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APPLICATION OF THE DISTRIBUTED TRANSFER FUNCTION METHOD AND THE RIGID FINITE ELEMENT METHOD FOR MODELLING OF 2-D AND 3-D SYSTEMS

RAFAŁ HEIN, CEZARY ORLIKOWSKI

Mechanical Engineering Department, Gdansk University of Technology
e-mail: rahe@pg.gda.pl, corlikow@pg.gda.pl

Summary. In the paper application of the Distributed Transfer Function Method and the Rigid Finite Element Method for modelling of 2-D and 3-D systems is presented. In this method an elastic body is divided into 1-D distributed parameter elements (strips or prisms). The whole body (divided into strips or prism) is described by a set of coupled partial differential equations. Solving this equations in the state space form it is possible to obtain the response of the system under any external excitations as well as to predict the system spectrum.

1. INTRODUCTION

In the analysis of the elastic two or three dimensional (2-D or 3-D) systems the Finite Element Method (FEM) is widely used. The discretization methods yields to a set of ordinary differential equations. However, to obtain accurate results it is necessary to apply a great number of finite elements and to solve high order model (a big number of the second order equations).

The Distributed Transfer Function Method (TDFM) is an alternative approach for the analysis of a class of such systems (linear, one-dimensional). Still distributed parameter systems are given in terms of linear partial differential equations, similar to lumped parameter systems they can also be described by the transfer function method. In this case the distributed transfer function is the corresponding mathematical model. It contains all information about a system and enables to obtain the response under any excitation and to predict the system spectrum. Distributed Transfer Function Method does not assume any approximation by lumping technique. The response of the system can be presented in an exact and closed form.

In the paper application of the Distributed Transfer Function Method and the Rigid Finite Element Method for modelling of 2-D and 3-D systems is proposed. This is an alternative approach to the known Strip Distributed Transfer Function Method (SDTFM). In this method an elastic body is divided into strips or prisms (1-D distributed parameter elements). Each strip/prism represents one-dimensional distributed system and it is described by appropriate second order partial differential equation. By application of the Distributed Transfer Function Method, the response of each strip/prism can be obtained in an exact and closed form. The whole body is then described by a set of coupled partial differential equations. Finally, the response related to the whole body is presented in a semi - exact, closed form. This method is an extension of the DTFM method for one-dimensional distributed parameter systems.

2. TRANSFER FUNCTION METHOD FOR 1-D DISTRIBUTED PARAMETER SYSTEM

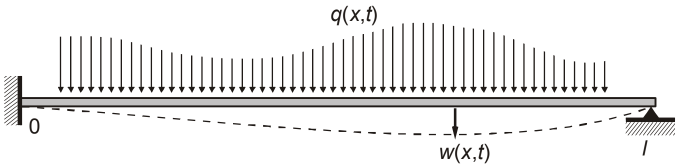

Fig. 1. One-dimensional distributed parameter system

Let us consider the distributed parameter system described by the one-dimensional, n-th order, linear partial differential equation after Laplace transformation with respect to time

$$\left(\sum_{i=0}^{n}\sum_{j=0}^{m} A_{ij}\frac{\partial^i}{\partial x^i}s^j\right)w(x,s)=q(x,s)\,, \tag{1}$$

with boundary conditions

$$M_i(w(0,s))+N_i w(l,s)=\gamma_i(s)\,, \qquad i=1,\ldots,n \tag{2}$$

and with zero initial conditions (for simplification), where: $q(x,t)$ – excitation, $w(x,t)$ – response, M_i , N_i – boundary condition operators, $\gamma_i(t)$ - known functions, s complex parameter.

Equations (1) ÷ (2) can be transformed into the state space form [3]

$$\frac{\partial}{\partial x}\boldsymbol{\eta}(x,s)=\boldsymbol{F}(s)\cdot\boldsymbol{\eta}(x,s)+\boldsymbol{u}(x,s)\,, \tag{3}$$

$$\boldsymbol{M}(s)\cdot\boldsymbol{\eta}(0,s)+\boldsymbol{N}(s)\cdot\boldsymbol{\eta}(l,s)=\boldsymbol{\gamma}(s)\,, \tag{4}$$

where: $\boldsymbol{\eta}(x,s)=col\left(w(x,s),\ \ \frac{\partial}{\partial x}w(x,s),\ \ \cdots,\ \ \frac{\partial^{n-1}}{\partial x^{n-1}}w(x,s)\right)$, $\boldsymbol{M}(s)$, $\boldsymbol{N}(s)$ – $n\times n$ matrices composed of M_i , N_i,

$$\boldsymbol{u}(x,s)=col\left(0,\ \ \cdots,\ \ 0,\ \ \frac{q(x,s)}{\sum_{j=1}^{m}A_{nj}s^j}\right)_{n\times 1},\ \boldsymbol{\gamma}(s)=\{\gamma_i(s)\}_{n\times 1}\,.$$

The solution of the equations (3) and (4) can be find in the following form [3]

$$\overline{\boldsymbol{\eta}}(x,s)=\int_0^l \boldsymbol{G}(x,\xi,s)\cdot\boldsymbol{q}(\xi,s)d\xi+\boldsymbol{H}(x,s)\cdot\boldsymbol{\gamma}(s)\,, \tag{5}$$

where:

$$\boldsymbol{G}(x,\xi,s)=\begin{cases}\mathbf{e}^{\boldsymbol{F}(s)x}\cdot(\boldsymbol{M}(s)+\boldsymbol{N}(s)\cdot\mathbf{e}^{\boldsymbol{F}(s)l})^{-1}\cdot\boldsymbol{M}(s)\cdot\mathbf{e}^{-\boldsymbol{F}(s)\xi} & \xi<x\\ -\mathbf{e}^{\boldsymbol{F}(s)x}\cdot(\boldsymbol{M}(s)+\boldsymbol{N}(s)\cdot\mathbf{e}^{\boldsymbol{F}(s)l})^{-1}\cdot\boldsymbol{N}(\mathbf{s})\cdot\mathbf{e}^{\boldsymbol{F}(s)(l-\xi)} & \xi>x\end{cases} \tag{6}$$

$$\boldsymbol{H}(x,s)=\mathbf{e}^{\boldsymbol{F}(s)x}\cdot\left(\boldsymbol{M}(s)+\boldsymbol{N}(s)\cdot\mathbf{e}^{\boldsymbol{F}(s)}\right)^{-1}, \tag{7}$$

$$\boldsymbol{G}(x,\xi,s)=\{G_{ik}(x,\xi,s)\}_{n\times n}\,, \tag{8}$$

$$\boldsymbol{H}(x,s)=\{H_{ik}(x,s)\}_{n\times n}\,. \tag{9}$$

Hence, the system response can be expressed in an integral form (5) with the integral kernel being the Green's function of the system. The transfer function of the distributed system $G_{ik}(x,\xi,s)$ is obtained by the Laplace transformation (with respect to time) of the Green's function. The transfer function contains all information about the system and enables to obtain the response of the system under any initial or external excitations as well as to predict the system spectrum and stability. The method can be also applied for modeling of complex distributed-lumped parameter systems [2, 3].

3. TRANSFER FUNCTION METHOD FOR 2-D AND 3-D DISTRIBUTED PARAMETER SYSTEMS

The DTFM can be extended to 2-D and 3-D continua [4, 5]. In [4, 5] to obtain the appropriate mathematical model DTFM and FEM methods are applied. In this paper a new, alternative approach is proposed. Instead of FEM, the Rigid Finite Element Method (RFEM) [6] is used. In the RFEM the idea of shape function in not applied. Comparing to conventional approach presented in [4, 5] the proposed method of modelling is much more simple and easier for implementation.

In this method of modelling the body is divided into strips (for 2-D system - Fig. 1b) and prism (for 3-D system - Fig. 1c). Each strip or prism represents one-dimensional distributed system and it is described by appropriate second order partial differential equation. However, these equations have also terms related to interactions between strip/prism. Hence, the given system can be described by a set of couplet (interactions between elements) second order partial differential equations.

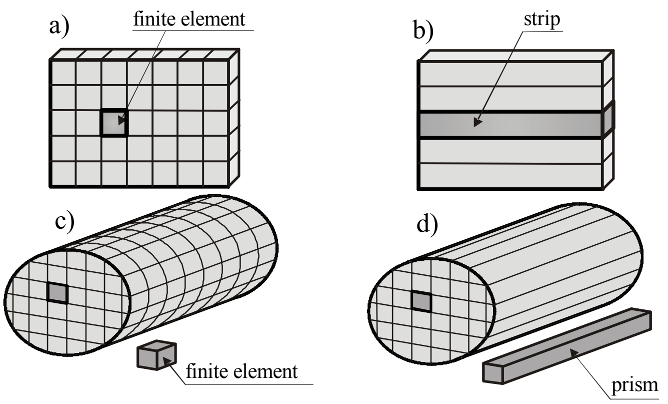

Fig. 2. Spatial discretization of 2D and 3D body: a), c) FEM, b), d) DTMF method

These differential equations can be presented in the following form:

$$[\boldsymbol{A}_{02}s^2 + \boldsymbol{A}_{20}\frac{\partial^2}{\partial x^2} + \boldsymbol{A}_{10}\frac{\partial}{\partial x} + \boldsymbol{A}_{00}]\boldsymbol{w}(x,s) = \boldsymbol{q}(x,s) \tag{10}$$

with boundary conditions

$$\boldsymbol{M}_i\boldsymbol{w}(0,s) + \boldsymbol{N}_i\boldsymbol{w}(l,s) = \boldsymbol{\gamma}_i(s), \qquad i=1,2, \tag{11}$$

where matrices $\boldsymbol{A}_{02}$, $\boldsymbol{A}_{20}$, $\boldsymbol{A}_{10}$, $\boldsymbol{A}_{00}$ contain differential equations coefficients, matrices $\boldsymbol{M}_i$, $\boldsymbol{N}_i$ are composed of boundary condition operators (2).

The equations (10, 11) may be written in the state space representation:

$$\frac{\partial}{\partial x}\boldsymbol{\eta}(x,s) = \boldsymbol{F}(s)\boldsymbol{\eta}(x,s) + \boldsymbol{u}(x,s), \tag{12}$$

$$\boldsymbol{M}(s)\boldsymbol{\eta}(0,s)+\boldsymbol{N}(s)\boldsymbol{\eta}(l,s)=\boldsymbol{\gamma}(s)\,, \tag{13}$$

where:

$$\boldsymbol{\eta}=col\big(\boldsymbol{w}(x,s),\quad \boldsymbol{w}'(x,s)\big),\ \boldsymbol{F}(s)=\begin{bmatrix}\boldsymbol{0} & \boldsymbol{I}\\ -\boldsymbol{A}_{20}^{-1}(\boldsymbol{A}_{02}\cdot s^2+\boldsymbol{A}_{00}) & -\boldsymbol{A}_{20}^{-1}\boldsymbol{A}_{10}\end{bmatrix},\ \boldsymbol{u}(x,s)=\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{A}_{20}^{-1}\boldsymbol{q}(x,s)\end{bmatrix},$$

matrices $\boldsymbol{M}$, $\boldsymbol{N}$ can be obtain from $\boldsymbol{M}_i$, $\boldsymbol{N}_i$, $\boldsymbol{I}$ – identity matrix.
The response of the system described by (12, 13) can be presented in the integral form (5). It enables to obtain the system eigenvalues, eigenfunctions, frequency response and the response to given harmonic excitation.

4. ILLUSTRATIVE EXAMPLE

As an simple illustrative example let us consider the membrane (Fig. 3) with the following parameters: m=10 kg/m^2 (mass per unit area), T_x=T_y=1000 N/m (force per unit length), a=b=1 m, Δy=1/5 m.

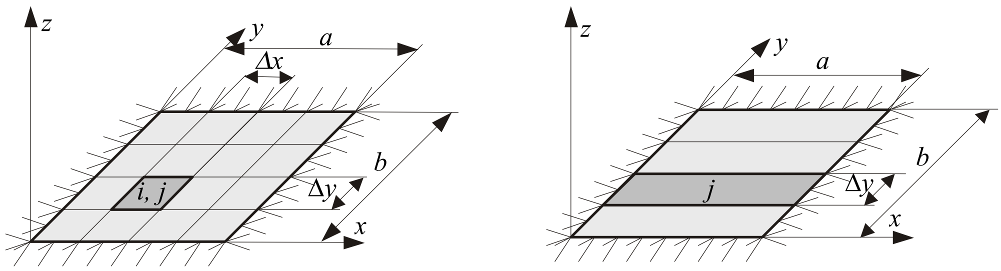

Fig. 3. Membrane as 2-D elastic body

The RFEM equation (in this case the difference equation) for i, j membrane element can be written as

$$ms^2\Delta x\Delta y z_{i,j}+T_x\Delta y\left[\frac{z_{i,j}-z_{i-1,j}}{\Delta x}+\frac{z_{i,j}-z_{i+1,j}}{\Delta x}\right]+T_y\Delta x\left[\frac{z_{i,j}-z_{i,j-1}}{\Delta y}+\frac{z_{i,j}-z_{i,j+1}}{\Delta y}\right]=f_{i,j}\Delta x\Delta y\,. \tag{14}$$

Dividing (14) by Δx one obtains

$$ms^2\Delta y z_{i,j}+T_x\Delta y\left[\frac{\dfrac{z_{i,j}-z_{i-1,j}}{\Delta x}+\dfrac{z_{i,j}-z_{i+1,j}}{\Delta x}}{\Delta x}\right]+\frac{T_y}{\Delta y}[(z_{i,j}-z_{i,j-1})+(z_{i,j}-z_{i,j+1})]=f_{i,j}\Delta y\,. \tag{15}$$

Assuming $\Delta x\to 0$, the equation (15) yields

$$ms^2\Delta y z_j(x)-T_x\Delta y z_j''(x)+\frac{T_y}{\Delta y}[(z_j(x)-z_{j-1}(x))+(z_j(x)-z_{j+1}(x))]=f_j(x)\Delta y\,. \tag{16}$$

For example, in the case of 4 strips (for j=1, 2, 3, 4) the equations (16) have the form:

$$ms^2\Delta y z_1-T_x\Delta y z_1''+\frac{T_y}{\Delta y}[(z_1-0)+(z_1-z_2)]=f_1\Delta y\,,$$

$$ms^2\Delta y z_2-T_x\Delta y z_2''+\frac{T_y}{\Delta y}[(z_2-z_1)+(z_2-z_3)]=f_2\Delta y\,,$$

$$ms^2\Delta y z_3 - T_x \Delta y z_3'' + \frac{T_y}{\Delta y}[(z_3 - z_2) + (z_3 - z_4)] = f_3 \Delta y\,,$$

$$ms^2\Delta y z_4 - T_x \Delta y z_4'' + \frac{T_y}{\Delta y}[(z_4 - z_3) + (z_4 - 0)] = f_4 \Delta y\,,$$

and boundary conditions:

$$z_1(0,s) = z_1(l,s) = z_2(0,s) = z_2(l,s) = z_3(0,s) = z_3(l,s) = z_4(0,s) = z_4(l,s) = 0\,.$$

Above equations can be written in the form of equations (10, 11), where:

$\boldsymbol{A}_{02} = diag(m\Delta y,\quad m\Delta y,\quad m\Delta y,\quad m\Delta y)$, $\boldsymbol{A}_{20} = diag(-T_x\Delta y,\quad -T_x\Delta y,\quad -T_x\Delta y,\quad -T_x\Delta y)$,
$\boldsymbol{A}_{10} = \boldsymbol{0}_{(4\times 4)}$,

$$\boldsymbol{A}_{00} = \begin{bmatrix} \frac{2T_y}{\Delta y} & -\frac{T_y}{\Delta y} & 0 & 0 \\ -\frac{T_y}{\Delta y} & \frac{2T_y}{\Delta y} & -\frac{T_y}{\Delta y} & 0 \\ 0 & -\frac{T_y}{\Delta y} & \frac{2T_y}{\Delta y} & -\frac{T_y}{\Delta y} \\ 0 & 0 & -\frac{T_y}{\Delta y} & \frac{2T_y}{\Delta y} \end{bmatrix},\ \boldsymbol{M}_1 = \boldsymbol{N}_2 = \boldsymbol{I}_{(4\times 4)},\ \boldsymbol{N}_1 = \boldsymbol{M}_2 = \boldsymbol{0}_{(4\times 4)},$$

and next in the form (12, 13), where

$$\boldsymbol{M} = \left[\begin{array}{c:c} \boldsymbol{M}_1 & \boldsymbol{0} \\ \hdashline \boldsymbol{0} & \boldsymbol{N}_1 \end{array}\right],\ \boldsymbol{N} = \left[\begin{array}{c:c} \boldsymbol{0} & \boldsymbol{N}_2 \\ \hdashline \boldsymbol{M}_2 & \boldsymbol{0} \end{array}\right].$$

Solving the state space equations related to the given membrane some results have been obtained. Fig. 4 presents frequency characteristics of the membrane from Fig. 3 (input force at x=0.1, y=0.5, and output displacement x=0.4, y=0.5). In the Tab. 1 natural frequencies of the investigated membrane, for different mathematical models, are presented.

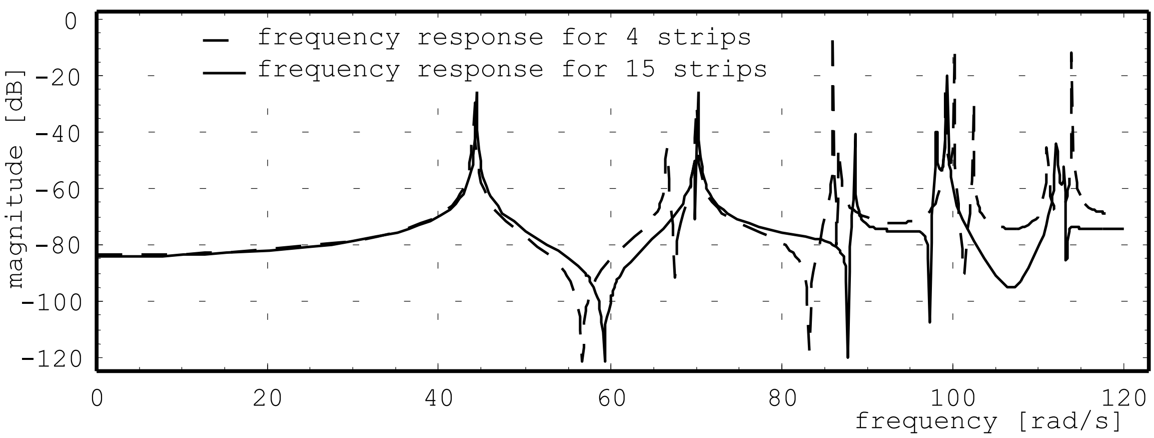

Fig. 4. Frequency response of membrane the membrane from Fig. 3.

Tab. 1. Natural frequencies ω [rad/s] of considered membrane

Exact frequencies	**RFEM** 16 finite elements	**Proposed method** 4 strips	16 strips
44.428	43.701	44.066	44.395
44.428	43.701	44.066	44.395
70.248	66.406	66.647	69.880
70.248	66.406	70.019	70.229
88.857	83.125	86.039	88.525
88.857	83.125	86.787	88.525
99.345	86.602	99.184	98.065
99.345	86.602	100.160	99.330
113.271	100.000	102.435	112.142
113.271	100.000	102.435	113.047
133.286	100.000	111.075	132.335
133.286	100.000	113.987	132.335

5. CONCLUSION

In the paper application of the Distributed Transfer Function Method and the Rigid Finite Element Method for modelling of 2-D and 3-D systems is proposed. In this method an elastic body is divided into strips or prisms. Each strip or prism represents one-dimensional distributed system and it is described by appropriate second order partial differential equation. By application of the Distributed Transfer Function Method, the response of each element can be obtained in an exact and closed form. The whole body (divided into strips or prism) is then described by a set of coupled partial differential equations. Simple, illustrative example prooves that proposed approach is relatively easy and convenient for computer coding.

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