PRESSURE FIELD IN A PLANE LAYER

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The paper presents results of analytical and numerical investigations of a plane acoustic waves propagation through a plane layer surrounded by another medium. The waves generation problem is discussed. The mathematical model of the pressure propagation in the layer is proposed. Propagated through the layer changes in pressure, the reflected and transmitted waves were studied. Some results of numerical investigations are presented.

INTRODUCTION

The wave generation and propagation inside layers with different physical properties is very important problem in practice. The problem considered in this paper is shown in Fig. 1. We assume that plane layer (region II) with density ρ_L and sound velocity c_L is placed between x=0 and x=L. This layer is surrounded by medium with density ρ_0 and sound velocity c_0 (regions I and III). In region I (x<0) the acoustic field consists of incident p_i and reflected p_r waves. In region III (x>L) only transmitted wave p_t is propagated. The pressure field in region II ($0 \le x \le L$) is formed by the right-going p_A and left-going p_B waves ($p = p_A + p_B$).

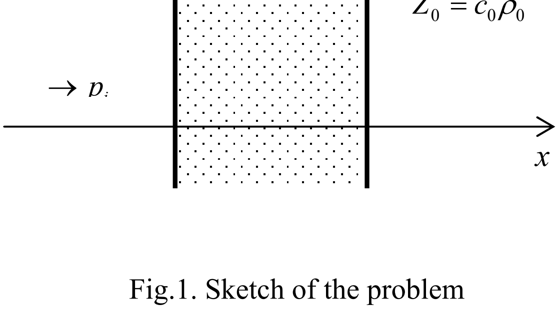

Fig.1. Sketch of the problem

The aim of this paper was theoretical analysis of plane wave travelling in inhomogeneous media. The mathematical model proposed in this paper was worked out on the basis of mechanics equations. The waves generation problem, presented earlier, allows to understand the problem and to verify the numerical model.

1. THEORETICAL EXAMINATION

The acoustic pressure and velocity at regions I, II and III are equal respectively:

$$
\begin{aligned}
p_I(x,t) &= p_i(x,t) + p_r(x,t) = p_0\left(e^{-ik_0x} + Ve^{ik_0x}\right)e^{i\omega t} \\
p_{II}(x,t) &= p_A(x,t) + p_B(x,t) = p_0\left(Ae^{-ik_Lx} + Be^{ik_Lx}\right)e^{i\omega t} \\
P_{III}(x,t) &= p_t(x,t) = p_0\left(We^{-ik_0(x-L)}\right)e^{i\omega t}
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
v_I(x,t) &= v_i(x,t) + v_r(x,t) = \left(p_i(x,t) - p_r(x,t)\right)/Z_0 \\
v_{II}(x,t) &= v_A(x,t) + v_B(x,t) = \left(p_A(x,t) - p_B(x,t)\right)/Z_L \\
v_{III}(x,t) &= v_t(x,t) = p_t(x,t)/Z_0
\end{aligned}
\tag{2}
$$

where $\omega = 2\pi f$ denotes angular frequency, $k_i = \omega / c_i$ (i=0, L) are wave numbers and Z_i - the acoustic impedance of the media. The pressure and velocities should be continuous at boundaries x=0 and x=L what leads to following equations

$$
\begin{aligned}
p_i(0,t) + p_r(0,t) &= p_A(0,t) + p_B(0,t) \\
v_i(0,t) + v_r(0,t) &= v_A(0,t) + v_B(0,t)
\end{aligned}
\tag{3}
$$

and

$$
\begin{aligned}
p_A(L,t) + p_B(L,t) &= p_t(L,t) \\
v_A(L,t) + v_B(L,t) &= v_t(L,t)
\end{aligned}
\tag{4}
$$

Substituting formulas (1), (2) into equations (3), (4) and solving obtained system, we find

$$
\begin{aligned}
V &= \frac{i(\varsigma^2 - 1)\sin\kappa}{2\varsigma\cos\kappa + i(1+\varsigma^2)\sin\kappa} \\
A &= 0.5\left[(1+\varsigma) + V(1-\varsigma)\right] \\
B &= 0.5\left[(1-\varsigma) + V(1+\varsigma)\right] \\
W &= Ae^{-i\kappa} + Be^{i\kappa}
\end{aligned}
\tag{5}
$$

where $\varsigma = Z_L / Z_0$ and $\kappa = k_L L$.

Using above model we can study only continuous waves propagation in medium I, II and III. When we would like to analyze acoustic field generation more carefully, more complex theoretical model is necessary. Now we assume that harmonic wave with amplitude equal p_0=1 is propagated in medium I and it is normally incident on boundary x=0. If it encounters this boundary, reflected and transmitted waves are generated. Let V_1 is the amplitude of reflected wave and A_1 amplitude of right-going wave. The continuity conditions of pressure and velocity at this boundary gives

$$V_1 = R_{12}$$
$$A_1 = 1 + R_{12}$$

where the reflection coefficient $R_{12} = \frac{Z_L - Z_0}{Z_L + Z_0}$. If the wave with amplitude A_1 incident at boundary $x=L$ the transmitted wave with amplitude W_1 and left-going wave with amplitude B_1 are generated. Now continuity conditions lead to

$$B_1 = R_{21}(1 + R_{12})e^{-i2\kappa}$$
$$W_1 = \left(1 - R_{12}^2\right)e^{-i\kappa}$$

where $R_{21} = -R_{12}$. Continuing this analysis step by step, we obtain

$$V_n = (1 - R_{12})B_{n-1}$$
$$A_n = R_{21}B_{n-1}$$
$$B_n = R_{21}e^{-i2\kappa}A_n$$
$$W_n = \left(1 - R_{12}\right)e^{-i\kappa}A_n$$

After some transformations, we get

$$\begin{aligned} &V_1 = R_{12}, V_n = q^{n-2}v_2 \ (n \geq 2) \\ &A_n = q^{n-1}a_1 \ (n \geq 1) \\ &B_n = q^{n-1}b_1 \ (n \geq 1) \\ &W_n = q^{n-1}w_1 \ (n \geq 1) \end{aligned} \tag{6}$$

with $q = R_{12}^2 e^{-i2\kappa}$, $a_1 = 1 + R_{12}$, $b_1 = R_{21}(1 + R_{12})e^{-i2\kappa}$, $v_2 = (1 - R_{12}^2)R_{21}e^{-i2\kappa}$, $w_1 = (1 - R_{12}^2)e^{-i\kappa}$. Equations (6) allows to investigate waves generation step by step. The sum of amplitudes A_n is a geometric series, moreover $\sum_{n=1}^{\infty} A_n = A$. Similar results when $n \to \infty$ we obtain for amplitudes B_n, V_n and W_n.

2. MATHEMATICAL MODEL

Till now the propagation of the plane waves with constant amplitudes in inhomogeneous media were analyzed. Now we will present more general theoretical model of the harmonic wave propagation in layer. The mathematical model of the pressure p propagation inside layer is built on the basis of the equation of continuity and the Euler equation. After some transformations, we obtain second order partial differential equation

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c_L^2}\frac{\partial^2 p}{\partial t^2} = 0 \tag{7}$$

The boundary conditions are defined in following way. If the media outside the layer with sound speed c_0 are regarded as linear then the incident, reflected and transmitted waves have the form

$$p_i(x,t) = p_i(t - x/c_0)$$
$$p_r(x,t) = p_r(t + x/c_0)$$
$$p_t(x,t) = p_t(t - x/c_0)$$

At boundary x=0 and x=L the pressure should be continuous and we have

$$p(x,0) = p_i(0,t) + p_r(0,t)$$
$$p(L,t) = p_t(L,t)$$

Moreover at boundary x=0 we can write

$$\frac{\partial p_i}{\partial x} + \frac{\partial p_r}{\partial x} = \frac{\partial p}{\partial x}$$
$$\frac{\partial p_r}{\partial t} = c_0 \frac{\partial p_r}{\partial x}$$

These conditions allow to eliminate function p_r. After calculations we obtain equation

$$\frac{\partial p}{\partial t} - c_0 \frac{\partial p}{\partial x} = 2\frac{\partial p_i}{\partial t}. \tag{8}$$

Similarly, at x=L we obtain

$$\frac{\partial p}{\partial t} + c_0 \frac{\partial p}{\partial x} = 0. \tag{9}$$

The finite-difference method was used to solve the problem (7)-(9).

3. RESULTS OF THEORETICAL AND NUMERICAL INVESTIGATIONS

First step of theoretical analysis was study of infinite waves propagation. Half-wave layer case is very important in practice. When the thickness of layer $L = N \cdot \lambda/2$ where $\lambda = c_L / f$ and N is natural number, the amplitude of the reflected wave V vanishes and we observe resonance effect. Amplitudes of transmitted and incident waves are equal. Figure 2a presents the amplitudes of reflected, transmitted, right-going and left-going waves for different values of parameter ς when thickness $L = \lambda/2$. In this situation all these amplitudes are real. Analogous results obtained when $L = \lambda/4$ shows Fig. 2b. Calculations were carried out using equations (5).

Analysis of waves generation using equation (6) is more difficult. Figure 3 presents absolute value of common ratio q as a function of parameter ς. This function shows influence of values of physical parameters on quickness of the wave amplitudes changes.

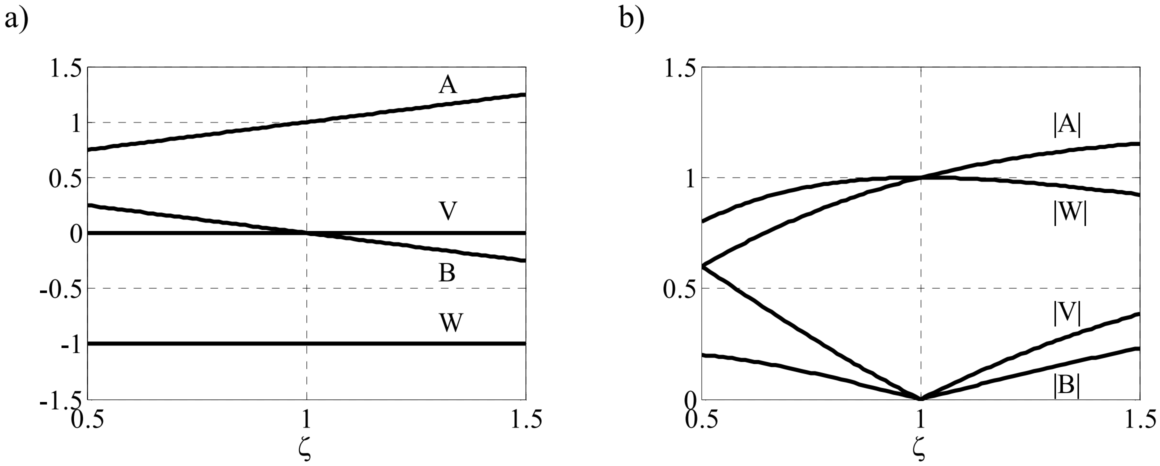

Fig.2. Amplitudes of different continuous waves: a) $L = \lambda / 2$, b) $L = \lambda / 4$

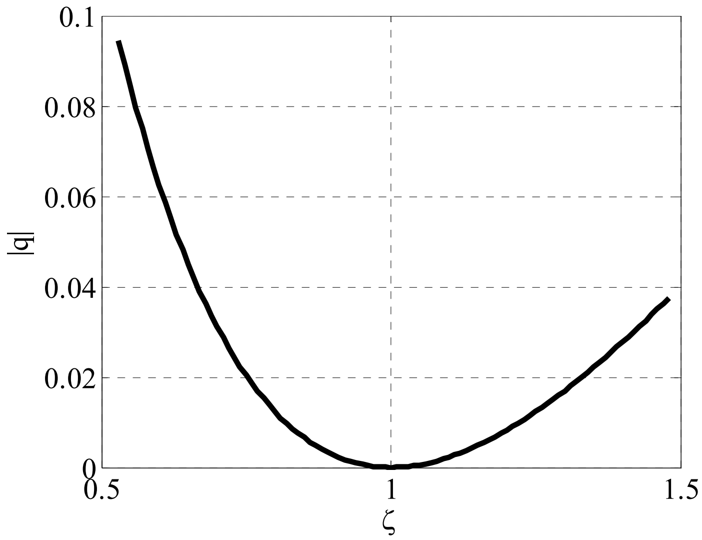

Fig.3. The absolute value of the common ratio q as function of parameter ς

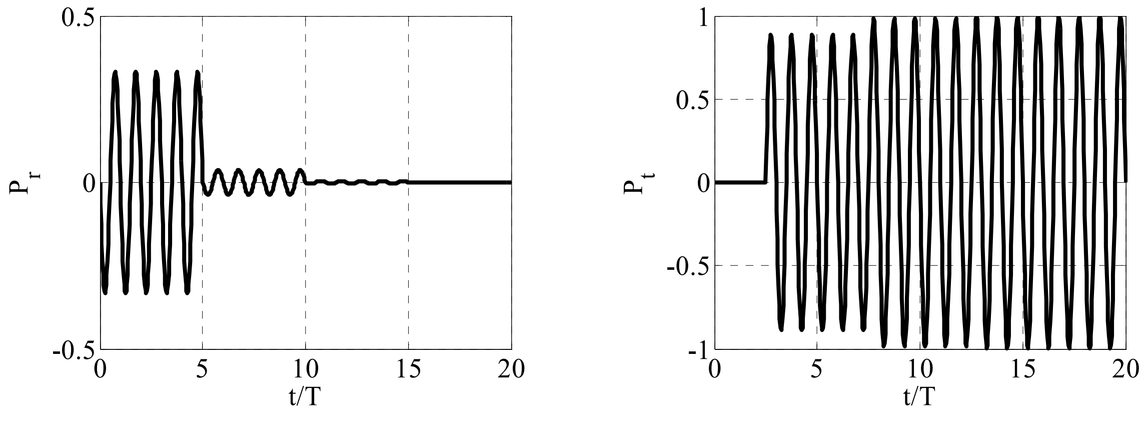

Fig.4. Reflected and transmitted wave calculated using equation (6): ς=0.5, $L = 2.5\lambda$

All presented in this paper results of theoretical and numerical investigations were carried out assuming that incident wave $p_i(0,t) = \sin\omega t$. Figure 4 presents reflected and transmitted wave obtained using equations (6) when ς=0.5 and $L = 2.5\lambda$.

Similar results we obtain solving numerically problem (7)-(9). The transmitted wave and pressure inside layer for t=20T (T - period of the incident wave) presents Fig. 5. Calculations were done for ς=0.93. Now changes of transmitted wave amplitude envelope are smaller than for ς=0.5. In this situation common ratio q is much smaller (see Fig. 3) and the amplitudes W_n converge to the limit 0 very quickly.

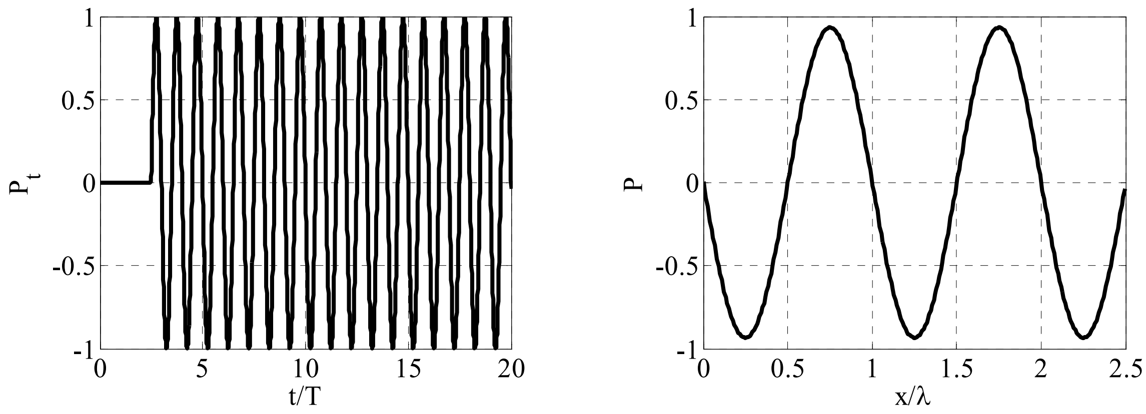

Fig.5. Transmitted wave and pressure inside layer (t=20T): ς=0.93, $L = 2.5\lambda$

Correct choice of values of numerical parameters is very important during numerical investigations. Figure 6 presents reflected wave obtain numerically for two different time step sizes Δt. In this situation the amplitudes of reflected waves, especially for $t > 10T$ when $\Delta t = T/32$, should be smaller. Precisely, when time elapses the value of reflected wave should be equal zero. Table 1 presents maximum absolute errors of reflected and transmitted waves obtained numerically solving problem (7)-(9) and using formulas (6) for different time step sizes.

a) b)

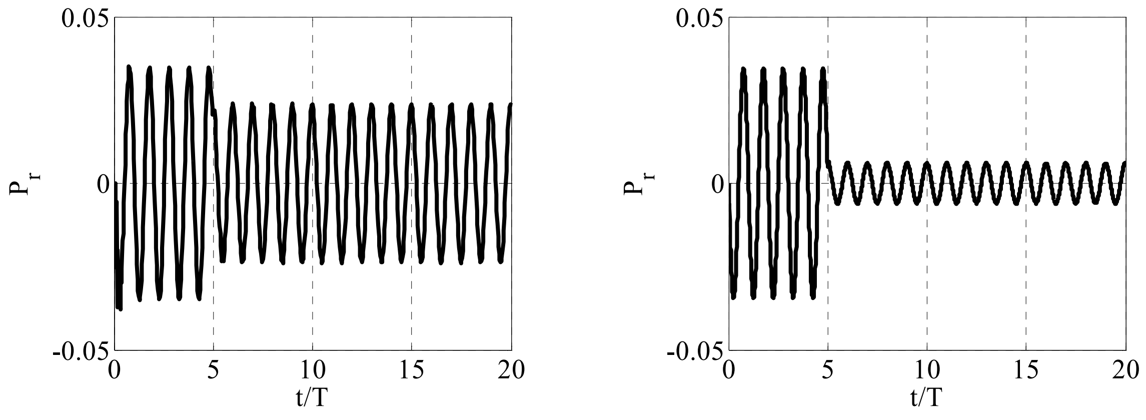

Fig.6. Reflected wave for different time step sizes: a) $\Delta t = T/32$, b) $\Delta t = T/128$

Tab.1. Maximum absolute error of reflected and transmitted waves

	$\Delta t = T/32$	$\Delta t = T/64$	$\Delta t = T/128$
Reflected wave	0.02413217	0.00981621	0.00632059
Transmitted wave	0.15111363	0.04162309	0.03859180

Using equations (5) we can analyze pressure field only for continuous waves. It means that only infinite harmonic signals with constant amplitudes is possible to investigate using this model. By means of equations (6) we can also study only signals with constant amplitudes but additionally we have possibility to investigate the finite signals. The numerical model gives us similar possibilities. Figure 7 presents the incident, reflected and transmitted waves calculated numerically assuming that the finite harmonic signal with the rectangular window (duration t=5T) is propagated in region I. Moreover we assume that the layer thickness $L = 5\lambda$.

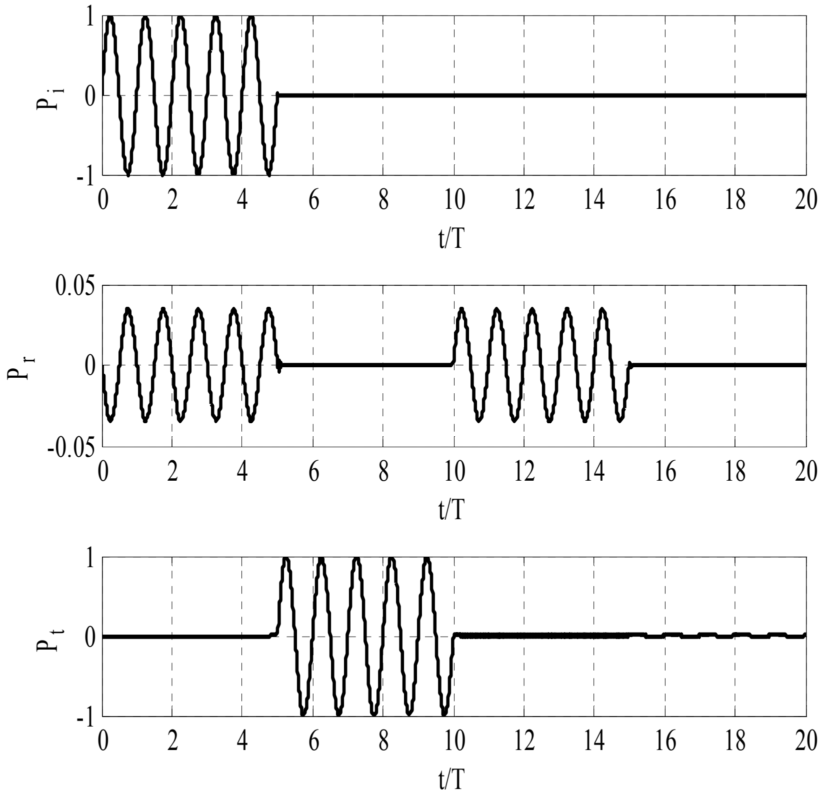

Fig.7. Incident, reflected and transmitted wave for finite signal: rectangular window

Presented till now results of theoretical investigations were carried out for signals with constant amplitudes. In practice, measured signals are not only finite but very often their amplitudes are not constant. Proposed in this paper numerical model allows to investigate waves propagation also in this situations. Figure 8 shows incident, reflected and transmitted waves when incident wave envelope is not constant. Values of all physical and numerical parameters were the same like previously, only incident wave was different.

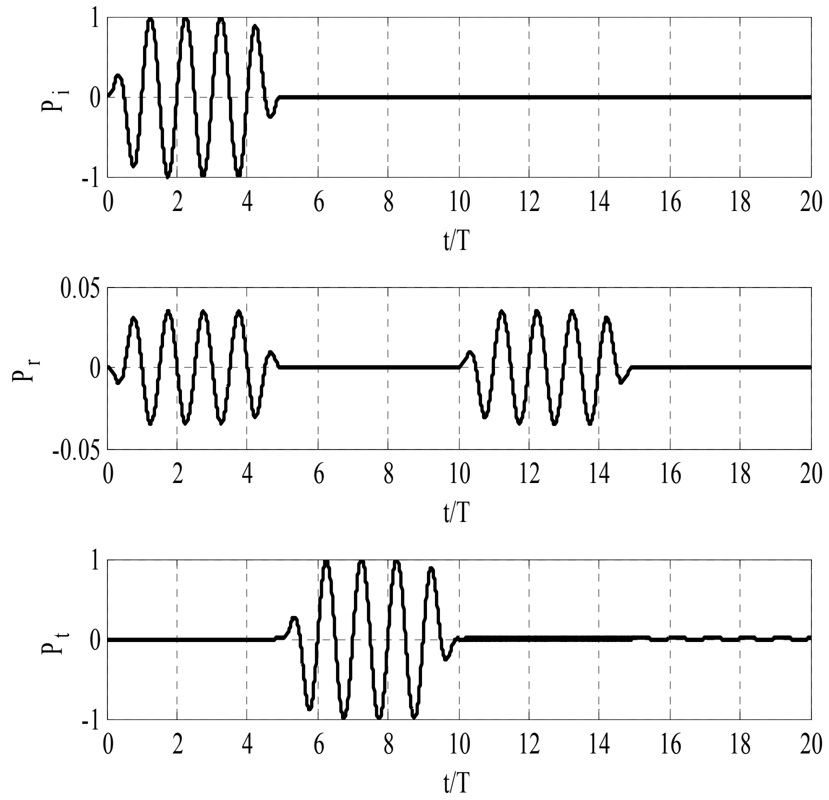

Fig.8. Incident, reflected and transmitted wave for finite signal

4. CONCLUSIONS

The plane acoustic waves generation and propagation in one-dimensional layer surrounded by medium with different physical properties were considered. Mathematical model and analytical formulas are presented. Some results of theoretical investigations are discussed. Proposed in this paper mathematical model can be used to study wave propagation for different infinite and finite signals propagated in media with different physical parameters. Theoretical models have much more limitations. They can be used only during investigation of propagation and generation waves with constant amplitudes. These theoretical models are considered in theoretical works. In this paper they illustrate the problem and are useful during verification of the numerical model. The finite-difference method was used to solve the problem numerically. Comparison of the results of numerical calculations with suitable analytical results confirms correctness of the mathematical model and computer programs. Proposed in this paper mathematical model does not cover all physical properties, for example attenuation. When they cannot be omitted, the model must be extended.

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