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# Minimal number of periodic points for smooth self-maps of $\mathbb{R}P^3$ ☆

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ABSTRACT

Let $f$ be a smooth self-map of 3-dimensional real projective space $\mathbb{R}P^3$ and $r$ be a fixed natural number. In this paper we determine the minimal number of $r$-periodic points in the smooth homotopy class of $f$.



## 1. Introduction

Nielsen type numbers, introduced in the 1980s by B. Jiang in [15], are topological invariants which estimate the minimal number of periodic points in the homotopy class of a given map. The invariant $NF_r(f)$ provides a lower bound for the number of $r$-periodic points ($r$ is fixed) in the homotopy class of $f$, a self-map of a compact manifold $M$. Later, in [12], it was proved that if $m = \dim M \geqslant 3$ then $NF_r(f)$ is the best such lower bound, i.e.

$$NF_r(f) = \min\{\#\operatorname{Fix}(g^r)\colon\ g \sim f\}. \tag{1.1}$$

In the recent paper [5] the authors defined the counterpart of such invariant in the smooth category. The new invariant, denoted as $NJD_r[f]$, gives the minimum in the formula (1.1) for smooth $f$, and for $g$ in its smooth homotopy class. (We may also consider continuous $f$ and search for the minimum over smooth $g$ in the continuous homotopy class. Both approaches are equivalent in the common domain, because every smooth homotopy may be approximated by a continuous one.)

For $r = 1$, the continuous and smooth category coincide, as was demonstrated by B. Jiang in [16], namely: $NF_1(f) = NJD_1[f] = N(f)$, where $N(f)$ denotes the Nielsen number of a smooth $f$. However, if $r > 1$ then the invariants are quite different. Obviously $NJD_r[f] \geqslant NF_r(f)$ and the inequality is usually sharp, which follows from the fact that in the smooth case there is an additional obstacle to minimizing the number of periodic points, in addition to the Reidemeister relations. This obstacle may be expressed in terms of local fixed point indices of iterates $\{\operatorname{ind}(f^n, x_0)\}_{n=1}^{\infty}$, where $x_0$ is a periodic

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point. As a result, to calculate $NJD_r[f]$ one must know also all forms of local indices of iterates of a smooth map in the given dimension $m$.

It is remarkable that the invariants differ in the simply-connected case. If $M$ is simply-connected then the Nielsen relation is trivial, and so $NF_r(f)$ is less or equal than 1. On the other hand, $NJD_r[f]$ (which is then denoted as $D_r[f]$) is usually greater than 1 [3].

Making use of the complete list of indices of a smooth map, provided in dimension 3 in [6], we were able to determine $D_r[f]$ for some simply-connected manifolds: $S^2 \times I$ [3], $S^3$ [4], and the two-holed 3-dimensional closed ball [2].

While finding the invariant $NF_r(f)$ is in general a difficult task (but in the last thirty years it was computed in many special cases, see for example [7–11,13,17–19]), computing $NJD_r[f]$ seems to be even more complicated. However, it is possible if the self-maps of the manifold have simple Reidemeister relations.

In this paper we determine $NJD_r^3[f]$ for all self-maps of 3-dimensional real projective space $\mathbb{R}P^3$. The case of even degree $\beta$ with arbitrary $r$ and odd $\beta$ with odd $r$ reduce to the simply-connected case, see Remark 2.1 and Theorem 4.9, respectively. The results for odd $\beta$ and even $r$ are gathered in the main Theorem 10.10.

## 2. Orbits of Reidemeister classes and degree of self-maps of $\mathbb{R}P^3$

Let $f$ be a self-map of 3-dimensional real projective space $\mathbb{R}P^3$, and let $n$ be a natural number. In this section we give some basic information about the degree and its relation to the Reidemeister classes of $f^n$, denoted by $\mathcal{R}(f^n)$, and the orbits of the Reidemeister classes of $f^n$, denoted by $\mathcal{OR}(f^n)$ (see [14] for the definition).

Interpreting $\mathbb{R}P^3$ as the quotient space of $S^3$ with identifying relation given by the antipodal action of $\mathbb{Z}_2$, we get the universal covering $p : S^3 \to \mathbb{R}P^3$, and thus the fundamental group $\pi_1\mathbb{R}P^3 = \mathbb{Z}_2$.

The space $\mathbb{R}P^3$ is oriented, thus for its self-map $f$ we may define the degree $\beta = \deg(f)$. Then the Lefschetz number $L(f)$ is related with $\beta$ by the formula $L(f) = 1 - \beta$. There is the following characterization of the Reidemeister classes with regard to the parity of $\beta$ [11].

- If $\beta$ is even then the homotopy group homomorphism $f_\# : \pi_1\mathbb{R}P^3 \to \pi_1\mathbb{R}P^3$ is the zero map and $\mathcal{R}(f^n) = \mathcal{OR}(f^n) = \{*\}$, a singleton set.
- If $\beta$ is odd then $f_\#$ is the isomorphism, hence $\mathcal{R}(f^n) = \mathcal{OR}(f^n) = \mathbb{Z}_2$ for all $n \in \mathbb{N}$.

**Remark 2.1.** If $\beta$ is even then $\mathcal{OR}(f^n) = \{*\}$ for all $n$, which reduces the problem of finding $NJD_r[f]$ to the simply-connected case [5]. That is, $NJD_r[f] = D_r[g]$, where $g$ is any self-map of $S^3$ of degree $\beta$. As a consequence, in this case we may apply the explicit formulae for $NJD_r[f]$ for even $\beta$ found in [4, Proposition 4.1, Theorems 4.2 and 4.7]. Therefore, in the rest of this paper we will consider the case of odd $\beta$.

## 3. Indices of iterates and Reidemeister graph

In this section we introduce the notion of the Reidemeister graph and describe it for self-maps of $\mathbb{R}P^3$ of odd degree. Also, we give a description of local indices of iterates in dimension 3. In Section 4 we will define the invariant $NJD_r[f]$ in terms of the Reidemeister graph and indices of iterates.

### 3.1. Reidemeister graph for $\mathbb{R}P^3$

In the set of orbits of the Reidemeister classes we define the natural map induced by inclusion of respective Nielsen classes. If $N^l \subset \mathrm{Fix}(f^l)$, $N^k \subset \mathrm{Fix}(f^k)$ are Nielsen classes representing the Reidemeister classes $A^l \in \mathcal{OR}(f^l)$ and $A^k \in \mathcal{OR}(f^k)$ respectively, then $N^l \subset N^k$ implies $i_{k,l}(A^l) = A^k$ (cf. [14]).

In the case of odd $\beta$ we know that $\mathcal{OR}(f^l) = \mathbb{Z}_2$. Let us denote $\mathcal{OR}(f^l) = \{l', l''\}$, $\mathcal{OR}(f^k) = \{k', k''\}$, where $l'$ and $k'$ correspond to the identity element in $\mathbb{Z}_2$.

The map $i_{k,l} : \mathcal{OR}(f^l) \to \mathcal{OR}(f^k)$ takes the following form (cf. [5])

$$i_{k,l}(l') = k', \tag{3.1}$$

$$i_{k,l}(l'') = \begin{cases} k'' & \text{if } \frac{k}{l} \text{ is odd,} \\ k' & \text{if } \frac{k}{l} \text{ is even.} \end{cases} \tag{3.2}$$

**Definition 3.1.** Let us consider the natural number $r$ and the set $\mathcal{OR}_r(f) = \bigcup_{k|r} \mathcal{OR}(f^k) = \bigcup_{k|r}\{k', k''\}$. In this set we introduce the partial order "$\preccurlyeq$" in the following way: $l^* \preccurlyeq k^*$, where $l^* \in \{l', l''\}$, $k^* \in \{k', k''\}$ if and only if

- $l|k$;
- $i_{k,l} : \{l', l''\} \to \{k', k''\}$ maps $l^*$ on $k^*$.

If $l^* \preccurlyeq k^*$ then we say that $l^*$ is preceding $k^*$. We use the notation $l^* \prec k^*$ if $l^* \preccurlyeq k^*$ but $l^* \neq k^*$.

**Definition 3.2.** We will interpret the partially ordered set of Reidemeister orbits as a directed graph (Hasse diagram). There is an edge from vertex $l^*$ to $k^*$ if and only if $l^* \preccurlyeq k^*$, with the convention that if $l^* \prec k^* \prec s^*$ then we omit the edge from $l^*$ to $s^*$, understanding that there is the connection between these two vertices through $k^*$.

We call this graph the graph of Reidemeister orbits for $f$ and denote it by $\mathcal{GOR}(f;r)$.

**Example 3.3.**

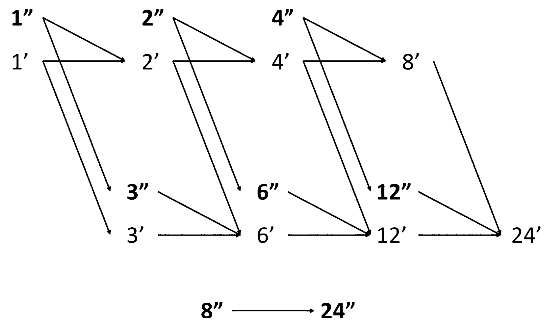


**Fig. 1.** Reidemeister graph for odd $\beta$, $r = 24$.

We will represent $r$ in the form $r = 2^R \cdot P$, where $R \geqslant 0$ and $P$ is odd.

**Remark 3.4.** The Reidemeister graph treated as an undirected graph has, for odd $\beta$, two separate connected components denoted as $V_s$ (small component) and $V_b$ (big component).

More precisely, if by $V$ we denote the set of all vertices in the graph, then we define two subsets of $V$ as below:

$$V_s = \{l'' \in V\colon 2^R | l\},$$
$$V_b = V \setminus V_s.$$

We see that they are the components of $V$: for both sets $V_s$ and $V_b$ there is a representative vertex which is connected to all other vertices in the set. In the first case it is $(2^R \cdot P)''$ and in the second $(2^R \cdot P)'$. Moreover, according to (3.1) and (3.2) there is no connection between any pair of vertices in which one belongs to $V_s$ and the other to $V_b$.

*3.2. Indices of iterates in dimension 3*

We give below (Theorem 3.6) the complete list of possible sequences of local indices of iterates of a smooth map in $\mathbb{R}^3$ [6].

**Definition 3.5.** For a given $d$ we define the basic sequence:

$$\operatorname{reg}_d(n) = \begin{cases} d & \text{if } d|n, \\ 0 & \text{if } d \nmid n. \end{cases}$$

**Theorem 3.6.** *There are seven kinds of local fixed point indices of iterates for smooth maps in dimension* 3:

(A) $c_A(n) = a_1 \operatorname{reg}_1(n) + a_2 \operatorname{reg}_2(n)$,
(B) $c_B(n) = \operatorname{reg}_1(n) + a_d \operatorname{reg}_d(n)$,
(C) $c_C(n) = -\operatorname{reg}_1(n) + a_d \operatorname{reg}_d(n)$,
(D) $c_D(n) = a_d \operatorname{reg}_d(n)$,
(E) $c_E(n) = \operatorname{reg}_1(n) - \operatorname{reg}_2(n) + a_d \operatorname{reg}_d(n)$,
(F) $c_F(n) = \operatorname{reg}_1(n) + a_d \operatorname{reg}_d(n) + a_{2d} \operatorname{reg}_{2d}(n)$, *where d is odd*,
(G) $c_G(n) = \operatorname{reg}_1(n) - \operatorname{reg}_2(n) + a_d \operatorname{reg}_d(n) + a_{2d} \operatorname{reg}_{2d}(n)$, *where d is odd*.

*In all cases* $d \geqslant 3$ *and* $a_i \in \mathbb{Z}$.

**Remark 3.7.** Theorem 3.6 provides also all forms of $\{\operatorname{ind}(f^n, P)\}_{n=1}^{\infty}$, where $P$ denotes an orbit with points of minimal period $p$. Namely, there are also seven types of such sequences, to obtain them it is enough to replace all expressions $\operatorname{reg}_d$ in the sequences (A)–(G) by $\operatorname{reg}_{pd}$ [3].

## 4. Definition of $NJD_r[f]$

### 4.1. Index function

We associate with each orbit of the Reidemeister classes $n^*$ (i.e. with each vertex) an integer $I(n^*)$, namely the fixed point index of this class, $I(n^*) = \operatorname{ind}(f^n, n^*)$. In this way we obtain a function $I$ defined on the graph $\mathcal{GOR}(f; r)$.

$\mathbb{R}P^3$ is a Jiang space, thus both Nielsen classes of the given self-map of $\mathbb{R}P^3$ have equal indices (cf. [15]), so $L(f^n) = I(n') + I(n'')$. As a result,

$$I(n') = I(n'') = \frac{1-\beta^n}{2}, \tag{4.1}$$

where $\beta$ denotes the degree of the map. This function we rewrite in the form of so-called *generalized periodic expansion*.

**Definition 4.1.** For each vertex $l^*$, where $l^* \in \{l', l''\}$, we define basic integer-valued function on the graph:

$$\operatorname{Reg}_{l^*}(n^*) = \begin{cases} l & \text{if } l^* \preccurlyeq n^*, \\ 0 & \text{otherwise.} \end{cases}$$

**Example 4.2.** $\operatorname{Reg}_{3'}(6') = 3$, $\operatorname{Reg}_{3'}(6'') = 0$.

Any integer-valued function, and in particular $I$, can be represented as a combination of basic functions $\operatorname{Reg}_{l^*}$:

$$I(n^*) = \sum_{l^* \preccurlyeq n^*} a_{l^*} \operatorname{Reg}_{l^*}(n^*). \tag{4.2}$$

**Remark 4.3.** This representation, called the general periodic expansion, is unique. Moreover, in the case discussed here, when the values $I(n^*)$ come from the fixed point index, the numbers $a_{l^*}$ turn out to be integers [5].

### 4.2. Attaching sequences at vertices

Now we consider $\Gamma$, one of the sequences (A)–(G) given in Theorem 3.6. This sequence is given as a combination of reg's, written $\Gamma = \sum_{d \in O} a_d \operatorname{reg}_d$. We will say that we attach $\Gamma$ at the (fixed) vertex $l^*$ if we define the following function $\Gamma_{l^*}$ on the Reidemeister graph:

$$\Gamma_{l^*}(n^*) = \sum_{l^* \preccurlyeq (dl)^*,\, d \in O} a_d \operatorname{Reg}_{(dl)^*}(n^*). \tag{4.3}$$

**Remark 4.4.** If we attach the sequence $\Gamma = \sum_{d \in O} a_d \operatorname{reg}_d$ at the vertex $l'$, we get

$$(*) \quad \Gamma_{l'} = \sum_{(dl)|r} a_d \operatorname{Reg}_{(dl)'}.$$

If we attach the same sequence $\Gamma$ at $l''$ of the big component, we obtain

$$(**) \quad \Gamma_{l''} = \sum_{2|d,\,(dl)|r} a_d \operatorname{Reg}_{(dl)'} + \sum_{2\nmid d,\,(dl)|r} a_d \operatorname{Reg}_{(dl)''}.$$

If we attach the same sequence $\Gamma$ at $l'' = (2^R \cdot s)''$ of the small component, we obtain

$$(***) \quad \Gamma_{l''} = \sum_{s|d|P} a_d \operatorname{Reg}_{(dl)''}.$$

In each expression $(*)$–$(***)$ the summation is taken only over $d \in O$.

**Example 4.5.** Let $r = 24$. If we attach the sequence $\Gamma = \operatorname{reg}_1 - \operatorname{reg}_2 - 2\operatorname{reg}_3 + 5\operatorname{reg}_6$ of the type (G) at $1''$, we get

$$\Gamma_{1''} = \operatorname{Reg}_{1''} - \operatorname{Reg}_{2'} - 2\operatorname{Reg}_{3''} + 5\operatorname{Reg}_{6'}.$$

If we attach the same sequence at $2''$ we obtain

$$\Gamma_{2''} = \operatorname{Reg}_{2''} - \operatorname{Reg}_{4'} - 2\operatorname{Reg}_{6''} + 5\operatorname{Reg}_{12'}.$$

**Definition 4.6.** We will say that a sequence $\Gamma$ of one of the types (A)–(F) attached at the vertex $l^*$ *realizes* $a_{k^*} \operatorname{Reg}_{k^*}$ (or $a_{k^*}$ for short) if this expression appears in the right-hand side of the formula (4.3).

### 4.3. Definition of minimal decomposition

Now we represent the index function $L$ as a sum of expressions (A)–(G) attached at some vertices:

$$I(n^*) = \sum_{l^* \preccurlyeq n^*} a_{l^*} \operatorname{Reg}_{l^*}(n^*) = \Gamma^1_{l_1^*}(n^*) + \cdots + \Gamma^s_{l_s^*}(n^*). \tag{4.4}$$

Each such decomposition determines the sum $l_1 + \cdots + l_s$, which we call *the decomposition number*.

**Definition 4.7.** We define the number $NJD_r[f]$ as the minimal decomposition number which can be obtained over all possible decompositions.

In the further part of the paper we will use the following notation: by $NJD_r[f, V_s]$ and $NJD_r[f, V_b]$ we will denote the minimal decomposition number taken over vertices in (4.4) separately for each component $V_s$ or $V_b$, respectively.

We will also consider minimal numbers for some special subsets of $\mathcal{GOR}(f; r)$.

The following theorem was proved in [5].

**Theorem 4.8.** *The number $NJD_r[f]$ satisfies*:

(1) *$NJD_r[f]$ is a homotopy invariant,*
(2) $\#\operatorname{Fix}(f^r) \geqslant NJD_r[f]$,
(3) *$f$ is homotopic to a smooth map $g$ realizing the number $NJD_r[f]$ i.e. $\#\operatorname{Fix}(g^r) = NJD_r[f]$.*

Let us notice that in the computation of $NJD_r[f]$ only the case of even $r$ needs careful analysis. If $r$ is odd then all $i_{k,l}$ are isomorphisms and the Reidemeister graph splits into two connected components with the same number of vertices.

Let $\zeta(r)$ be the number of all divisors of $r$ (including 1). Applying simply-connected methods to each of them, we get the following theorem proved in [5]:

**Theorem 4.9.** *Let $f : \mathbb{R}P^3 \to \mathbb{R}P^3$ be a smooth map of odd degree $\beta$, and let $r$ be odd, then*

$\underline{\text{for } \beta = \pm 1,}$

$$NJD_r[f] = \begin{cases} 0 & \text{if } \beta = 1, \\ 2 & \text{if } \beta = -1; \end{cases}$$

$\underline{\text{for } |\beta| \geqslant 3,}$

$$NJD_r[f] = \begin{cases} 2\zeta(r) - 2 & \text{if } -2\zeta(r) + 3 \leqslant \beta \leqslant 2\zeta(r) - 1, \\ 2\zeta(r) & \text{otherwise.} \end{cases}$$

### 4.4. Geometrical interpretation of $NJD_r[f]$

In this subsection we will provide the geometrical background of the invariant $NJD_r[f]$ and compare the continuous and smooth categories.

If $f$ is a continuous map, then the minimization of the number of the points in $\operatorname{Fix}(g^r)$ for all $g$ homotopic to $f$ is a classical problem. In the continuous category this minimal number is given by $NF_r(f)$ – the invariant introduced by Jiang in [15]. Let us briefly sketch the definition of $NF_r(f)$. A subset $S \subset \mathcal{OR}_r(f)$ is called *a Preceding System* if each essential orbit in $\mathcal{OR}_r(f)$ is preceded by an orbit in $S$. $S$ is called *a Minimal Preceding System* (MPS) if the sum of the depths of elements in $S$

$$\sum_{H \in S} \mathrm{d}(H)$$

is minimal. The invariant $NF_r(f)$ is defined as the minimum of the above sum i.e. the sum of depth of orbits in an MPS (see [14] for details).

In fact, $NF_r(f)$ has a clear geometrical interpretation, namely in each $H \in S$ we can create an orbit with $d(H)$ elements and remove other periodic points in the homotopy class of $f$ [12], so that the number of elements in the created orbits realizes $NF_r(f)$.

We may think of $\mathcal{OR}_r(f)$ as the Reidemeister graph. Then, in order to calculate $NF_r(f)$ it is enough to sum up all the contributions which come from the minimal elements $S$ of the graph (i.e. vertices that has no other preceding vertices).

Notice that for calculating $NF_r(f)$ we do not care about the values of indices at the created orbits. Indeed, the only information we need is whether the indices are non-zero (the class is essential) or not.

Now, let us discuss the definition of $NJD_r[f]$.

First of all let us mention that in the smooth category, unlike the continuous one, a sequence of local fixed point indices of iterates at an isolated periodic point $x_0 \in \mathbb{R}^N$ takes very special forms [1], let us call them special expressions. For example any such sequence must be periodic. The list of all possible *special expressions* in dimension $N = 3$ was given in Theorem 3.6.

Now, in order to calculate $NJD_r[f]$ we attach some sequences at the vertices of the graph. This is equivalent to creating orbits with the special expressions as the indices of iterates. In the smooth case we have to realize, by the sum of these special expressions, all the Lefschetz numbers of iterates (for $\mathbb{R}P^3$ given by the coefficients $a_{l^*}$ of the formula (4.2)).

If, during the calculation of $NJD_r[f]$, we attach in each $H$ of a given MPS some special expression, that may not be enough, because some coefficients $a_{l^*}$ may not be realized. As a consequence, usually $NJD_r[f] > NF_r(f)$ and the equality holds only in very special situations.

## 5. Coefficients of general periodic expansions

For even $r$ we give explicit formulae for the coefficients of the expansion

$$I(n^*) = \sum_{l^* \preccurlyeq n^*} a_{l^*} \operatorname{Reg}_{l^*}(n^*),$$

where $I(n^*) = \frac{1-\beta^n}{2}$, for a map $f : \mathbb{R}P^3 \to \mathbb{R}P^3$ of odd degree $\beta$.

Let $\mu$ denote the classical Möbius function, i.e. $\mu : \mathbb{N} \to \mathbb{Z}$ is defined by the following three properties: $\mu(1) = 1$, $\mu(k) = (-1)^r$ if $k$ is a product of $r$ different primes, $\mu(k) = 0$ otherwise.

We will also consider a generalized version of the Möbius function on $\mathcal{GOR}(f; r)$ which we denote by $\mu^\diamond$. We recall that for a set partially ordered by $\preccurlyeq$, the interval $[B, A]$ is the set of elements $C$ satisfying $B \preccurlyeq C \preccurlyeq A$.

**Definition 5.1.** Let $\mu^\diamond$ be the Möbius function on the partially ordered set $(\mathcal{GOR}(f; r), \preccurlyeq)$, i.e. $\mu^\diamond : \operatorname{Int}(\mathcal{GOR}(f; r)) \to \mathbb{Z}$, where $\operatorname{Int}(\mathcal{GOR}(f; r))$ denotes the set of all intervals in $\mathcal{GOR}(f; r)$, and $\mu^\diamond$ is defined by two properties:

- $\mu^\diamond[B, B] = 1$,
- $\mu^\diamond[B, A] = -\sum_{\{C:\ B \preccurlyeq C \prec A\}} \mu^\diamond[B, C]$.

**Lemma 5.2.** *([5]) Let $k^*, n^*$ be two vertices of $\mathcal{GOR}(f; r)$. If $k^* \preccurlyeq n^*$, then $\mu^\diamond[k^*, n^*] = \mu(\frac{n}{k})$.*

**Theorem 5.3.** *Assume $r = 2^R \cdot P$, $R \geqslant 1$, where $P$ is odd, then*

$$a_{p'} = a_{p''} = \frac{1}{2p} \sum_{l|p} \mu\left(\frac{p}{l}\right)\left(1 - \beta^l\right) \tag{5.1}$$

*for each $p|P$.*

$$a_{(2^s p)''} = \frac{1}{2^{s+1} p} \sum_{l|p} \mu\left(\frac{p}{l}\right)\left(1 - \beta^{2^s l}\right) \tag{5.2}$$

*for each $p|P$ and $1 \leqslant s \leqslant R$.*

$$a_{(2^s p)'} = \frac{-1}{2^{s+1} p} \sum_{l|p} \mu\left(\frac{p}{l}\right)\left(1 - \beta^{2^{s-1} l}\right)^2 \tag{5.3}$$

*for each $p|P$ and $1 \leqslant s \leqslant R$.*

**Proof.** By (4.2) and Remark 4.3 for every $k^*$ a unique integer number $a_{k^*}$ is defined such that for any given $n^*$ the following equality holds

$$I(n^*) = \sum_{k^* \preccurlyeq n^*} a_{k^*} \operatorname{Reg}_{k^*}(n^*). \tag{5.4}$$

Then, by the Möbius inversion formula for partially ordered sets, we find that (5.4) is equivalent, for $k^* \preccurlyeq n^*$, to

$$a_{n^*} n = \sum_{k^* \preccurlyeq n^*} \mu^\diamond[k^*, n^*] I(k^*), \tag{5.5}$$

or by Lemma 5.2, to

$$a_{n^*}=\frac{1}{n}\sum_{k^*\preccurlyeq n^*}\mu\left(\frac{n}{k}\right)I\left(k^*\right). \tag{5.6}$$

Now, the parts (1) and (2) of the theorem follow easily from the application of the formula (5.6) for appropriate $n^*$.

Now we present the calculations for part (3) which are a bit more complicated. Applying the formula (5.6) for $n^*=(2^s p)'$ we get

$$a_{(2^s p)'}=\frac{1}{2^s p}\sum_{k^*\preccurlyeq(2^s p)'}\mu\left(\frac{2^s p}{k}\right)I\left(k^*\right).$$

By (3.1) and (3.2) we can establish the form of elements which precede $(2^s p)'$ in the order $\preccurlyeq$. These are $k'$ such that $k|2^s p$ and $k''$ such that $\frac{2^s p}{k}$ is even. Taking into account that $\{k|2^s p\colon \frac{2^s p}{k} \text{ is even}\}=\{k\colon k|2^{s-1}p\}$, we get

$$a_{(2^s p)'}=\frac{1}{2^s p}\left[\sum_{k|2^s p}\mu\left(\frac{2^s p}{k}\right)I\left(k'\right)+\sum_{k|2^{s-1}p}\mu\left(\frac{2^s p}{k}\right)I\left(k''\right)\right]. \tag{5.7}$$

Using the formula (4.1) for $I(k^*)$ and rearranging the above sum we get

$$\begin{aligned} a_{(2^s p)'}2^s p&=\sum_{k|2^s p}\mu\left(\frac{2^s p}{k}\right)\frac{1-\beta^k}{2}+\sum_{k|2^s p}\mu\left(\frac{2^s p}{k}\right)\frac{1-\beta^k}{2}-\sum_{2^s|k|2^s p}\mu\left(\frac{2^s p}{k}\right)\frac{1-\beta^k}{2}\\ &=\sum_{k|2^s p}\mu\left(\frac{2^s p}{k}\right)\left(1-\beta^k\right)-\sum_{l|p}\mu\left(\frac{p}{l}\right)\frac{1-\beta^{2^s l}}{2}. \end{aligned} \tag{5.8}$$

As $\mu(\frac{2^s p}{k})=0$ for $k|2^{s-2}$ we get

$$\begin{aligned} \sum_{k|2^s p}\mu\left(\frac{2^s p}{k}\right)\left(1-\beta^k\right)&=\sum_{2^{s-1}|k|2^s p}\mu\left(\frac{2^s p}{k}\right)\left(1-\beta^k\right)=\sum_{l|2p}\mu\left(\frac{2p}{l}\right)\left(1-\beta^{2^{s-1}l}\right)\\ &=\sum_{l|p}\mu\left(\frac{2p}{l}\right)\left(1-\beta^{2^{s-1}l}\right)+\sum_{2|l|2p}\mu\left(\frac{2p}{l}\right)\left(1-\beta^{2^{s-1}l}\right)\\ &=-\sum_{l|p}\mu\left(\frac{p}{l}\right)\left(1-\beta^{2^{s-1}l}\right)+\sum_{l|p}\mu\left(\frac{p}{l}\right)\left(1-\beta^{2^s l}\right), \end{aligned} \tag{5.9}$$

where in the last equality we use the fact that $\mu$ is a multiplicative function.

Finally, we obtain

$$a_{(2^s p)'}2^s p=\sum_{l|p}\mu\left(\frac{p}{l}\right)\left[\frac{1-\beta^{2^s l}}{2}-\left(1-\beta^{2^{s-1}l}\right)\right]=-\sum_{l|p}\mu\left(\frac{p}{l}\right)\frac{(1-\beta^{2^{s-1}l})^2}{2}, \tag{5.10}$$

which gives the formula (3). $\square$

**Corollary 5.4.** *Applying Theorem* 5.3, *formula* (5.2) *for* $p=1$ *we get*

$$a_{(2^s)''}=\frac{1-\beta^{2^s}}{2^{s+1}}.$$

**Lemma 5.5.** *If* $|\beta|\geqslant 3$ *then* $a_{n^*}\neq 0$.

**Proof.** Let $n=2^s p$, where $p$ is odd, $s\geqslant 0$. We see from the formulas (5.2) and (5.3) that $a_{(2^R p)^*}\neq 0$, for $p=1$. Assume that $p\geqslant 3$.

We have the following estimate

$$|a_{n^*}|\alpha=\left|\sum_{l|p}\mu\left(\frac{p}{l}\right)A_l\right|\geqslant|A_p|-(2\sqrt{p}-1)|A_{p/q}|, \tag{5.11}$$

where $q$ is the smallest prime divisor of $p$, $\alpha \neq 0$ and $A_l$ are the respective numbers taken from the formulas (5.1)–(5.3) depending on the form of $n^*$. To obtain the inequality in (5.11) we used the fact that the number of divisors of $p$ is not greater then $2\sqrt{p}$.

We will show that $|a_{n^*}| > 0$, which is equivalent to the following inequality:

$$|A_p| > (2\sqrt{p} - 1)|A_{p/q}|. \tag{5.12}$$

In the case of $n^* = p'$, by (5.1) $A_l = 1 - \beta^l$, so the formula (5.12) takes the form:

$$\left|\frac{\beta^p - 1}{\beta^{(p/q)} - 1}\right| > 2\sqrt{p} - 1. \tag{5.13}$$

As $p \geqslant 3$ and $|\beta| \geqslant 3$ we have

$$\begin{aligned}\left|\frac{\beta^p - 1}{\beta^{(p/q)} - 1}\right| &\geqslant \frac{|\beta|^p - 1}{|\beta|^{(p/q)} + 1} \geqslant \frac{|\beta|^p - 1}{|\beta|^{(p/3)} + 1}\\ &\geqslant \frac{\frac{26}{27}|\beta|^p}{\frac{4}{3}|\beta|^{(p/3)}} = \frac{13}{18}|\beta|^{(2p/3)}.\end{aligned} \tag{5.14}$$

Thus, it is enough to check whether

$$\frac{13}{18}|\beta|^{(2p/3)} > 2\sqrt{p}, \tag{5.15}$$

which implies $|a_{p'}| > 0$.

Denote $\tilde{\beta} = |\beta|$ and consider the map $G(\tilde{\beta}, p) = \frac{13}{18}\tilde{\beta}^{(2p/3)} - 2\sqrt{p}$. We see that it is positive for $(\tilde{\beta}, p) = (3, 3)$ and that its partial derivatives are positive for all $\tilde{\beta}, p \geqslant 3$, which gives the inequality (5.15). This ends the proof that $a_{p'} \neq 0$.

Now the proof for the rest $n^*$ is very similar and follows from the formula (5.13). Namely, in order to show that $a_{(2^s p)''} \neq 0$ and $a_{(2^s p)'} \neq 0$ (i.e. that the formula (5.12) holds for respective values of $A_l$), we just use (5.13) but instead of $\beta$ we consider $\beta^{2^s}$ or $\beta^{2^{s-1}}$, respectively. □

**Corollary 5.6.** *Calculating $NJD_r[f]$ one must take into account that for $|\beta| \geqslant 3$ each coefficient $a_{l^*}$ in* (4.4) *is non-zero* (*by Lemma* 5.5) *and should be realized by some sequence* (A)–(G) *attached at one of vertices.*

## 6. Minimal decompositions

Now we prove that to calculate $NJD_r[f]$ it is enough to confine ourselves to some special decompositions.

**Lemma 6.1.** *Let $I(n^*) = \Gamma^1_{l_1^*}(n^*) + \cdots + \Gamma^s_{l_s^*}(n^*)$ be a decomposition of Lefschetz numbers. Then there always exists another decomposition $I(n^*) = \Gamma^1_{k_1^*}(n^*) + \cdots + \Gamma^t_{k_t^*}(n^*)$, where $k_i = 1', 2', (2^j)''$, $j = 0, 1, \ldots$, with a smaller or equal decomposition number.*

**Proof.** Let $l_i = 2^j \cdot k$, where $k \geqslant 3$ is odd. If $\Gamma_{l_i'}$ is one of the sequences (A)–(G) attached at $l_i'$, then we may (cf. Lemma 4.8 [3]) replace it by no more than 3 sequences attached at $1'$.

Now let us consider $\Gamma_{l_i''}$ which is one of the following sequences attached at $l_i''$:

(1) (A)–(C),
(2) (D),
(3) (E)–(F),
(4) (G).

We will consider all the above cases in the reverse order.

**Case (4).** Let $\Gamma$ be of the type (G) i.e. $\Gamma = \mathrm{reg}_1 - \mathrm{reg}_2 + a_d \,\mathrm{reg}_d + a_{2d} \,\mathrm{reg}_{2d}$. Then, to obtain $\Gamma_{l_i''}$ we will attach the following three sequences:

(C) of the form $-\mathrm{reg}_1 + \mathrm{reg}_k$ at $(2^j)''$,
(F) of the form $\mathrm{reg}_1 + a_d \,\mathrm{reg}_{dk} + a_{2d} \,\mathrm{reg}_{2dk}$ at $(2^j)''$,
(D) of the form $-\mathrm{reg}_{2 \cdot 2^j \cdot k}$ at $1'$.

We then get by Remark 4.4 (we recall that $d$ is odd here):

$$\begin{aligned}\Gamma_{l_i''} &= \mathrm{Reg}_{l_i''} - \mathrm{Reg}_{(2\cdot l_i)'} + a_d \mathrm{Reg}_{(d\cdot l_i)''} + a_{2d} \mathrm{Reg}_{(2d\cdot l_i)'} \\ &= \mathrm{Reg}_{(2^j\cdot k)''} - \mathrm{Reg}_{(2\cdot 2^j\cdot k)'} + a_d \mathrm{Reg}_{(d\cdot 2^j\cdot k)''} + a_{2d}\, \mathrm{Reg}_{(2d\cdot 2^j\cdot k)'} \\ &= (-\mathrm{Reg}_{(2^j)''} + \mathrm{Reg}_{(2^j\cdot k)''}) + (\mathrm{Reg}_{(2^j)''} + a_d \mathrm{Reg}_{(d\cdot 2^j\cdot k)''} + a_{2d}\, \mathrm{Reg}_{(2d\cdot 2^j\cdot k)'}) + (-\mathrm{Reg}_{(2\cdot 2^j\cdot k)'}).\end{aligned} \tag{6.1}$$

**Case (3).** In the case (F) in which $\Gamma = \mathrm{reg}_1 + a_d\, \mathrm{reg}_d + a_{2d}\, \mathrm{reg}_{2d}$ is attached at $l_i''$ we will minimize the decomposition number using two sequences: (C) $-\mathrm{reg}_1 + \mathrm{reg}_k$ and (F) $\mathrm{reg}_1 + a_d\, \mathrm{reg}_{dk} + a_{2d}\, \mathrm{reg}_{2dk}$ both attached at $(2^j)''$, then again by Remark 4.4 we obtain

$$\begin{aligned}\Gamma_{l_i''} &= \mathrm{Reg}_{l_i''} + a_d\, \mathrm{Reg}_{(d\cdot l_i)''} + a_{2d} \mathrm{Reg}_{(2d\cdot l_i)'} \\ &= \mathrm{Reg}_{(2^j\cdot k)''} + a_d \mathrm{Reg}_{(d\cdot 2^j\cdot k)''} + a_{2d} \mathrm{Reg}_{(2d\cdot 2^j\cdot k)'} \\ &= (-\mathrm{Reg}_{(2^j)''} + \mathrm{Reg}_{(2^j\cdot k)''}) + (\mathrm{Reg}_{(2^j)''} + a_d \mathrm{Reg}_{(d\cdot 2^j\cdot k)''} + a_{2d} \mathrm{Reg}_{(2d\cdot 2^j\cdot k)'}).\end{aligned} \tag{6.2}$$

In a similar way by attaching two sequences (C) and (F) at $(2^j)''$ we replace the sequence of type (E) attached at $l_i''$.

**Cases (1) and (2).** Finally, in the first case we get a smaller decomposition by attaching the sequence (D) at $(2^j)''$ two times. A similar conclusion can be drawn for the second case where we use the sequence (D) only once at $(2^j)''$. □

**Corollary 6.2.** *Note that according to Lemma* 6.1 *any minimal decomposition can be transformed to another, in which we attach sequences only at vertices indexed by powers of* 2*. Therefore, in the rest of this paper we will assume that all minimal decompositions have such a form.*

**Lemma 6.3.**

$$NJD_r[f] = NJD_r[f, V_s] + NJD_r[f, V_b].$$

**Proof.** Let $I(n^*) = \sum_{l^*\preccurlyeq n^*} a_{l^*} \mathrm{Reg}_{l^*}(n^*) = \sum_{v^*\in V_s} a_{v^*} \mathrm{Reg}_{v^*}(n^*) + \sum_{w^*\in V_b} a_{w^*} \mathrm{Reg}_{w^*}(n^*)$. We recall that $r = 2^R \cdot P$, where $P$ is odd. The vertices in $V_s$ have the form $(2^R \cdot h)''$, where $h$ is odd.

Let us attach the sequence

$$\Gamma = \sum_{d\in O} a_d\, \mathrm{reg}_d \tag{6.3}$$

at the vertex $v^* \in V_s$. By the form of $V_s$, $v^* = v''$ and $v = 2^R \cdot h$, where $h$ is odd. As a result, we get $\Gamma_{v^*}(n^*) = \sum_{d\in O} a_d\, \mathrm{Reg}_{(v\cdot d)^*}(n^*)$ and

(i) either $(v \cdot d)\nmid r$, then $\mathrm{Reg}_{(v\cdot d)^*}(n^*) = 0$,
(ii) or $(v \cdot d)|r$ and then $d$ is odd, thus by Remark 4.4$(**)$ $(v \cdot d)^* = (2^R \cdot h \cdot d)'' \in V_s$.

Similarly, let $w^* \in V_b$, so either $w^* = w'$, or $w^* = (2^g \cdot h)''$, where $g < R$ and $h$ is odd. If we attach the sequence (6.3) at the vertex $w^*$ we get $\Gamma_{w^*}(n^*) = \sum_{d\in O} a_d \mathrm{Reg}_{(w\cdot d)^*}(n^*)$ and

- if $(w \cdot d)\nmid r$ then $\mathrm{Reg}_{(w\cdot d)^*}(n^*) = 0$,
- if $(w \cdot d)|r$ and $d$ is even then by Remark 4.4$(*)$ and $(**)$ we get that $(w \cdot d)^* = (w \cdot d)' \in V_b$,
- $(w \cdot d)|r$ and $d$ is odd then again by Remark 4.4 and the fact that $g < R$ we obtain $(w \cdot d)^* = (2^g \cdot h \cdot d)^* \in V_b$.

In a conclusion we observe that the problem of finding $NJD_r[f, V_s]$ and $NJD_r[f, V_b]$ are independent, because decompositions are taken over disjoint sets of vertices. □

In the next part of the paper we will determine the minimal decomposition number for even $r$ and a map $f$ of odd degree $\beta$ by finding the numbers $NJD_r[f, V_b]$ and $NJD_r[f, V_s]$.

## 7. Special cases of $\beta = \pm 1$, $r$ arbitrary and $r = 2^R$, $\beta$ odd

Let us consider the cases of $\beta = \pm 1$ with arbitrary $r$. In the case $\beta = 1$, by Theorem 5.3 all $a_n^* = 0$, thus $NJD_r[f] = 0$. In the case $\beta = -1$, we calculate the coefficients $a_n^*$ by the formulas (5.1)–(5.3). Taking into account that $\sum_{l|p} \mu(\frac{p}{l}) = 0$ for $p > 1$ we get that $a_{1'} = a_{1''} = 1$ and $a_{2'} = 1$ and all other coefficients are equal to zero. As a consequence, using the sequence (A) (attached at $1'$ and $1''$) twice we find that $NJD_r[f] = 2$.

Now we assume that $r = 2^R$ and $\beta$ is odd, $|\beta| \geqslant 3$. Let us attach at each vertex $1'', 2'', 4'', \dots, (2^{R-1})''$ sequences of the type (A). In this way we also realize $a_B \mathrm{Reg}_B$ for $B \in \{2', 4', \dots, (2^R)'\}$. If we add two more sequences of the type (A): one at $1'$ and one at $(2^R)''$ we get the decomposition of whole index function on the graph.

As a result,

$$NJD_r[f] \leqslant 1 + \sum_{s=0}^{R-1} 2^s + 2^R = 2^{R+1}. \tag{7.1}$$

On the other hand, by [11] $NF_{2^R}(f) = 2^{R+1}$. Now (7.1) and the inequality $NF_r(f) \leqslant NJD_r[f]$ imply

$$NJD_r[f] = 2^{R+1} = 2r. \tag{7.2}$$

Before we consider the general case $r = 2^R \cdot P$, $R \geqslant 1$, where $P > 1$ is odd, we will analyze a special example illustrating the problem.

**Example 7.1.** Let $r = 24 = 2^3 \cdot 3$ and $\beta = 3$ (see Fig. 1). We seek the minimal decomposition in this case. The example corresponds to the case $|\frac{1-\beta}{2}| < \zeta(P)$, $|\frac{1-\beta^{2^R}}{2^{R+1}}| \geqslant \zeta(P)$ of Theorem 10.10.

By the straightforward calculation we check that each coefficient $a_{l^*}$ is non-zero.

The procedure of finding $NJD_r[f]$ will be divided into two steps: first we cover the $a_{l''}$ coefficients in the index function $I(n^*)$, second we realize $a_{l'}$. Let us start with realizing $a_{8''}$ and $a_{24''}$. We see at once that both of them can be covered attaching at $8''$ sequences (A) and (D) respectively. This gives us the partition $8+8$ to $NJD_r[f]$. We can now proceed with $a_{2''}$, $a_{6''}$, $a_{4''}$, $a_{12''}$. The first two we can realize using sequences (A) and (B)–(G) attached at $2''$, the last two using the same sequences at $4''$. It is easy to check that using sequence (A) at $2''$ and $4''$ we cover also $a_{4'}$ and $a_{8'}$. However, for covering $a_{6''}$ and $a_{12''}$ it is more convenient to use sequences (F)–(G), because then we get the extra realization of the $a_{12'}$ and $a_{24'}$. What is left is to realize $a_{1''}$ and $a_{3''}$. In our example $a_{1''} = -1$ thus we can use one sequence (C) attached at $1''$ to realize $a_{1''}$ and $a_{3''}$. Notice that the contribution to $NJD_r[f]$ so far is $(8+8)+(2+2+4+4)+(1) = 29$.

This shows that after the first step only a few coefficients of the form $a_{l'}$ are left: $a_{1'}$, $a_{2'}$, $a_{3'}$ and $a_{6'}$. We can realize all of them attaching at $1'$ sequences (A) and (F)–(G) respectively for the first pair and for the second one.

The above decomposition is minimal in this case and $NJD_r[f] = 31$. We will prove this in the following sections.

## 8. Determination of $NJD_r[f, V_s]$

In the rest of the paper (till the final Theorem 10.10) we assume that $|\beta| > 1$.

**Theorem 8.1.**

$$NJD_r[f, V_s] = \begin{cases} 2^R(\zeta(P) - 1) & \text{if } |\frac{1-\beta^{2^R}}{2^{R+1}}| < \zeta(P), \\ 2^R \zeta(P) & \text{otherwise.} \end{cases}$$

**Proof.** According to Remark 4.4($*{*}*$), attaching at some vertex $l'' = (2^R \cdot s)''$ one of the sequences (A)–(F) gives

$$\Gamma_{l''} = \sum_{s|d|P} a_d \mathrm{Reg}_{(dl)''}.$$

As $P$ is odd, $d$ must also be odd, so we will in fact attach only sequences of one of the types (A)–(D) and only at $(2^R)''$ by Lemma 6.1. We will need at least $\zeta(P) - 1$ of them, because each of them has only one $a_k \mathrm{reg}_k$ with $k > 1$ and so realizes one $a_{(2^R \cdot k)''} \mathrm{Reg}_{(2^R \cdot k)''}$ (cf. the similar proof in [3, Theorem 4.10]). On the other hand, notice that by Corollary 5.4 the coefficient at $\mathrm{Reg}_{(2^R)''}$ is equal to $\frac{1-\beta^{2^R}}{2^{R+1}}$, so it can be obtained as a sum of $(\zeta(P) - 1)$ sequences $\pm \mathrm{reg}_1$ which are the parts of the sequences (B)–(D) if and only if

$$\left| \frac{1-\beta^{2^R}}{2 \cdot 2^R} \right| \leqslant \zeta(P) - 1,$$

otherwise we must use one additional sequence of the type (A) attached at $(2^R)''$ to realize it. $\square$

## 9. Upper bounds for $NJD_r[f, V_b]$

In this section we will find the upper bound for the decomposition number of the big component (Lemma 9.4).

We consider the sum $S = S_1 + S_2$, where

$$S_1 = \sum_{p|P} a_{p'}\mathrm{Reg}_{p'} + a_{p''}\mathrm{Reg}_{p''} + a_{(2p)'}\mathrm{Reg}_{(2p)'},$$
$$S_2 = \sum_{s=1}^{R-1}\sum_{p|P} a_{(2^s p)''}\mathrm{Reg}_{(2^s p)''} + a_{(2^{s+1}p)'}\mathrm{Reg}_{(2^{s+1}p)'}. \tag{9.1}$$

In two lemmas given below we present some decompositions of the sum $S$. Their decomposition numbers are, by the definition, upper bounds of $NJD_r[f, V_b]$. Later we will show that, under the corresponding assumptions, these bounds are equal to $NJD_r[f, V_b]$.

**Lemma 9.1.** *The minimal decomposition number for*

$$S_2 = \sum_{s=1}^{R-1}\sum_{p|P} a_{(2^s p)''}\mathrm{Reg}_{(2^s p)''} + a_{(2^{s+1}p)'}\mathrm{Reg}_{(2^{s+1}p)'}$$

*is less than or equal to* $(2^R - 2)\zeta(P)$.

**Proof.** We will find a decomposition of $S_2$ whose decomposition number is equal to $(2^R - 2)\zeta(P)$.

$$\begin{aligned} S_2 &= \sum_{s=1}^{R-1}\sum_{p|P} a_{(2^s p)''}\mathrm{Reg}_{(2^s p)''} + a_{(2^{s+1}p)'}\mathrm{Reg}_{(2^{s+1}p)'} \\ &= \sum_{s=1}^{R-1}\sum_{1\neq p|P} \mathrm{Reg}_{(2^s)''} + a_{(2^s p)''}\mathrm{Reg}_{(2^s p)''} + a_{(2^{s+1}p)'}\mathrm{Reg}_{(2^{s+1}p)'} && (F)_{/(2^s)''} \\ &\quad + \sum_{s=1}^{R-1}\big(a_{(2^s)''} - \zeta(P) + 1\big)\mathrm{Reg}_{(2^s)''} + a_{(2^{s+1})'}\mathrm{Reg}_{(2^{s+1})'}, && (A)_{/(2^s)''} \end{aligned}$$

where on the right-hand side of the above formula we indicated that the first sum is realized by the sequences of the type (F) attached at $(2^s)''$ and the second by (A) attached at $(2^s)''$.

Thus, the decomposition number in this case cannot be greater than

$$\sum_{s=1}^{R-1}\sum_{1\neq p|P} 2^s + \sum_{s=1}^{R-1} 2^s = \big(\zeta(P) - 1\big)\sum_{s=1}^{R-1} 2^s + \sum_{s=1}^{R-1} 2^s = \big(2^R - 2\big)\zeta(P). \qquad \square$$

Before we continue, we recall that $|\beta| > 1$ and $a_{1'} = a_{1''} = \frac{1-\beta}{2}$. Depending on the value of $\frac{1-\beta}{2}$ in relation to $\beta$ and $P$, none, one or both coefficients $a_{1'}$ and $a_{1''}$ may be realized as a part of other sequences which are necessary to realize some coefficients $a_{k^*}$ with $k > 1$. In the following lemma we explore this problem for a part of the Reidemeister graph determined by $S_1$.

**Lemma 9.2.** *Minimal decomposition number for*

$$S_1 = \sum_{p|P} a_{p'}\mathrm{Reg}_{p'} + a_{p''}\mathrm{Reg}_{p''} + a_{(2p)'}\mathrm{Reg}_{(2p)'}$$

*is less or equal to*

$$\begin{cases} 2\zeta(P) & \text{if } \left|\frac{1-\beta}{2}\right| \geqslant \zeta(P), \\ 2\zeta(P) - 1 & \text{if } \left|\frac{1-\beta}{2}\right| < \zeta(P) \text{ and } \beta > 0, \\ 2\zeta(P) - 1 & \text{if } \left|\frac{1-\beta}{2}\right| < \zeta(P),\ \frac{3+\beta^2}{2^2} > \zeta(P) \text{ and } \beta < 0, \\ 2\zeta(P) - 2 & \text{if } \left|\frac{1-\beta}{2}\right| < \zeta(P),\ \frac{3+\beta^2}{2^2} \leqslant \zeta(P) \text{ and } \beta < 0. \end{cases}$$

**Proof.** In each case we will find decompositions with the appropriate decomposition number.

**Case (1).** $|\frac{1-\beta}{2}| \geqslant \zeta(P)$.

$$\begin{aligned} S_1 &= \sum_{p|P} a_{p'}\mathrm{Reg}_{p'} + a_{p''}\mathrm{Reg}_{p''} + a_{(2p)'}\mathrm{Reg}_{(2p)'} \\ &= \sum_{1\neq p|P} \mathrm{Reg}_{1''} + a_{p''}\mathrm{Reg}_{p''} + a_{(2p)'}\mathrm{Reg}_{(2p)'} && (F)_{/1''} \\ &\quad + \sum_{1\neq p|P} a_{p'}\mathrm{Reg}_{p'} && (D)_{/1'} \\ &\quad + (a_{1'}\mathrm{Reg}_{1'} + a_{2'}\mathrm{Reg}_{2'}) && (A)_{/1'} \\ &\quad + \big(\big(a_{1''} - \zeta(P) + 1\big)\mathrm{Reg}_{1''} + 0\cdot\mathrm{Reg}_{2'}\big). && (A)_{/1''} \end{aligned}$$

Thus, the decomposition number is equal to $2\zeta(P)$.

**Case (2).** $|\frac{1-\beta}{2}| < \zeta(P)$ and $\beta > 0$.

Let

$$DIV_P = \{p\colon\ p|P \text{ and } p > 1\}.$$

We can then find two subsets $P_1$ and $P_2$ of $DIV_P$ which satisfy $P_1 \cap P_2 = \emptyset$, $P_1 \cup P_2 = DIV_P$ and $|P_1| = |\frac{1-\beta}{2}| = |a_{1'}|$.

Then

$$\begin{aligned} S_1 &= \sum_{p|P} a_{p'}\mathrm{Reg}_{p'} + a_{p''}\mathrm{Reg}_{p''} + a_{(2p)'}\mathrm{Reg}_{(2p)'} \\ &= \sum_{1\neq p|P} \mathrm{Reg}_{1''} + a_{p''}\mathrm{Reg}_{p''} + a_{(2p)'}\mathrm{Reg}_{(2p)'} && (F)_{/1''} \\ &\quad + \sum_{p\in P_1} -\mathrm{Reg}_{1'} + a_{p'}\mathrm{Reg}_{p'} && (C)_{/1'} \\ &\quad + \sum_{p\in P_2} a_{p'}\mathrm{Reg}_{p'} && (D)_{/1'} \\ &\quad + \big(\big(a_{1''} - \zeta(P) + 1\big)\mathrm{Reg}_{1''} + a_{2'}\mathrm{Reg}_{2'}\big). && (A)_{/1''} \end{aligned}$$

Thus, the decomposition number in this case is equal to $2\zeta(P) - 1$.

**Case (3).** $|\frac{1-\beta}{2}| < \zeta(P)$, $\frac{3+\beta^2}{2^2} > \zeta(P)$ and $\beta < 0$.

In this case $a_{1'} = \frac{1-\beta}{2} > 0$. Thus, if we use the decomposition from Case (2), but instead of the sequences (C) we use (B), we get the decomposition with decomposition number equal to $2\zeta(P) - 1$.

**Case (4).** $|\frac{1-\beta}{2}| < \zeta(P)$, $\frac{3+\beta^2}{2^2} \leqslant \zeta(P)$ and $\beta < 0$.

Notice that for $\beta \leqslant -3$ there is: $\frac{(1-\beta)^2}{4} > \frac{3+\beta^2}{2^2} > \frac{1-\beta}{2}$. We consider two subcases (4a) $\frac{(1-\beta)^2}{4} \geqslant \zeta(P) \geqslant \frac{\beta^2+3}{4}$ and (4b) $\zeta(P) > \frac{(1-\beta)^2}{4}$.

**Subcase (4a).** First we prove the following lemma:

**Lemma 9.3.** *Let $\frac{(1-\beta)^2}{4} \geqslant \zeta(P) \geqslant \frac{\beta^2+3}{4}$. Then the inequalities*

$$1 \leqslant \frac{(1-\beta)^2}{4} - \big(\zeta(P) - 1\big) \leqslant \zeta(P) - 1$$

*hold for all integers $|\beta| \geqslant 3$.*

**Proof.** Since the first inequality follows immediately from the first inequality of assumption, we concentrate on the second one. It may be rewritten as

$$\frac{(1-\beta)^2}{4} \leqslant 2\big(\zeta(P) - 1\big),$$

or

$$\frac{\beta^2 - 2\beta + 9}{8} \leqslant \zeta(P).$$

Since by the assumption $\frac{\beta^2+3}{4} \leqslant \zeta(P)$, it remains to observe that the inequality

$$\frac{\beta^2-2\beta+9}{8} \leqslant \frac{\beta^2+3}{4}$$

is equivalent to

$$0 \leqslant \beta^2+2\beta-3$$

and the last inequality holds for $|\beta| \geqslant 3$. □

For the sake of simplicity let us first consider the subcase of Case (4) defined by two additional assumptions:

(4.a1) The number $\zeta(P)-1$ is even.
(4.a2) The difference $\frac{(1-\beta)^2}{4} - (\zeta(P)-1)$ is also even.

We will indicate a decomposition of $S_1$ with the decomposition number equal to $2\zeta(P)-2$. We divide $DIV_P$, into two subsets: $P_3$ and $P_4$ which satisfy $P_3 \cap P_4 = \emptyset$, $P_3 \cup P_4 = DIV_P$ and $|P_3| = |P_4|$.

We fix subsets $\bar{P}_3 \subset P_3$, $\bar{P}_4 \subset P_4$ both of cardinality

$$\frac{1}{2} \cdot \left(\frac{(1-\beta)^2}{4} - \bigl(\zeta(P)-1\bigr)\right).$$

This is possible due to Lemma 9.3. Now we consider the decomposition of

$$S_1 = \sum_{p|P} a_{p'}\mathrm{Reg}_{p'} + a_{p''}\mathrm{Reg}_{p''} + a_{(2p)'}\mathrm{Reg}_{(2p)'} \quad \text{into six sums:}$$

$$\begin{aligned}
S_1 = {} & \sum_{p\in P_3} \mathrm{Reg}_{1''} - \mathrm{Reg}_{2'} + a_{p''}\mathrm{Reg}_{p''} + a_{(2p)'}\mathrm{Reg}_{(2p)'} && (G)_{/1''}\\
& + \sum_{p\in P_4} \mathrm{Reg}_{1'} - \mathrm{Reg}_{2'} + a_{p'}\mathrm{Reg}_{p'} + a_{(2p)'}\mathrm{Reg}_{(2p)'} && (G)_{/1'}\\
& + \sum_{p\in \bar{P}_4} \mathrm{Reg}_{1''} - \mathrm{Reg}_{2'} + a_{p''}\mathrm{Reg}_{p''} && (E)_{/1''}\\
& + \sum_{p\in P_4\setminus\bar{P}_4} (-\mathrm{Reg}_{1''}) + a_{p''}\mathrm{Reg}_{p''} && (C \text{ or } D)_{/1''}\\
& + \sum_{p\in \bar{P}_3} \mathrm{Reg}_{1'} - \mathrm{Reg}_{2'} + a_{p'}\mathrm{Reg}_{p'} && (E)_{/1'}\\
& + \sum_{p\in P_3\setminus\bar{P}_3} (-\mathrm{Reg}_{1'}) + a_{p'}\mathrm{Reg}_{p'} && (C \text{ or } D)_{/1'}
\end{aligned}$$

where the parentheses in sums (4) and (6) mean that we may choose as each summand either (C) or (D).

We notice that the sums realize coefficients $a_{p'}$, $a_{p''}$, $a_{(2p)'}$ for $p \neq 1$. Moreover, their contribution to the coefficient $a_{2'}$ is

$$-\bigl(|P_3|+|P_4|\bigr) - \bigl(|\bar{P}_4|+|\bar{P}_3|\bigr) = -\bigl(\zeta(P)-1\bigr) - \left(\frac{(1-\beta)^2}{4} - \bigl(\zeta(P)-1\bigr)\right) = \frac{-(1-\beta)^2}{4},$$

as required.

To complete the proof of the fact that the decomposition has the decomposition number equal to $2\zeta(P)-2$ it remains to show that we can choose the sequences (C) and (D) in sums (4), (6) so that we obtain the coefficients $a_{1'} = a_{1''} = \frac{1-\beta}{2}$. The sums (2), (5) and (1), (3) give the contribute $\frac{1}{2} \cdot \frac{(1-\beta)^2}{4}$ to each of $a_{1'}$, $a_{1''}$ respectively. This may be greater than the required $\frac{1-\beta}{2}$. If we can use $\frac{1}{2} \cdot \frac{(1-\beta)^2}{4} - \frac{1-\beta}{2}$ sequences (C) in each sum (4) and (6) then this contribution drops to the required $\frac{1-\beta}{2}$. This is possible if

$$\frac{1}{2} \cdot \frac{(1-\beta)^2}{4} - \frac{1-\beta}{2} \leqslant |P_3| - |\bar{P}_3|. \tag{9.2}$$

But this means

$$\frac{1}{2} \cdot \frac{(1-\beta)^2}{4} - \frac{1-\beta}{2} \leqslant \frac{\zeta(P)-1}{2} - \frac{1}{2} \cdot \left(\frac{(1-\beta)^2}{4} - \bigl(\zeta(P)-1\bigr)\right),$$

$$\frac{(1-\beta)^2}{4} - \frac{1-\beta}{2} + 1 \leqslant \zeta(P)$$

and the last inequality is equivalent to the assumption $\frac{3+\beta^2}{4} \leqslant \zeta(P)$. This completes the proof under the assumption that the conditions (4.a1) and (4.a2) hold.

The above approach can be easily modified to cover the remaining cases (we drop the auxiliary assumptions (4.a1) and (4.a2)). We can choose appropriately adjusted subsets $P_3$, $P_4$, $\bar{P}_3$, $\bar{P}_4$ with $||P_4|-|P_3|| \leqslant 1$ and $||\bar{P}_4|-|\bar{P}_3|| \leqslant 1$ and repeat the same reasoning.

**Subcase (4b).** We choose subsets $P_3$, $P_4$, $\bar{P}_3$, $\bar{P}_4$ with $P_3 \cap P_4 = \emptyset$, $P_3 \cup P_4 = DIV_P$, $||P_4|-|P_3|| \leqslant 1$ and $||\bar{P}_4|-|\bar{P}_3|| \leqslant 1$ such that $|\bar{P}_3|+|\bar{P}_4| = \frac{(1-\beta)^2}{4}$.

Now we consider the following decomposition of $S_1$:

$$\begin{aligned}
S_1 = {} & \sum_{p\in\bar{P}_3} \mathrm{Reg}_{1''} - \mathrm{Reg}_{2'} + a_{p''}\mathrm{Reg}_{p''} + a_{(2p)'}\mathrm{Reg}_{(2p)'} && (G)_{/1''}\\
& + \sum_{p\in\bar{P}_4} \mathrm{Reg}_{1'} - \mathrm{Reg}_{2'} + a_{p'}\mathrm{Reg}_{p'} + a_{(2p)'}\mathrm{Reg}_{(2p)'} && (G)_{/1'}\\
& + \sum_{p\in P_3\setminus\bar{P}_3} \mathrm{Reg}_{1''} + a_{p''}\mathrm{Reg}_{p''} + a_{(2p)'}\mathrm{Reg}_{(2p)'} && (F)_{/1''}\\
& + \sum_{p\in P_4\setminus\bar{P}_4} \mathrm{Reg}_{1'} + a_{p'}\mathrm{Reg}_{p'} + a_{(2p)'}\mathrm{Reg}_{(2p)'} && (F)_{/1'}\\
& + \sum_{p\in P_3} (-\mathrm{Reg}_{1''}) + a_{p''}\mathrm{Reg}_{p''} && (C \text{ or } D)_{/1''}\\
& + \sum_{p\in P_4} (-\mathrm{Reg}_{1'}) + a_{p'}\mathrm{Reg}_{p'} && (C \text{ or } D)_{/1'}
\end{aligned}$$

where the parentheses in sums (5) and (6) mean that we may choose as each summand either (C) or (D).

By the same reasoning as in Subcase (4a), to show that the decomposition number is equal to $2\zeta(P)-2$, it is enough to choose the sequences (C) and (D) in the sums (5) and (6) so that $a_{1'} = a_{1''} = \frac{1-\beta}{2}$. The sums (1)–(4) contribute $|P_3|$ to $a_{1'}$ and $|P_4|$ to $a_{1''}$. This may be greater than the required $\frac{1-\beta}{2}$, but we can use $|P_3| - \frac{(1-\beta)}{2}$ sequences (C) in the sum (5) and $|P_4| - \frac{(1-\beta)}{2}$ in (6) to establish the equality. The assumption of Subcase (4b) implies that both differences determining the number of sequences (C) are non-negative. This ends the proof of Subcase (4b) and the proof of Lemma 9.2. □

We recall that we are assuming that $r = 2^R \cdot P$ with $R \geqslant 1$ and $P$ odd.

**Lemma 9.4.**

$$NJD_r[f, V_b] \leqslant \begin{cases} 2^R\zeta(P) & \text{if } |\frac{1-\beta}{2}| \geqslant \zeta(P),\\ 2^R\zeta(P)-1 & \text{if } |\frac{1-\beta}{2}| < \zeta(P) \text{ and } \beta > 0,\\ 2^R\zeta(P)-1 & \text{if } |\frac{1-\beta}{2}| < \zeta(P),\ \frac{3+\beta^2}{2^2} > \zeta(P) \text{ and } \beta < 0,\\ 2^R\zeta(P)-2 & \text{if } |\frac{1-\beta}{2}| < \zeta(P),\ \frac{3+\beta^2}{2^2} \leqslant \zeta(P) \text{ and } \beta < 0. \end{cases}$$

**Proof.** We again represent $S = S_1 + S_2$ (cf. formula (9.1)). We realize the sum $S_1$ and $S_2$ separately with the decompositions given in Lemmas 9.2 and 9.1 respectively, and this completes the proof. □

## 10. Lower bounds for $NJD_r[f, V_b]$

We will consider two cases: $\zeta(P)$ small (Subsection 10.1) and $\zeta(P)$ large (Subsection 10.2). In the first case we will obtain the lower bound, and thus the exact value of $NJD_r[f, V_b]$ in Theorem 10.5. In the second case $NJD_r[f, V_b]$ will be given in Theorems 10.8 and 10.9.

**Lemma 10.1.** *Let us fix a number $s = 1, \dots, R-1$.*

*A minimal decomposition for*

$$\sum_{p|P,\, p\neq 1} a_{(2^s p)''}\mathrm{Reg}_{(2^s p)''}$$

*must contain at least $\zeta(P)-1$ sequences attached at $(2^s)''$.*

**Proof.** It is sufficient to notice that to realize $a_{(2^s p)''}\mathrm{Reg}_{(2^s p)''}$ (for $p \neq 1$) we need a sequence (B)–(G) attached at $(2^s)''$. Moreover, such a sequence cannot realize $a_{(2^s p_1)''}\mathrm{Reg}_{(2^s p_1)''}$ and $a_{(2^s p_2)''}\mathrm{Reg}_{(2^s p_2)''}$ for $p_1 \neq p_2$. □

### 10.1. $\zeta(P)$ is small

In this subsection we assume that $|\frac{1-\beta}{2}| = |a_{1''}| \geqslant \zeta(P)$. Notice that this also implies that $|\frac{1-\beta^{2^k}}{2^{k+1}}| = |a_{(2^k)''}| \geqslant \zeta(P)$ for $k \geqslant 1$.

We will consider some different parts of the big component. For some parts we are able to give a minimal realization, for the other parts we obtain only a lower bound for $NJD_r[f, V_b]$. As a result, for the whole big component we get the lower bound.

**Lemma 10.2.** *Let us fix $0 < s \leqslant R-1$ and assume that $|\frac{1-\beta}{2}| \geqslant \zeta(P)$. Then the minimal decomposition number for*

$$\sum_{p|P} a_{(2^s p)''} \operatorname{Reg}_{(2^s p)''} \tag{10.1}$$

*is equal to $2^s\zeta(P)$.*

**Proof.** It follows from Lemma 10.1 that at least $\zeta(P)-1$ sequences must be attached at $(2^s)''$. On the other hand, we have the following realization: $\zeta(P)-1$ sequences of the type (D) and one sequence (A), all attached at $(2^s)''$.

It remains to show that $\zeta(P)-1$ sequences cannot realize the sum and we need one more sequence based at $(2^s)''$. Assume to the contrary that these $\zeta(P)-1$ sequences realize the sum (10.1). Since $a_{(2^s)''} < 0$ and the decomposition is minimal, $|a_{(2^s)''}|$ of them must be sequences of the type (C). However, this is impossible by the assumption that $|a_{(2^s)''}| \geqslant \zeta(P)$, so we have not enough sequences (C) to do that. As a result, we have to attach one additional sequence (A) at $(2^s)''$ to realize $a_{(2^s)''}$. □

**Lemma 10.3.** *Let $|\frac{1-\beta}{2}| \geqslant \zeta(P)$. The minimal decomposition number for*

$$\sum_{s=1}^{R-1}\sum_{p|P} a_{(2^s p)''} \operatorname{Reg}_{(2^s p)''} \tag{10.2}$$

*is equal to $(2^R-2)\zeta(P)$.*

**Proof.** By Lemma 10.2 the minimal decomposition number is $\leqslant \sum_{s=1}^{R-1} 2^s\zeta(P) = (2^R-2)\zeta(P)$. In fact we have equality here, since there is no vertex preceding both $(2^s \cdot p)''$ and $(2^{\bar{s}} \cdot \bar{p})''$ for $s \neq \bar{s}$. □

**Lemma 10.4.** *Let $|\frac{1-\beta}{2}| \geqslant \zeta(P)$. The minimal decomposition number for*

$$\sum_{p|P} a_{p'} \operatorname{Reg}_{p'} + a_{p''} \operatorname{Reg}_{p''} \tag{10.3}$$

*is greater or equal to $2\zeta(P)$.*

**Proof.** In order to realize the sum (10.3) we may attach sequences at $1'$ or $1''$. Notice that no vertex precedes both $p'$ and $q''$. Thus, by the same argument as in the proof of Lemma 10.2, we must use at least $2(\zeta(P)-1)$ sequences to realize the coefficients $a_{p'}$ and $a_{p''}$ of the sum (10.3) for $p > 1$, and two sequences of the type (A) to realize the coefficients $a_{1'}$ and $a_{1''}$. □

**Theorem 10.5.** *Let $|\frac{1-\beta}{2}| \geqslant \zeta(P)$, then*

$$NJD_r[f, V_b] = 2^R\zeta(P).$$

**Proof.** $\leqslant$ follows from Lemma 9.4. It remains to show that the decomposition number of each decomposition of the big component is at least $2^R\zeta(P)$.

In every "global" minimal decomposition we must realize the sum (10.3) by $2\zeta(P)$ sequences in the way indicated in the proof of Lemma 10.4. However, each such realization does not affect the realization of any part of the sum (10.2), which follows from the fact that in the oriented Reidemeister graph there is no path from the vertices $p'$ (or $p''$) to $(2^s p)''$ ($s \geqslant 1$). As a result, by Lemmas 10.4 and 10.3 we get

$$NJD_r[f, V_b] \geqslant 2\zeta(P) + \left(2^R-2\right)\zeta(P) = 2^R\zeta(P). \qquad \square$$

*10.2. $\zeta(P)$ large*

Now we assume that $|\frac{1-\beta}{2}| = |a_{1''}| < \zeta(P)$.

**Lemma 10.6.** *Let $|\frac{1-\beta}{2}| < \zeta(P)$. The minimal decomposition number for*

$$\sum_{p|P} a_{p'}\mathrm{Reg}_{p'} + a_{p''}\mathrm{Reg}_{p''} + a_{(2p)'}\mathrm{Reg}_{(2p)'} \tag{10.4}$$

*is greater than or equal to*

$$\begin{cases} 2\zeta(P)-1 & \text{if } \beta > 0, \\ 2\zeta(P)-1 & \text{if } \dfrac{3+\beta^2}{2^2} > \zeta(P) \text{ and } \beta < 0, \\ 2\zeta(P)-2 & \text{if } \dfrac{3+\beta^2}{2^2} \leqslant \zeta(P) \text{ and } \beta < 0. \end{cases}$$

**Proof.** Let us notice that each sequence (A)–(G) attached at $1'$ or $1''$ realizes at most one $a_{p^*}\mathrm{Reg}_{p^*}$ for $p \neq 1$. Thus, we need at least $2\zeta(P)-2$ sequences, which completes the proof in the case of $\frac{3+\beta^2}{2^2} \leqslant \zeta(P)$ and $\beta < 0$.

Suppose that there is a decomposition of the sum (10.4) using $2\zeta(P)-2$ sequences for $\beta > 0$. These $2\zeta(P)-2$ sequences must also realize the remaining coefficients. To realize $a_{(2p)'}\mathrm{Reg}_{(2p)'}$ (for $p \neq 1$) we need a sequence of the type $(F)$ or $(G)$ attached at $1'$ or $1''$, hence we need $\zeta(P)-1$ such sequences. But the coefficient at $1^*$ in each such sequence is $+1$ while $a_{1^*}$ is negative. Thus we also need $\zeta(P)-1-a_{1'}-a_{1''}$ sequences of the type (C). Thus we need at least

$$\big(\zeta(P)-1\big) + \big(\zeta(P)-1-a_{1'}-a_{1''}\big) = 2\big(\zeta(P)-1\big) - a_{1'} - a_{1''} > 2\zeta(P)-2$$

sequences, which contradicts the assumption that the minimal decomposition number is equal to $2\zeta(P)-2$.

We have to consider the last case, where $\frac{3+\beta^2}{2^2} > \zeta(P)$ and $\beta < 0$. Suppose, contrary to our claim, that $2\zeta(P)-2$ sequences will realize the sum (10.4), in particular the coefficients $a_{1^*}$ and $a_{2'}$. To realize $a_{2'} = -\frac{(1-\beta)^2}{2^2}$ we have to attach exactly $\frac{(1-\beta)^2}{2^2}$ sequences (E) or (G) (the coefficient at $2'$ in each such sequence is $-1$). On the other hand, the coefficient at $1^*$ in each sequence (E) or (G) is $+1$. Moreover, we have the inequality $a_{1'} + a_{1''} \leqslant |a_{2'}|$ for $\beta \leqslant -3$, which implies that we need to attach in addition at least $|a_{2'}| - (a_{1'} + a_{1''})$ sequences (C). Thus, we need at least

$$\begin{aligned} |a_{2'}| + |a_{2'}| - (a_{1'} + a_{1''}) &= 2 \cdot \frac{(1-\beta)^2}{2^2} - (1-\beta) \\ &= 2\left(\frac{3+\beta^2}{2^2}\right) - 2 > 2\zeta(P)-2 \end{aligned} \tag{10.5}$$

sequences (E), (G) or (C) which contradicts the assumption that the minimal decomposition number is equal to $2\zeta(P)-2$. □

The proof of Theorem 10.8 will follow from the next key lemma.

**Lemma 10.7.** *Let $|\beta| > 3$. There exists a minimal decomposition of the big component satisfying*:

at each vertex $(2^s)''$, for $s = 1, \dots, R-1$, there are attached (at least) $\zeta(P)$ sequences.

**Proof.** Let $s = 0, \dots, R-1$ be the minimal number satisfying:

*there is a minimal decomposition in which at each $(2^{s+1})'', \dots, (2^{R-1})''$ there are attached (at least) $\zeta(P)$ sequences.*

Then the thesis of Lemma 10.7 is equivalent to $s = 0$. Contrary to our claim, we assume that $s \geqslant 1$ and we will get either a contradiction or another minimal decomposition in which also at $(2^s)''$ (at least) $\zeta(P)$ sequences are attached. In any case we get a contradiction, which implies $s = 0$.

It follows from the proof of Lemma 10.1 that $\zeta(P)-1$ sequences must be attached at $(2^s)''$. Since $a_{(2^s)''} < 0$ (cf. Corollary 5.4) and the decomposition is minimal, $|a_{(2^s)''}|$ of them must be sequences of the type (C). We assume here that $\zeta(P)-1 \geqslant |a_{(2^s)''}|$, because otherwise we cannot realize $|a_{(2^s)''}|$ and we get a contradiction. If we use these $|a_{(2^s)''}|$ sequences of the type (C) to realize $a_{(2^s)''}$ (during the realization of $a_{(2^s p)''}$), then the corresponding coefficients $a_{(2^{s+1}p)'}$ cannot be realized by them. To realize each $a_{(2^{s+1}p)'}$, where $p > 1$, we can use:

- sequence (B)–(E) at $1'$, or
- sequence (B)–(E) at $(2^g)''$, where $g < s$, or
- sequence (B)–(E) at $2'$ for $s > 1$, or
- sequence (B)–(G) at $2'$ for $s = 1$.

Let us notice that any sequence attached at $1'$, $2'$ or $(2^g)''$, where $g < s$, can realize two different $a_{(2^{s+1}p_1)'}$ and $a_{(2^{s+1}p_2)'}$, where $p_1, p_2 > 1$, so there must be $|a_{(2^s)''}|$ such additional sequences.

Let us make one more observation: a sequence (F) or (G) attached at $2'$ can realize $a_{(2^s p)'}$ and $a_{(2^{s+1}p)'}$ only if $s = 1$. By $\xi_{2'}$ we will denote the number of the sequences attached at $2'$ which realize $a_{(2^{s+1}p)'}$.

We will now modify the initial decomposition in the following way.

**Remove:**

(I) All $\zeta(P) - 1$ sequences attached at $(2^s)''$,
(II) $\xi_{2'}$ sequences attached at $2'$,
(III) $|a_{(2^s)''}| - \xi_{2'}$ sequences attached at $1'$ or $(2^g)''$.
Parts (II)–(III) embrace the sequences that realize $a_{(2^{s+1}p)'}$.

While removing the set of sequence (I)–(III) we decrease the decomposition number by at least

$$\bigl(2^s\bigl(\zeta(P) - 1\bigr)\bigr) + (2\xi_{2'}) + \bigl(|a_{(2^s)''}| - \xi_{2'}\bigr) = 2^s\bigl(\zeta(P) - 1\bigr) + |a_{(2^s)''}| + \xi_{2'}. \tag{10.6}$$

**Add:**

(1a) One sequence (A) attached at $(2^s)''$ to realize $a_{(2^s)''}$ and $a_{(2^{s+1})'}$,
(1b) $\zeta(P) - 1$ sequences (F) attached at $(2^s)''$ to realize $a_{(2^s p)''}$ and $a_{(2^{s+1}p)'}$,
(2) $s$ sequences (A) attached at $(2^g)''$ for each $0 \leqslant g \leqslant s - 1$ to realize $a_{(2^g)''}$ and $a_{(2^{g+1})'}$,
(3) one sequence (A) attached at $1'$ to realize $a_{1'}$ and $a_{2'}$,
(4) $\xi_{2'}$ sequences (D) attached at $1'$ to realize $a_{(2p)'}$ (if applicable).

By adding the set of sequences (1)–(4) we increase the decomposition number by exactly

$$\bigl(2^s\bigr) + \bigl(2^s\bigl(\zeta(P) - 1\bigr)\bigr) + \bigl(2^s - 1\bigr) + (1) + (\xi_{2'}) = 2^s\bigl(\zeta(P) - 1\bigr) + 2^{s+1} + \xi_{2'}. \tag{10.7}$$

During the *remove* phase we possibly changed the value of the following coefficients:

(i) $a_{(2^s p)''}$, $a_{(2^{s+1}p)'}$, where $p|P$,
(ii) $a_{(2^g)''}$ and $a_{(2^{g+1})'}$, where $g < s$,
(iii) $a_{1'}$,
(iv) $a_{(2^s p)'}$, if $s = 1$.

However, during the *add* phase we restored the original values, as the items (1)–(4) correspond to (i)–(iv).

Note that we have actually proved that after the modification we realize at least the same coefficients that were realized in the initial decomposition. Comparing the change of the decomposition number (see (10.2) and (10.7)) we get

$$\begin{aligned} &2^s\bigl(\zeta(P) - 1\bigr) + |a_{(2^s)''}| + \xi_{2'} \geqslant 2^s\bigl(\zeta(P) - 1\bigr) + 2^{s+1} + \xi_{2'}, \\ &\frac{\beta^{2^s} - 1}{2^{s+1}} \geqslant 2^{s+1} \end{aligned} \tag{10.8}$$

which is true for $|\beta| \geqslant 5$ and $s \geqslant 1$.

A sharp inequality would mean that the new decomposition has a lower decomposition number which contradicts the minimality of the given one. In the case of equality we get a new decomposition with $\zeta(P)$ sequences at $(2^s)''$ and get the contradiction, which proves our theorem. □

Under the assumption $|\frac{1-\beta}{2}| < \zeta(P)$ we give in Theorems 10.8 and 10.9 the value of $NJD_r[f, V_b]$ for $|\beta| > 3$ and $|\beta| = 3$, respectively.

**Theorem 10.8.** *Let* $|\frac{1-\beta}{2}| < \zeta(P)$.

$$NJD_r[f, V_b] = \begin{cases} 2^R\zeta(P) - 1 & \text{if } \beta > 3, \\ 2^R\zeta(P) - 1 & \text{if } \frac{3+\beta^2}{2^2} > \zeta(P) \text{ and } \beta < -3, \\ 2^R\zeta(P) - 2 & \text{if } \frac{3+\beta^2}{2^2} \leqslant \zeta(P) \text{ and } \beta < -3. \end{cases}$$

**Proof.** $\leqslant$ follows from Lemma 9.4. To prove the opposite inequality we consider a decomposition of the big component.

We again let $S = S_1 + S_2$ (cf. the formula (9.1)), where

$$S_1 = \sum_{p|P} a_{p'}\mathrm{Reg}_{p'} + a_{p''}\mathrm{Reg}_{p''} + a_{(2p)'}\mathrm{Reg}_{(2p)'},$$

$$S_2 = \sum_{s=1}^{R-1}\sum_{p|P} a_{(2^s p)''}\mathrm{Reg}_{(2^s p)''} + a_{(2^{s+1} p)'}\mathrm{Reg}_{(2^{s+1} p)'}.$$

First we consider the sum $S_2$.

The total contribution to the decomposition number of the set of sequences attached at $2'', (2^2)'', \dots, (2^{R-1})''$ must be at least

$$\sum_{s=1}^{R-1} 2^s\zeta(P) = \bigl(2^R - 2\bigr)\zeta(P)$$

(Lemma 10.7). On the other hand, these sequences do not realize coefficients $a_{p'}, a_{p''}, a_{(2p)'}$ from $S_1$. By Lemma 10.6, the minimal decomposition number of $S_1$ is at least

$$\begin{aligned}
&2\zeta(P) - 1 \quad \text{for } \beta > 3,\\
&2\zeta(P) - 1 \quad \text{for } \frac{3+\beta^2}{2^2} > \zeta(P) \text{ and } \beta < -3,\\
&2\zeta(P) - 2 \quad \text{for } \frac{3+\beta^2}{2^2} \leqslant \zeta(P) \text{ and } \beta < -3.
\end{aligned}$$

This completes the proof in all cases. □

**Theorem 10.9.** *Let* $|\frac{1-\beta}{2}| < \zeta(P)$.

$$NJD_r[f, V_b] = \begin{cases} 2^R\zeta(P) - 1 & \text{if } \beta = 3,\\ 2^R\zeta(P) - 1 & \text{if } \frac{3+\beta^2}{2^2} > \zeta(P) \text{ and } \beta = -3,\\ 2^R\zeta(P) - 2 & \text{if } \frac{3+\beta^2}{2^2} \leqslant \zeta(P) \text{ and } \beta = -3.\end{cases}$$

**Proof.** $\leqslant$ follows from Lemma 9.4. Our proof for the opposite inequality starts with the observation that the formula (10.8) given in Lemma 10.7 is true for $\beta = \pm 3$ and $s \geqslant 2$. Thus we can follow the proof of Lemma 10.7 to get a minimal decomposition satisfying the condition that at each vertex $(2^s)''$, for $s = 2, \dots, R-1$, there are attached at least $\zeta(P)$ sequences. The contribution to $NJD_r[f, V_b]$ for this part will be at least

$$\sum_{s=2}^{R-1} 2^s\zeta(P) = 2^R\zeta(P) - 2^2\zeta(P). \tag{10.9}$$

However, these sequences do not realize $a_{p'}$, $a_{p''}$, $a_{(2p)'}$, $a_{(2p)''}$, $a_{(2^2 p)'}$, where $p|P$. We will prove that the minimal decomposition number for a realization of the above coefficients is at least

$$\begin{aligned}
&2^2\zeta(P) - 1 \quad \text{for } \beta = 3,\\
&2^2\zeta(P) - 1 \quad \text{for } \frac{3+\beta^2}{2^2} > \zeta(P) \text{ and } \beta = -3,\\
&2^2\zeta(P) - 2 \quad \text{for } \frac{3+\beta^2}{2^2} \leqslant \zeta(P) \text{ and } \beta = -3.
\end{aligned} \tag{10.10}$$

This will complete the proof, because adding both contributions (10.9) and (10.10) to the minimal decomposition number gives the required value of $NJD_r[f, V_b]$.

To realize coefficients $a_{(2p)''}$ we must attach at least $\zeta(P) - 1$ sequences at $2''$ (Lemma 10.1). We have the following two alternatives depending on how many sequences $m$ are attached at $2''$.

$\boldsymbol{m = \zeta(P)}$. By Lemma 10.6 the minimal decomposition number for the coefficients $a_{p'}$, $a_{p''}$, $a_{(2p)'}$, which were not realized by the sequences attached at $2''$, is at least

$$2\zeta(P)-1 \quad \text{for } \beta=3,$$

$$2\zeta(P)-1 \quad \text{for } \frac{3+\beta^2}{2^2} > \zeta(P) \text{ and } \beta=-3,$$

$$2\zeta(P)-2 \quad \text{for } \frac{3+\beta^2}{2^2} \leqslant \zeta(P) \text{ and } \beta=-3.$$

If we add the decomposition number for these two phases we will get (10.10).

$\boldsymbol{m = \zeta(P) - 1}$. The coefficient $a_{2''}$ will be realized by the sequences of the type (C).

We recall that

$$a_{2''} = \frac{1-(-3)^2}{2^2} = -2.$$

We will consider three cases.

**Case (1).** $\zeta(P)=2$. We need at least two sequences (C) at $2''$ to realize $a_{2''}$. However, under the given assumption, we can attach only one $(1=\zeta(P)-1)$ sequence at $2''$ which gives the contradiction.

**Case (2).** $\zeta(P)=3$. Let $1<p_1<p_2$ and $p_1|P$, $p_2|P$. We can use only two sequences of the type (C) at $2''$ to realize $a_{2''}=-2$, $a_{(2p_1)''}$, $a_{(2p_2)''}$. Then we have to realize coefficients $a_{p_1'}$, $a_{p_2'}$, $a_{p_1''}$, $a_{p_2''}$, $a_{(2^2)'}$, $a_{(2^2p_1)'}$, $a_{(2^2p_2)'}$ with a contribution at least 7 to the decomposition number. Finally the minimal decomposition number in this case is at least $4+7=11=2^2\zeta(P)-1$.

**Case (3).** $\zeta(P)\geqslant 4$. We need at least two sequences of the type (C) to realize $a_{2''}$. If we used only two sequences of the type (C), the rest of the coefficients $a_{(2p)''}$ with $p|P$, $p>1$ will be realized together with the corresponding $a_{(2^2p)'}$ by sequences (F)–(G) at $2''$, but this would change the coefficient $a_{2''}$. This implies that there are in fact at least three sequences of the type (C) at $2''$, so at least three coefficients $a_{(2^2p)'}$ need to be realized separately.

Thus, sequences at $2''$ do not realize (at least) the following coefficients $a_{p'}$, $a_{p''}$, $a_{(2^2p_1)'}$, $a_{(2^2p_2)'}$, $a_{(2^2p_3)'}$, where $1\neq p|P$ and $p_1$, $p_2$, $p_3$ are some fixed divisors of $P$. Let us notice that coefficients $a_{p'}$, $a_{p''}$ cannot be realized together with $a_{(2^2p)'}$ using one sequence. As a consequence, we need at least $2(\zeta(P)-1)$ sequences at $1^*$ to realize $a_{p'}$, $a_{p''}$ and three additional sequences for $a_{(2^2p_1)'}$, $a_{(2^2p_2)'}$, $a_{(2^2p_3)'}$. Finally, taking into account that (by the assumption) we used $\zeta(P)-1$ sequences at $2''$, the minimal decomposition number is at least $2(\zeta(P)-1)+2(\zeta(P)-1)+3=2^2\zeta(P)-1$, which is more than in the case $m=\zeta(P)$. □

Now we are in a position to prove the main result of this paper.

**Theorem 10.10.** *Let $f:\mathbb{R}P^3\to\mathbb{R}P^3$ be a smooth map of odd degree $\beta$, and let $r=2^R\cdot P$ where $P$ is odd and $R\geqslant 1$, then*

*for $\beta=\pm 1$,*

$$NJD_r[f]=\begin{cases}0 & \text{if } \beta=1,\\ 2 & \text{if } \beta=-1;\end{cases}$$

*for $\beta\geqslant 3$,*

$$NJD_r[f]=\begin{cases}2^{R+1}\zeta(P) & \text{if } |\frac{1-\beta}{2}|\geqslant\zeta(P),\\ 2^{R+1}\zeta(P)-1 & \text{if } |\frac{1-\beta}{2}|<\zeta(P),\ |\frac{1-\beta^{2^R}}{2^{R+1}}|\geqslant\zeta(P),\\ 2^{R+1}\zeta(P)-2^R-1 & \text{if } |\frac{1-\beta^{2^R}}{2^{R+1}}|<\zeta(P);\end{cases}$$

*for $\beta\leqslant -3$,*

$$NJD_r[f]=\begin{cases}2^{R+1}\zeta(P) & \text{if } |\frac{1-\beta}{2}|\geqslant\zeta(P),\\ 2^{R+1}\zeta(P)-1 & \text{if } |\frac{1-\beta}{2}|<\zeta(P),\ \frac{3+\beta^2}{2^2}>\zeta(P),\ |\frac{1-\beta^{2^R}}{2^{R+1}}|\geqslant\zeta(P),\\ 2^{R+1}\zeta(P)-2 & \text{if } |\frac{1-\beta}{2}|<\zeta(P),\ \frac{3+\beta^2}{2^2}\leqslant\zeta(P),\ |\frac{1-\beta^{2^R}}{2^{R+1}}|\geqslant\zeta(P),\\ 2^{R+1}\zeta(P)-2^R-2 & \text{if } |\frac{1-\beta^{2^R}}{2^{R+1}}|<\zeta(P).\end{cases}$$

**Proof.** The special cases of $\beta=\pm 1$ and $r=2^R$ were discussed in Section 7 (notice that for $|\beta|\geqslant 3$ and $P=1$ the condition $|\frac{1-\beta}{2}|\geqslant\zeta(P)$ is satisfied).

By Theorem 8.1 we know the number $NJD_r[f,V_s]$. Theorems 10.5, 10.8 and 10.9 provide the value of $NJD_r[f,V_b]$. Finally, by Lemma 6.3 $NJD_r[f]=NJD_r[f,V_s]+NJD_r[f,V_b]$ which completes the proof. □

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