $N[v] = N(v) \cup \{v\}$. For a set S of vertices the *open neighborhood* $N(S)$ is defined as the union of open neighborhoods $N(v)$ of vertices $v \in S$, the *closed neighborhood* $N[S]$ of S is $N[S] = N(S) \cup S$. The *degree* $d_G(v) = d(v)$ of a vertex v in G is the number of edges incident to v in G; by our definition of a graph, this is equal to $|N(v)|$. A *leaf* in a graph is a vertex of degree one, however a *stem* is a vertex adjacent to a leaf.

In the present paper we continue the study of restrained domination. Problems related to restrained domination in graphs appear in [1–5]. A set $S \subseteq V$ is a *restrained dominating set*, denoted **RDS**, if every vertex in $V - S$ is adjacent to a vertex in S and another vertex in $V - S$. The *restrained domination number* of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a restrained dominating set of G. We will call a set S a γ_r*-set* if S is a restrained dominating set of cardinality $\gamma_r(G)$. The solutions

G of the graph-equations $\gamma_r(G) = n$ and $\gamma_r(G) = n-2$ are well-known (see [1]). In this paper we describe all graphs G satisfying

$$\gamma_r(G) = n - 3. \tag{1}$$

For this purpose we need the following statements.

Fact 1. *Let a graph G be a subgraph of a graph G', written as $G \subseteq G'$. If S is a* **RDS** *in G then $S' = S \cup (V(G') - V(G))$ is a* **RDS** *in G'.*

Proof. By definition of **RDS** and the equality $V(G) - S = V(G') - S'$, we obtain the thesis of this statement. □

Fact 2. *If $G \subseteq G'$ and $\gamma_r(G) = n - k$, then $\gamma_r(G') \le n' - k$, where $n = |V(G)|$, $n' = |V(G')|$, $0 \le k \le n-1$, $k \ne 1$.*

Proof. Assume that S is a γ_r-*set* in G; thus $|V(G) - S| = k$. It follows from Fact 1 that $S' = S \cup (V(G') - V(G))$ is a **RDS** in G'. Moreover, we have $|S'| = n' - |V(G') - S'| = n' - |V(G) - S| = n' - k$. Hence we obtain $\gamma_r(G') \le |S'| = n' - k$. □

2 The equation $\gamma_r = n - 3$

The main purpose of this paper is to find all graphs G for which $\gamma_r(G) = n - 3$. Assume at first that G is a tree. In this case we have the following result.

Theorem 2.1. *If T is a tree of order $n \ge 4$, then $\gamma_r(T) = n - 3$ if and only if T is obtained from P_5 or P_6 by adding zero or more leaves to the stems of the path and adding either* (1) *at least one leaf or* (2) *exactly one stem to v_3 for $P_5 = (v_1, v_2, v_3, v_4, v_5)$ and to exactly one of v_i, $i = 3, 4$, for $P_6 = (v_1, v_2, v_3, v_4, v_5, v_6)$.*

Proof. Let T be a tree described above. We shall verify that $\gamma_r(T) = n-3$. Denote by x the stem appearing in (2). If T is obtained from P_5 by the above construction, then $V(T) - \{v_2, v_3, v_4\}$ in case (1) and $V(T) - \{v_2, v_3, v_4\}$, $V(T) - \{v_2, v_3, x\}$, $V(T) - \{x, v_3, v_4\}$ in case (2) are γ_r-sets of size $n - 3$. However, if T is obtained from P_6, then $V(T) - \{v_2, v_3, v_4\}$ (for $i = 3$), $V(T) - \{v_3, v_4, v_5\}$ (for $i = 4$) in case (1) and $V(T) - \{v_2, v_3, v_4\}$, $V(T) - \{v_2, v_3, x\}$ (for $i = 3$), $V(T) - \{v_3, v_4, v_5\}$, $V(T) - \{x, v_4, v_5\}$ (for $i = 4$), $V(T) - \{v_3, v_4, x\}$ (for $i = 3, 4$) in case (2) are γ_r-sets of size $n - 3$.

Conversely, let T be a tree of order n such that $\gamma_r = n - 3$. If $\operatorname{diam}(T) \ge 6$, then T contains an induced $P_7 = (v_1, v_2, v_3, v_4, v_5, v_6, v_7)$. But then $V(T) - \{v_2, v_3, v_5, v_6\}$ is a **RDS** in T of size $n - 4$, a contradiction. Thus, consider the following cases.

Case 1: $\operatorname{diam}(T) = 5$.

Then T contains an induced $P_6 = (v_1, v_2, v_3, v_4, v_5, v_6)$.

Case 1.1: Neither v_3 nor v_4 have neighbors not on P_6.

Then $V(T) - \{v_2, v_3\}$, $V(T) - \{v_3, v_4\}$, $V(T) - \{v_4, v_5\}$ are γ_r-sets of size $n - 2$, which contradicts the assumption that $\gamma_r(T) = n - 3$.

Case 1.2: Both v_3, v_4 have neighbors not on the path.

Then $V(T) - \{v_2, v_3, v_4, v_5\}$ is a **RDS** of T, therefore $\gamma_r(T) \leq n-4$, a contradiction.

Case 1.3: Exactly one of the vertices v_3, v_4, say v_3, has neighbors not on P_6.

With respect to the condition $\operatorname{diam}(T) = 5$, new neighbors of v_3 can be only leaves or stems.

Case 1.3.1: All neighbors of v_3 not on P_6 are leaves.

In this case, $V(T) - \{v_2, v_3, v_4\}$ is a γ_r-set of T of size $n-3$.

Case 1.3.2: Neighbors of v_3 not on P_6 are at least one leaf and stems $w_1, \ldots, w_m$, $m \geq 1$.

Then $V(T) - \{v_2, v_3, v_4, w_1, \ldots, w_m\}$ is a **RDS** of T, so $\gamma_r(T) \leq n-4$, a contradiction.

Case 1.3.3: Neighbors of v_3 not on P_6 are only stems $w_1, w_2, \ldots, w_m$, $m \geq 2$.

But then $V(T) - \{v_2, v_3, v_4, w_1, \ldots, w_{m-1}\}$ is a **RDS** of T; hence $\gamma_r(T) \leq n-4$, which is a contradiction.

Case 1.3.4: A unique neighbor of v_3 not on P_6 is a stem w_1.

We now can deduce that $V(T)-\{v_2, v_3, v_4\}$, $V(T)-\{v_2, v_3, w_1\}$, $V(T)-\{v_3, v_4, w_1\}$ are γ_r-sets in T, and thus $\gamma_r(T) = n-3$.

One can obtain similar cases when v_4 has neighbors not on P_6.

Case 2: $\operatorname{diam}(T) = 4$.

Then T has an induced $P_5 = (v_1, v_2, v_3, v_4, v_5)$.

Case 2.1: The open neighborhood of v_3 in T is $N(v_3) = \{v_2, v_4\}$.

In this position $V(T) - \{v_2, v_3\}$, $V(T) - \{v_3, v_4\}$ are γ_r-sets in T of size $n-2$, which contradicts the assumption that $\gamma_r(T) = n-3$.

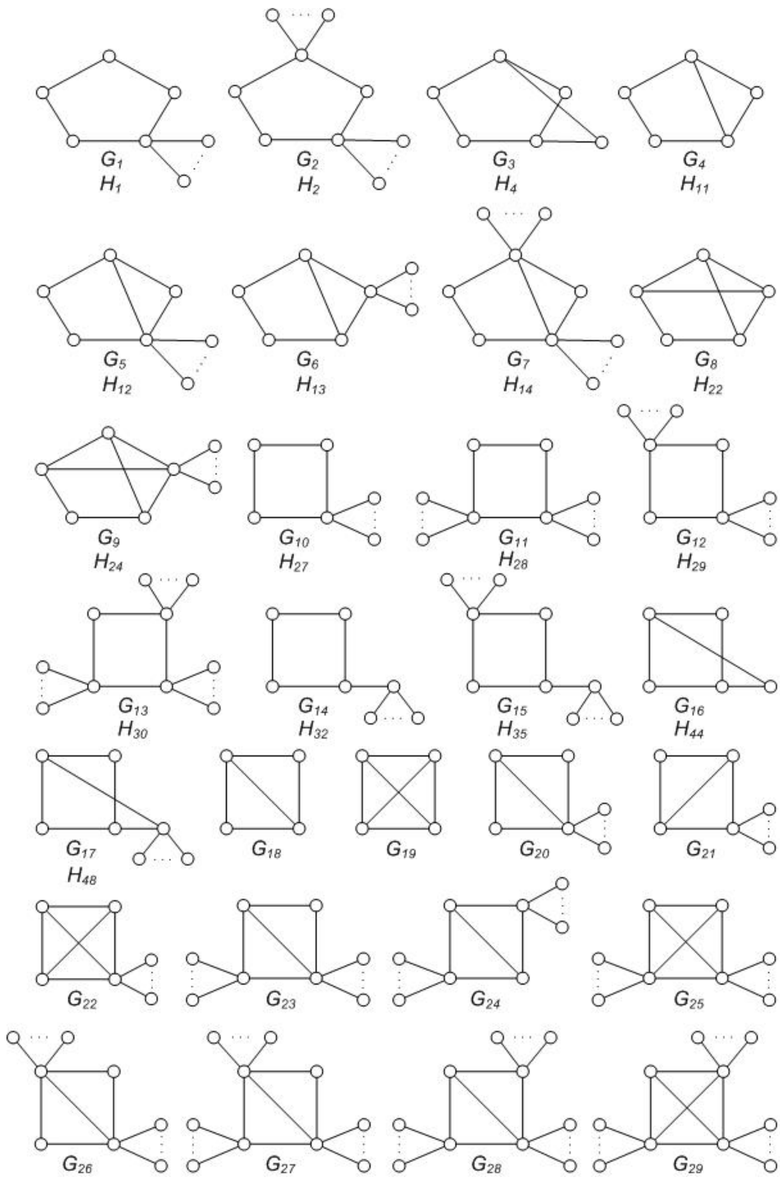
Case 2.2: Vertex v_3 has neighbors not on P_5.

According to Case 2, new neighbors of v_3 can only be leaves or stems.

Now let us replace P_6 by P_5 in cases 1.3.1, 1.3.2, 1.3.3, 1.3.4. In this way we obtain cases 2.2.1, 2.2.2, 2.2.3, 2.2.4, respectively.

The proof is complete. $\square$

Now consider connected graphs which contain a cycle. Let $\mathcal{G}$ be the collection of graphs in Figure 1.



G_1 H_1
G_2 H_2
G_3 H_4
G_4 H_{11}
G_5 H_{12}
G_6 H_{13}
G_7 H_{14}
G_8 H_{22}
G_9 H_{24}
G_{10} H_{27}
G_{11} H_{28}
G_{12} H_{29}
G_{13} H_{30}
G_{14} H_{32}
G_{15} H_{35}
G_{16} H_{44}
G_{17} H_{48}
G_{18}
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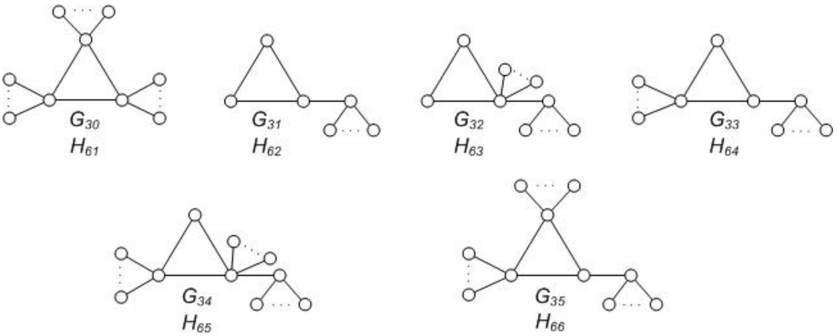

Figure 1: Graphs in family $\mathcal{G}$.

We shall show that the graphs in family $\mathcal{G}$ are the unique connected graphs containing a cycle and satisfying (1).

Theorem 2.2 *Let G be a connected graph of order $n \geq 4$ containing a cycle. Then $\gamma_r(G) = n - 3$ if and only if $G \in \mathcal{G}$.*

Proof. Let G be a connected graph of order n containing a cycle and satisfying (1). We shall prove that $G \in \mathcal{G}$. Our aim is to find all graphs G for which (1) holds. For finding graphs G we shall apply the following method.

Procedure (P): *For a connected H such that $\gamma_r(H) = n - 2$ or $\gamma_r(H) = n - 3$ find a connected G, $H \subseteq G$, satisfying* (1). *We start with $H := C_i$, $i = 3, 4, 5$.*

Note that by Fact 2 in the above procedure it suffices to consider $\gamma_r(H) = n - 2$ or $n - 3$.

We first observe that G cannot have a cycle of length at least 6. In fact, if $v_1, v_2, \ldots, v_m$, where $m \geq 6$, are consecutive vertices on a cycle, then $V(G) - \{v_1, v_2, v_4, v_5\}$ is a **RDS** for G and $\gamma_r(G) \leq n - 4$, which is a contradiction. Thus, consider the following cases.

Case 1: C_5 is the longest cycle in G.

Then in (P) we put $H := C_5$ for and find G satisfying (1). Note that $\gamma_r(H) = n - 2$.

Case 1.1: Assume that there exists an induced C_5 in G.

Case 1.1.1: Assume that the induced C_5 has neighbors not on the cycle, i.e. $I = N[V(C_5)] - V(C_5) \neq \emptyset$.

Case 1.1.1.1: Suppose all $v \in I$ are leaves.

Let H_1 be a graph obtained from C_5 by adding one or more leaves to one vertex of the cycle.

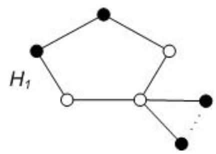

Figure 2: $H_1 \cong G_1$ satisfies (1).

Figure 2 illustrates the graph H_1, where the shaded vertices form a γ_r-set. We shall continue to use this convention in our proof. For H_1 we have $\gamma_r(H_1) = n - 3$ and $H_1 \cong G_1$, where $G_1 \in \mathcal{G}$. Now, let us add leaves to at least two vertices of C_5.

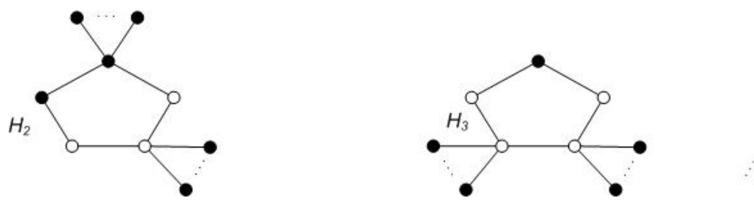

Figure 3: $H_2 \cong G_2$ satisfies (1) but H_3 does not.

Hence, by Fact 2 and case exhaustion, we can conclude that any graph obtained from C_5 by adding leaves to at least 3 vertices does not satisfy (1).

Case 1.1.1.2: Assume that there exists $v \in I$ which is not a leaf.

Case 1.1.1.2.1: Let $|N(v) \cap V(C_5)| \geq 2$.

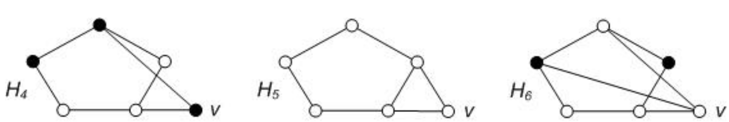

Figure 4: For $H_4 \cong G_3$ (1) holds; for H_5 and H_6 (1) does not hold.

It follows from Fact 2 and $C_6 \subseteq H_5$ that (1) does not hold for H_5. Now, by setting in (P) $H := H_4$ and adding a new vertex, we have the graphs in Fig. 5, which do not satisfy (1).

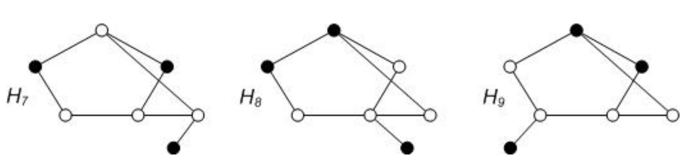

Figure 5: H_7, H_8, H_9 do not satisfy (1).

Case 1.1.1.2.2: Let $|N(v) \cap V(C_5)| = 1$.

Then we have the graph shown in Fig. 6, which does not satisfy (1).

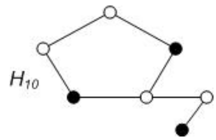

Figure 6: $\gamma_r(H_{10}) = n - 4$.

We now can deduce that H_1, H_2 and H_4 are the unique connected graphs which contain the induced C_5 and satisfy (1).

We next shall reapply the procedure (P) by adding chords to C_5.

Case 1.2: Each C_5 in G has a chord.

Case 1.2.1: Some C_5 has exactly one chord.

Case 1.2.1.1: Let us consider the graph which consists of C_5 with exactly one chord. In Fig. 7 below, $\gamma_r(H_{11}) = n - 3$.

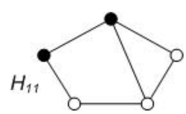

Figure 7: (1) holds for $H_{11} \cong G_4$.

Setting in (P) $H := H_{11}$ we now obtain graphs G by adding new vertices.

Case 1.2.1.2: $I = N[V(H_{11})] - V(H_{11}) \neq \emptyset$.

Case 1.2.1.2.1: Assume that all $v \in I$ are leaves.

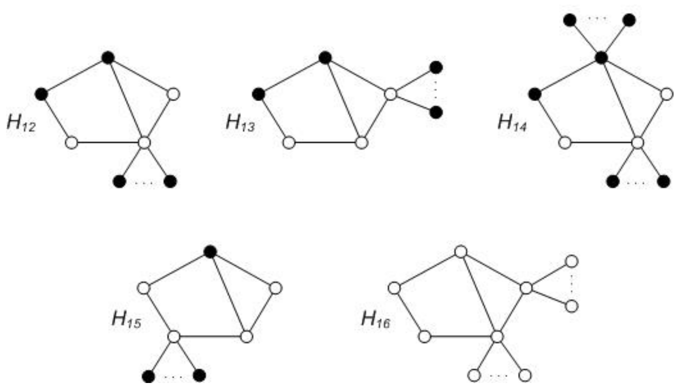

Figure 8: $H_{12} \cong G_5$, $H_{13} \cong G_6$, $H_{14} \cong G_7$ satisfy (1) and H_{15}, H_{16} do not.

It is easy to deduce that among all graphs obtained from H_{11} by adding leaves, the unique graphs satisfying (1) are H_{12}, H_{13} and H_{14} of Fig. 8. Note that (1) does not hold for H_{15}. Furthermore, $H_3 \subseteq H_{16}$ and hence (1) does not hold for H_{16}.

Case 1.2.1.2.2: Assume that there exists $v \in I$ which is not a leaf.

Case 1.2.1.2.2.1: $|N(v) \cap V(H_{11})| \geq 2$.

In the procedure (P) we put $H := H_{12}$, H_{13} or H_{14} with one leaf. In this way we obtain H_{17}, H_{18}, H_{19} and H_{20} from H_{12} and H_{21} from H_{13} or H_i, $i = 17, 18, 19, 20$, from H_{14} (see Fig. 9).
Clearly, H_{19}, H_{20} and H_{21} are the supergraphs of C_6 and hence (1) is false. Furthermore, $\gamma_r(H_{17}) = \gamma_r(H_{18}) = n - 4$.

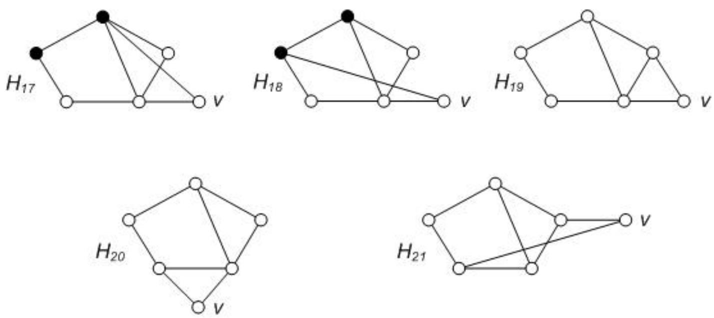

Figure 9: For $H_{17}, \ldots, H_{21}$ obtained from H_{12} and H_{13}, the equality (1) is false.

Case 1.2.1.2.2.2: $|N(v) \cap V(H_{11})| = 1$.

Then we have a supergraph of H_{10} and hence (1) does not hold.

Case 1.2.2: Each C_5 has at least two chords.

Case 1.2.2.1: Let us consider graphs with the vertex set $V(C_5)$.

One can see that it suffices to consider the two graphs of Fig. 10. We have that $\gamma_r(H_{22}) = n - 3$ and $\gamma_r(H_{23}) = n - 4$.

Figure 10: (1) holds for $H_{22} \cong G_8$ and fails for H_{23}.

Case 1.2.2.2: There exists a vertex not on C_5, i.e. $I = N[V(C_5)] - V(C_5) \neq \emptyset$.

Then in (P) we put $H := H_{22}$.

Case 1.2.2.2.1: Suppose all $v \in I$ are leaves.

Then we only consider graphs based on H_{12}, H_{13}, H_{14}. It follows that (1) holds for H_{24}, but not for H_{25} and H_{26}. Note that $H_{25} \subseteq H_{26}$ (see Fig. 11).

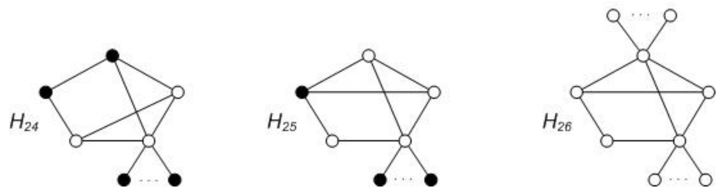

Figure 11: $H_{24} \cong G_9$ satisfies (1) and H_{25}, H_{26} do not.

Case 1.2.2.2.2: Assume that there exists $v \in I$ which is not a leaf.

In this case we obtain supergraphs of H_{10}, H_{17}, H_{18}, H_{19}, H_{20} or H_{21}, and (1) does not hold.

Case 2: C_4 is the longest cycle in G.

Case 2.1: Assume that there exists an induced C_4 in G.

Since $\gamma_r(C_4) = n - 2$ we now put $H := C_4$ in (P) and we have the following cases.

Case 2.1.1: Assume that the induced C_4 has neighbors not on the cycle, i.e. $I = N[V(C_4)] - V(C_4) \neq \emptyset$.

Case 2.1.1.1: Suppose all $v \in I$ are leaves.

Then (1) holds for H_{27}, H_{28}, H_{29} and H_{30} of Fig. 12.

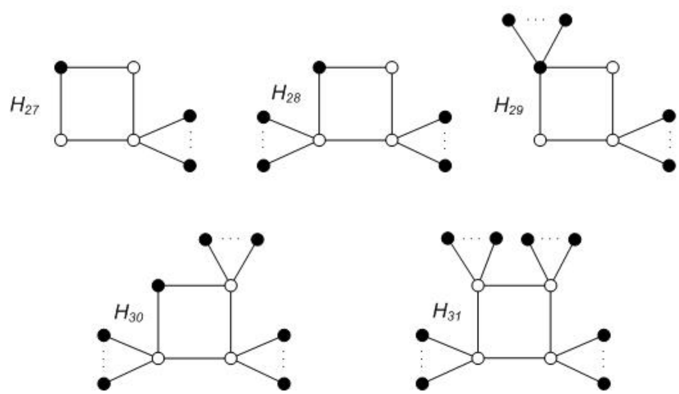

Figure 12: (1) holds for $H_{27} \cong G_{10}$, $H_{28} \cong G_{11}$, $H_{29} \cong G_{12}$, $H_{30} \cong G_{13}$ and fails for H_{31}.

Case 2.1.1.2: Assume that there exists $v \in I$ which is not a leaf.

Case 2.1.1.2.1: Suppose that $|N(v) \cap V(C_4)| = 1$.

At first we study the graphs of Fig. 13. For H_{32} and H_{35}, (1) holds.

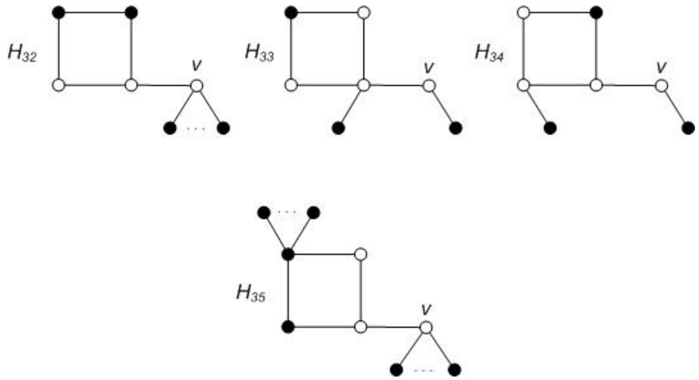

Figure 13: Only $H_{32} \cong G_{14}$ and $H_{35} \cong G_{15}$ satisfy (1).

We now consider supergraphs of H_{32} and H_{35}, depicted in Fig. 14. Let H_0 be any connected graph. Notice that $C_6 \subseteq H_{38}$, $H_{39} \cong H_4$, $H_{36} \subseteq H_{41}$, where H_0 is of order 1, and $H_{37} \subseteq H_i$, $i = 42, 43$.

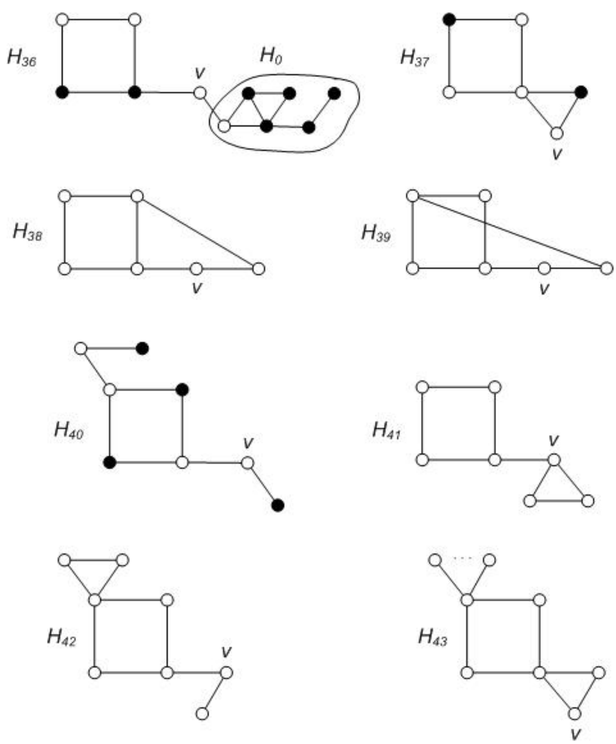

Figure 14: Only $H_{39} \cong G_3$ satisfies (1).

Case 2.1.1.2.2: Suppose that $|N(v) \cap V(C_4)| \geq 2$.

According to case 2 it suffices to consider the respective equality.

At first we check (1) for the graphs H_{44}, H_{45}, H_{46} and H_{47} of Fig. 15, which are supergraphs of H_{27}, H_{28}, H_{29}, H_{30}. It follows that (1) is true for $H_{44} \cong G_{16}$, but false for H_{45}, H_{46} and H_{47}. Note that $C_6 \subseteq H_{46}$.

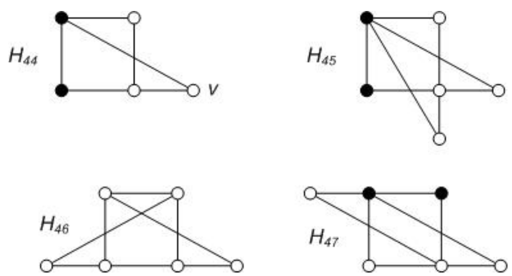

Figure 15: (1) is true for $H_{44} \cong G_{16}$ and is false for H_{45}, H_{46} and H_{47}.

Next, according to the procedure (P), we investigate supergraphs of $H := H_{44}$. Observe that it suffices to consider the graphs H_{48}, H_{49} and H_{50} of Fig. 16. For $H_{48} \cong G_{17}$ (1) holds, but for H_{49} and H_{50} it does not.

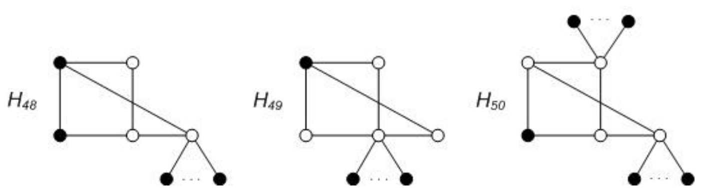

Figure 16: For $H_{48} \cong G_{17}$ (1) holds but for H_{49}, H_{50}, it does not.

We now put $H := H_{48}$ in (P) and study supergraphs of H_{48} (Fig. 17). It follows that (1) fails for H_{51} and H_{52}. Note that $H_{37} \subseteq H_{52}$ and $H_{51} \subseteq H_{52}$.

Figure 17: (1) fails for H_{51} and H_{52}.

We next consider supergraphs of H_{32} and H_{35} (Fig. 18). Since $H_{53} \cong H_{48}$, (1) holds. For H_{54} and H_{55}, it does not. Note that H_{54} and H_{55} are supergraphs of H_{49}.

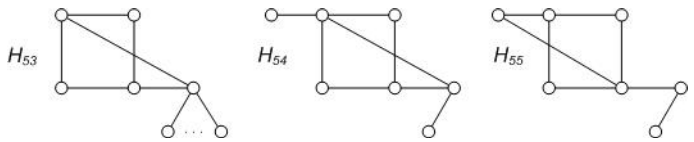

Figure 18: H_{53}, H_{54} and H_{55}.

Case 2.2: Each C_4 in G has a chord.

Let us consider the family $\mathcal{G}'$ of graphs obtained from C_4 by adding one or two chords and zero or more leaves to at least three vertices. It is easy to verify that for each $G \in \mathcal{G}'$, (1) holds, since an endvertex of a chord and the leaves form a γ_r-set. In this way, we obtain the graphs G_{18}, G_{19},..., G_{29} in the family $\mathcal{G}$.

We now must determine whether supergraphs of H_i, $i = 18, \dots, 29$, satisfy (1). One can see that it suffices to consider the graphs H_{56}, H_{57} and H_{58}, for which (1) is false. See Fig. 19.

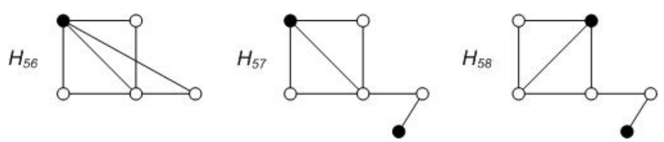

Figure 19: (1) is false for H_{56}, H_{57} and H_{58}.

It follows that each supergraph H' of H_i ($i = 18, \dots, 29$), such that $H' \notin \mathcal{G}'$, contains H_{31}, H_{37}, H_{56}, H_{57}, H_{58} or C_m, where $m \geq 5$.

Case 3: C_3 is the longest cycle in G.

Observe that in this case we have $I = N[V(C_3)] - V(C_3) \neq \emptyset$ and $|N(v) \cap V(C_3)| = 1$ for each $v \in I$. Otherwise we could obtain a cycle C_m, $m \geq 4$, in G, contrary to our assumption in this case. Therefore, let us study the graphs of Fig. 20.

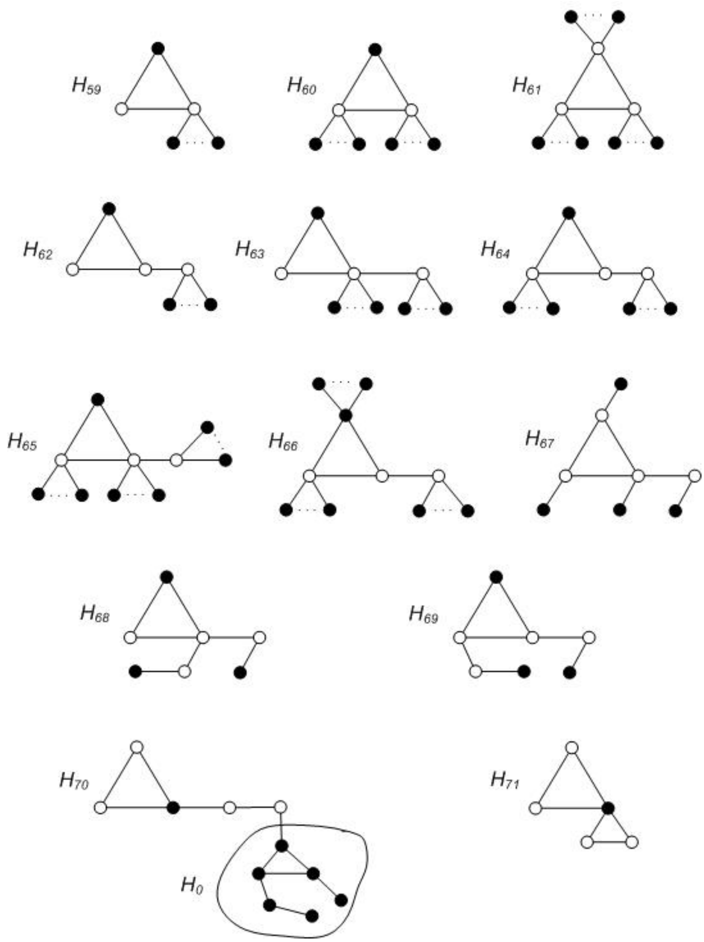

Figure 20: $H_i \cong G_{i-31}$, $i = 61, \ldots 66$, are the unique graphs for which (1) is true (H_0 is any connected graph).

Conversely, let G be any graph of the family $\mathcal{G}$. It follows from the former investigations that (1) holds for G.

This completes the proof of the theorem. $\square$.

We end this paper with the following statement.

Corollary 2.1. *If G is a graph of order $n \geq 4$, then $\gamma_r(G) = n - 3$ if and only if exactly one of the components of G is isomorphic to a graph given in Theorems* 2.1 *or* 2.2 *and every other component is a star or K_1.*

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