

## PARITY VERTEX COLOURING OF GRAPHS

PIOTR BOROWIECKI

*Department of Algorithms and System Modeling*  
*Faculty of Electronics, Telecommunications and Informatics*  
*Gdańsk University of Technology*  
*Narutowicza 11/12, 80–233 Gdańsk, Poland*  
**e-mail:** pborowie@eti.pg.gda.pl

KRISTÍNA BUDAJOVÁ

*Faculty of Aeronautics*  
*Technical University Košice*  
*Rampová 7, SK–04121 Košice, Slovak Republic*  
**e-mail:** kristina.budajova@tuke.sk

STANISLAV JENDROL’\*

*Institute of Mathematics*  
*P.J. Šafárik University Košice*  
*Jesenná 5, SK–04154 Košice, Slovak Republic*  
**e-mail:** stanislav.jendrol@upjs.sk

AND

STANISLAV KRAJČI

*Institute of Computer Sciences*  
*P.J. Šafárik University Košice*  
*Jesenná 5, SK–041 54 Košice, Slovak Republic*  
**e-mail:** stanislav.krajci@upjs.sk

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### Abstract

A parity path in a vertex colouring of a graph is a path along which each colour is used an even number of times. Let  $\chi_p(G)$  be the least number of colours in a proper vertex colouring of  $G$  having no parity path. It is proved that for any graph  $G$  we have the following tight bounds  $\chi(G) \leq \chi_p(G) \leq |V(G)| - \alpha(G) + 1$ , where  $\chi(G)$  and  $\alpha(G)$  are the chromatic number and the independence number of  $G$ , respectively. The bounds are improved for trees. Namely, if  $T$  is a tree with diameter  $\text{diam}(T)$  and radius  $\text{rad}(T)$ , then  $\lceil \log_2(2 + \text{diam}(T)) \rceil \leq \chi_p(T) \leq 1 + \text{rad}(T)$ . Both bounds are tight. The second thread of this paper is devoted to relationships between parity vertex colourings and vertex rankings, i.e. a proper vertex colourings with the property that each path between two vertices of the same colour  $q$  contains a vertex of colour greater than  $q$ . New results on graphs critical for vertex rankings are also presented.

**Keywords:** parity colouring, graph colouring, vertex ranking, ordered colouring, tree, hypercube, Fibonacci number.

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## 1. INTRODUCTION

The graphs we consider are finite, simple and undirected. Let the *usage* of a colour on a path be the number of times it appears along the path. The *parity path* is a path for which every colour has even usage. We define the *parity vertex colouring* to be a proper vertex colouring having no parity path. The *parity vertex chromatic number*  $\chi_p(G)$  is the minimum number of colours in a parity vertex colouring of  $G$ . Observe that using distinct colours on all vertices produces a parity vertex colouring while paths on two vertices force  $\chi_p(G) \geq \chi(G)$ , where  $\chi(G)$  is the chromatic number of  $G$ . For any terminology and notations not defined here the readers are referred to [4].

The work on parity colouring of graphs was initiated in [2] and it began by studying which graphs embed in the hypercube. Both problems are closely related and the question of embeddings is motivated by the fact that hypercube is a common and one of the most efficient architectures for parallel computation [8]. Inspired by the recent paper of Bunde, Milans, West and Wu [2], who studied parity edge colourings, we begin the study of parity vertex colourings. A significant part of this paper is also devoted to the intriguing relationship between parity colouring and *ranking of vertices*,



i.e. a colouring with the property that each path between two vertices of the same colour  $q$  contains a vertex of colour greater than  $q$ . The study of vertex rankings is motivated by parallel Cholesky factorization of matrices [9] and applications in VLSI layout [10]. It is also worth pointing out that every parity colouring is a *conflict-free colouring*, i.e. a colouring in which every path uses some colour exactly once. Conflict-free colourings were recently studied due to their theoretical and practical importance, e.g. for frequency assignment in cellular networks [5].

## 2. FUNDAMENTAL BOUNDS ON THE PARITY VERTEX CHROMATIC NUMBER

**Theorem 1.** *Let  $G = (V, E)$  be an  $n$ -vertex graph with the chromatic number  $\chi(G)$  and the independence number  $\alpha(G)$ . Then*

$$\chi(G) \leq \chi_p(G) \leq n - \alpha(G) + 1.$$

**Proof.** Two adjacent vertices form a path, therefore they must have different colours, hence  $\chi_p(G) \geq \chi(G)$ . It is easy to see that  $\chi_p(K_n) = n = \chi(K_n)$ . Hence the lower bound is tight.

Let  $S$  be a maximum independent set of  $G$ , i.e.  $|S| = \alpha(G)$ . Let us colour the vertices of  $G$  in the following manner. The vertices of  $S$  are coloured with the same colour, say 1. The vertices of  $V - S$  are coloured with different colours from the set  $\{2, 3, \dots, n - \alpha(G) + 1\}$ . Because every path on at least two vertices contains a vertex from the set  $V - S$  this colouring is a parity vertex colouring. ■

The tightness of the upper bound follows from Theorem 2.

**Lemma 1.** *For the union  $G \cup H$  of any two vertex disjoint graphs  $G$  and  $H$  we have*

$$\chi_p(G \cup H) = \max\{\chi_p(G), \chi_p(H)\}.$$

Let  $G + H$  be a *join* of two graphs  $G$  and  $H$  defined as follows  $V(G + H) = V(G) \cup V(H)$  and  $E(G + H) = E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}$ .

**Theorem 2.** *For any graphs  $G$  and  $H$  we have*

$$\chi_p(G + H) = \min\{\chi_p(G) + |V(H)|, \chi_p(H) + |V(G)|\}.$$



**Proof.** ( $\leq$ ) Let  $\chi_p(G) = r$ . Whenever  $G$  is coloured using colours  $1, \dots, r$ , it is possible to colour  $H$  with colours  $r + 1, \dots, r + |V(H)|$ . Consequently,  $\chi_p(G+H) \leq \chi_p(G) + |V(H)|$ . Similarly, we get  $\chi_p(G+H) \leq \chi_p(H) + |V(G)|$ . From the above it follows that  $\chi_p(G+H) \leq \min\{\chi_p(G) + |V(H)|, \chi_p(H) + |V(G)|\}$ .

( $\geq$ ) If  $\chi_p(G) = |V(G)|$  and  $\chi_p(H) = |V(H)|$ , then  $\chi_p(G+H) = |V(H)| + |V(G)| \geq \min\{\chi_p(G) + |V(H)|, \chi_p(H) + |V(G)|\}$ . Now, suppose that  $\chi_p(G) < |V(G)|$ . Then there exist two vertices of  $G$ , say  $u$  and  $v$ , coloured with the same colour. Therefore each vertex of  $H$  has to be coloured with a different colour. Otherwise, for any  $x, y \in V(H)$  there would be a parity path  $(u, x, v, y)$ . Moreover, because of join, colours used for  $V(H)$  have to be different from all colours used for  $V(G)$ . Consequently,  $\chi_p(G+H) = \chi_p(G) + |V(H)| \geq \min\{\chi_p(G) + |V(H)|, \chi_p(H) + |V(G)|\}$ . ■

From Theorem 2 we immediately have the following corollary for complete bipartite graphs.

**Corollary 1.** Let  $K_{r,s}$  be a complete bipartite graph such that  $r \leq s$ . Then

$$\chi_p(K_{r,s}) = r + 1.$$

Similar reasoning can be applied to find the parity vertex chromatic number of complete  $k$ -partite graphs.

**Corollary 2.** Let  $K_{r_1, \dots, r_k}$  be a complete  $k$ -partite graph. Then

$$\chi_p(K_{r_1, \dots, r_k}) = \sum_{i=1}^k r_i - \max_{1 \leq i \leq k} r_i + 1.$$

**Proof.** By induction on  $k$ . ■

**Theorem 3.** Let  $S$  be a cut-set of a graph  $G$  and let  $H_1, \dots, H_r$  be the components of  $G[V - S]$ . Then

$$\chi_p(G) \leq \max_{1 \leq i \leq r} \{\chi_p(H_i)\} + |S|.$$

**Proof.** Assuming that vertices of each  $H_i$  are coloured with consecutive colours starting from 1, it is enough to colour each vertex of  $S$  with a different colour, which has not been used in  $V(H_1) \cup \dots \cup V(H_r)$ . ■

The bound is tight, i.e. there exist graphs for which the equality holds, e.g.  $K_1 + (K_p \cup K_q)$ ,  $p, q \geq 1$ . Notice that the cut-set  $S$  in Theorem 3 may be assumed to be minimal. A recursive application of theorem gives an algorithm, which in general does not have to be efficient. However, for some classes of graphs the desired upper bound can be computed in polynomial time.

In [2] it was proved that for every path  $P_m$  on  $m$  vertices  $\chi'_p(P_m) = \lceil \log_2 m \rceil$ . Since  $P_n$  is the line graph of the path  $P_{n+1}$  we obtain the following.

**Theorem 4.** *Let  $P_n$  be an  $n$ -vertex path. Then*

$$\chi_p(P_n) = \lceil \log_2(n+1) \rceil.$$

To be able to formulate our next results, we recall some definitions. The *distance* between two vertices  $u$  and  $v$  of a graph  $G$ ,  $\text{dist}(u, v)$ , is defined to be the length of the shortest path between  $u$  and  $v$ . The *eccentricity*  $e(v)$  of a vertex  $v$  in the graph  $G$  is the distance from  $v$  to a vertex furthest from  $v$ , i.e.  $e(v) = \max\{\text{dist}(v, u) | u \in V(G)\}$ . The *radius*  $\text{rad}(G)$  of a connected graph  $G$  is defined as  $\text{rad}(G) = \min\{e(v) | v \in V(G)\}$ , and the *diameter*  $\text{diam}(G)$  of a connected graph  $G$  is defined by  $\text{diam}(G) = \max\{e(v) | v \in V(G)\}$ . A vertex  $c$  is called *central* if its eccentricity equals  $\text{rad}(G)$ , i.e.  $e(c) = \text{rad}(G)$ . The following theorem strengthens Theorem 1 for trees.

**Theorem 5.** *Let  $T$  be a tree. Then*

$$\lceil \log_2(2 + \text{diam}(T)) \rceil \leq \chi_p(T) \leq 1 + \text{rad}(T).$$

**Proof.** The lower bound is a consequence of the above Theorem 4 when considering the longest path in  $T$ . To show the upper bound it is sufficient to find a colouring with  $1 + \text{rad}(T)$  colours having the required property that every path in  $T$  uses some colour an odd number of times. It is well known that every tree  $T$  has exactly one central vertex  $c$  if  $\text{diam}(T)$  is even and exactly two central vertices  $c$  and  $c^*$  which are adjacent if  $\text{diam}(T)$  is odd. Define the following  $k$ -colouring  $\varphi : V(T) \rightarrow \{1, \dots, k\}$  of the vertices of  $T$ , where  $k = 1 + \text{rad}(T)$ . In both cases we colour vertex  $c$  with the colour  $\varphi(c) = k$  and every vertex  $v$  such that  $\text{dist}(c, v) = j$  with the colour  $\varphi(v) = j$ . Clearly,  $j \leq k - 1 = \text{rad}(T)$ . Next, we need to show that every path  $Q$  in  $T$  uses some colour an odd number of times. In fact, we argue that on every path  $Q$  there is a vertex which uses some colour exactly once.



Namely, if  $Q$  is a path in  $T$ , then the colour of the vertex in  $Q$  that is closest to  $c$  appears only once along  $Q$ . ■

The tightness of the lower bound follows from Theorem 4 while the upper bound is tight by Theorem 6.

**Theorem 6.** *For all  $r \in \mathbb{N}$  there is a tree  $T_r$  with the radius  $r = \text{rad}(T_r)$  such that  $\chi_p(T_r) = r + 1$ . Moreover, for  $r > 0$  there are infinitely many such trees.*

**Proof.** Define (by induction) a sequence  $(T_r : r \in \mathbb{N})$  of trees in the following way:

1.  $T_0$  consists of only one vertex (its root  $t_0$ ) and no edge,
2. By induction, there exists a rooted tree  $T_{r-1}$  with radius  $r - 1$  and parity vertex chromatic number at least  $r$ . Let  $T_r$  be the rooted tree obtained from  $2^{r-1} + 1$  copies of  $T_{r-1}$  by introducing a new vertex  $t_r$  to serve as the root of  $T_r$  and adding edges between  $t_r$  and the roots of the copies of  $T_{r-1}$ .

We show that the parity vertex chromatic number of  $T_r$  is at least  $r + 1$ . Suppose for a contradiction that  $\varphi$  is a parity vertex colouring of  $T_r$  that uses only  $r$  colours. Let  $a = \varphi(t_r)$ . We consider two cases.

First suppose that the colouring  $\varphi$  does not use  $a$  in any copy  $S$  of  $T_{r-1}$ . In this case, the induction hypothesis implies that  $\varphi$  uses at least  $r$  colours on vertices in  $S$ , none of which is  $a$ . It follows that  $\varphi$  uses at least  $r + 1$  colours on  $T_r$ , a contradiction.

Otherwise,  $\varphi$  uses  $a$  in each copy of  $T_{r-1}$ . For each  $1 \leq j \leq 2^{r-1} + 1$  let  $v_j$  be some vertex in the  $j$ -th copy of  $T_{r-1}$  with  $\varphi(v_j) = a$  and let  $P_j$  be the shortest path from  $t_r$  to  $v_j$ . Because there are  $2^{r-1} + 1$  paths of the form  $P_j$ , the pigeonhole principle implies that there exist two paths  $P_p$  and  $P_q$  which agree in the parity of the usage of each colour, with the possible exception of  $a$ . Let  $P$  be a path formed by a concatenation of  $P_p$  with (the reverse of)  $P_q$ . A path  $P'$  obtained by the removal of one of the endvertices of  $P$  is a parity path. But now  $P'$  contradicts that  $\varphi$  is a parity vertex colouring of  $T_r$ .

It is easy to construct an infinite family of trees of radius  $r$  for every  $r > 0$ . Take  $s$  copies of  $T_{r-1}$  for arbitrary  $s > 2^{r-1}$  instead of  $2^{r-1} + 1$  and proceed as above. ■

3.  $d$ -DIMENSIONAL CUBES

It would be interesting to determine the exact values of parity vertex chromatic number for some families of graphs. Very intriguing candidates are  $d$ -dimensional cubes  $Q_d, d \geq 0$ . It is easy to see that  $\chi_p(Q_0) = 1, \chi_p(Q_1) = 2$  and  $\chi_p(Q_2) = 3$ . For the next value of  $d$  we have

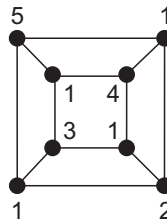


Figure 1. Parity vertex colouring of 3-dimensional cube  $Q_3$ .

**Lemma 2.**  $\chi_p(Q_3) = 5$ .

*Proof.* From Figure 1 it follows that  $\chi_p(Q_3) \leq 5$ . Because  $Q_3$  contains  $P_8$  as a subgraph, it follows from Theorem 4 that  $\chi_p(Q_3) \geq 4$ . Without loss of generality let us suppose that there is a parity vertex colouring using four colours. Since the graph  $Q_3$  is hamiltonian and has exactly eight vertices there are at least three vertices of  $Q_3$  coloured with the same colour, say 1. Let the colour 1 be used exactly three times. These vertices form an independent set in  $Q_3$ . Because any maximal independent set of  $Q_3$  has four vertices, the fourth vertex of this set is coloured with different colour, say 2. If any other vertex has colour 2 it is only a neighbour of all three vertices coloured 1. The remaining three vertices of  $Q_3$  must have mutually distinct colours. Otherwise, a parity cycle on four vertices appears. A contradiction that four colours are enough.

Let colour 1 be used four times. Vertices which are coloured with this colour form a maximal independent set of  $Q_3$ . If there is another colour used twice, then this colour cannot be used on the same cycle on four vertices. However, for each two vertices from remaining four ones, coloured with the same colour, there exists a common cycle of length 4 which contains a parity path  $P_4$  as a subgraph. Hence the remaining vertices must be coloured with different colours, a contradiction. ■

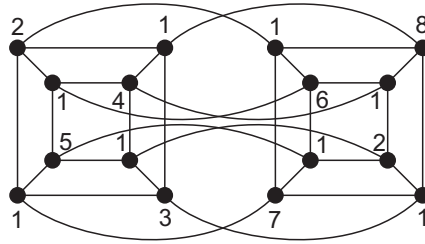


Figure 2. Parity vertex colouring of 4-dimensional cube  $Q_4$ .

Concerning graph  $Q_4$ . From Figure 2 we have that  $\chi_p(Q_4) \leq 8$ . Moreover, we have checked by computer that  $\chi_p(Q_4) > 7$ , hence  $\chi_p(Q_4) = 8$ . We strongly believe that the following question has an affirmative answer.

**Problem 1.** Is it true that  $\chi_p(Q_d) = F_{d+2}$ , where  $F_i$  is the  $i$ -th Fibonacci number?

#### 4. PARITY AND RANKING

A *vertex ranking* of a graph is a proper vertex colouring by a linearly ordered set of colours such that for every path in the graph with end vertices of the same colour there is a vertex on this path with a higher colour (see [3] for a survey on rankings). The vertex ranking problem asks for a vertex ranking of a given graph  $G$  which has the minimum number of colours. This number denoted by  $\chi_r(G)$  is the *vertex ranking number* of  $G$ . Notice that in any connected graph  $G$  there exists exactly one vertex coloured with the maximum colour  $\chi_r(G)$ .

**Theorem 7.** *Every vertex ranking of graph  $G$  is a parity vertex colouring of  $G$  and consequently we have  $\chi(G) \leq \chi_p(G) \leq \chi_r(G)$ .*

**Proof.** Assume on the contrary that  $\varphi$  is a vertex ranking of  $G$ , which is not a parity vertex colouring, i.e.  $G$  contains a parity path  $P$ . Let  $r$  be the maximum colour used on  $P$ . Since  $P$  contains at least two vertices coloured  $r$ , choose two of them, say  $u, v$  in such a way that no other vertex of colour  $r$  lies on  $P$  between  $u$  and  $v$ . Following the definition of ranking, a vertex of colour greater than  $r$  exists for each path between  $u$  and  $v$ , which contradicts the maximality of  $r$ . ■



From Theorem 7 it also follows that whenever  $G$  is a graph for which  $\chi_r(G) = \chi(G)$ , then  $\chi_p(G) = \chi(G)$ . Moreover, in [1] it was proved that any graph  $G$  for which  $\chi_r(G) = \chi(G)$  satisfies  $\chi(G) = \omega(G)$ , where  $\omega(G)$  denotes the clique number of  $G$ . Hence whenever  $\chi_r(G) = \chi(G)$ , then  $\chi_p(G) = \omega(G)$ . There exist graphs for which  $\chi_p(G) = \omega(G)$  and  $\chi_r(G) > \omega(G)$ . Namely, the graph  $K_n^+$  is obtained from  $K_n$ , having vertex set  $\{v_1, v_2, \dots, v_n\}$ , by addition of  $n$  new vertices  $u_i$  and  $n$  new edges  $v_i u_i, i \in \{1, 2, \dots, n\}$ . It is not hard to see that for  $n \geq 3$  we have  $\chi_p(K_n^+) = n = \omega(K_n^+) < \chi_r(K_n^+) = n+1$ .

*Trivially perfect graphs* can be characterized by forbidding  $C_4$  and  $P_4$  as the induced subgraphs [6]. If we require the equality of  $\chi_p$  and  $\chi$  to hold for every subgraph, then we can prove the following theorem which is also known to hold for vertex rankings [1].

**Theorem 8.** *For a graph  $G = (V, E)$  the following conditions are equivalent:*

- (i)  $G$  is trivially perfect,
- (ii)  $\chi_r(G[A]) = \chi(G[A])$  for every  $A \subseteq V(G)$ ,
- (iii)  $\chi_p(G[A]) = \chi(G[A])$  for every  $A \subseteq V(G)$ .

**Proof.** The equivalence of (i) and (ii) is proved in [1]. The implication (iii) $\Rightarrow$ (i) is obvious since  $\chi_p(C_4) = \chi_p(P_4) = 3$  while  $\chi(C_4) = \chi(P_4) = 2$ . It remains to prove the implication (i) $\Rightarrow$ (iii). Let  $G$  be trivially perfect and assume that there exists a subset  $A$  such that  $\chi_p(G[A]) > \chi(G[A])$ . By the equivalence of (i) and (ii) we have  $\chi_p(G[A]) > \chi_r(G[A])$ , a contradiction by Theorem 7. ■

Note that arguments used within the proof hold also for conflict-free colouring.

The class of *cographs*, known also as the class of  $P_4$ -free graphs, is the smallest class of graphs fulfilling the following conditions:

1. The graph  $K_1$  is a cograph.
2. If  $G_1, G_2$  are vertex disjoint cographs, then
  - (a) their union  $G_1 \cup G_2$  is a cograph,
  - (b) their join  $G_1 + G_2$  is a cograph.

To see that the analogue of Theorem 2 holds for rankings, observe that whenever two vertices of  $G_1$  have the same colour  $q$  in some ranking of  $G_1$ , then all vertices in  $V(G_2)$  have to be coloured differently with colours larger



than  $q$ . Now, since analogues of Lemma 1 and Theorem 2 hold as well for parity vertex colourings as for vertex rankings, we immediately have the following theorem.

**Theorem 9.** *If  $G$  is a cograph, then  $\chi_p(G) = \chi_r(G)$ .*

With the formulas given in Lemma 1 and Theorem 2 we can compute the parity vertex chromatic number of any cograph in polynomial time.

Further analysis of the relationships between ranking and parity chromatic number leads to the following intriguing problem which turned out to be challenging even for basic classes of graphs like trees.

**Problem 2.** For what classes of graphs there exists  $c \in \mathbb{N}$  such that

$$\chi_r(G) - \chi_p(G) \leq c?$$

It seems that critical and minimal graphs could be used to solve this problem. Graphs critical for vertex ranking are analyzed in the next section and we use them to state the lower bound for trees.

## 5. RANKING CRITICAL GRAPHS

A graph  $G$  is said to be *ranking  $k$ -critical* if  $\chi_r(G) = k$  but  $\chi_r(G - v) < k$  for every vertex  $v \in V(G)$ . *Ranking  $k$ -minimal* graphs are those graphs  $G$  for which  $\chi_r(G) = k$  but  $\chi_r(G - e) < k$  for any edge  $e \in E(G)$ . The *parity  $k$ -critical* and *parity  $k$ -minimal* graphs are defined analogously. Ranking minimal graphs were analyzed by Katchalski *et al.* in [7] (the authors called them  $k$ -critical). The following theorem strengthens the result of Katchalski *et al.* ([7] Proposition 2.1).

**Theorem 10.** *Let  $G$  be any connected graph such that  $\chi_r(G) = k$  and let  $H_i$ ,  $i \in \{1, \dots, p\}$  be all ranking  $k$ -critical subgraphs of  $G$ . Then  $X = \bigcap_{i=1}^p V(H_i) \neq \emptyset$ . Moreover, if  $\tilde{\varphi}(v) = k$  for some ranking  $\tilde{\varphi}$  then  $v \in X$  and for any vertex  $w \in X$  there exists a ranking  $\varphi$  such that  $\varphi(w) = k$ .*

**Proof.** If there existed ranking  $k$ -critical subgraphs  $H_i$  and  $H_j$  such that  $V(H_i) \cap V(H_j) = \emptyset$ , then for any ranking  $\varphi$  of  $G$  we would have two vertices  $v_i \in V(H_i)$  and  $v_j \in V(H_j)$  coloured with the maximum colour  $k$ , a contradiction. For the same reason any vertex coloured  $k$  must belong to  $X$ .

Now, observe that whenever  $v \in X$ ,  $\chi_r(H_i - v) = k - 1$  for all  $1 \leq i \leq p$  and consequently  $\chi_r(G - v) = k - 1$ . Any  $(k - 1)$ -ranking  $\varphi$  of  $G - v$  can be easily extended to a  $k$ -ranking of  $G$  using colour  $k$  for the vertex  $v$ . ■

The operation of adding the edge between vertices of two disjoint ranking  $(k - 1)$ -minimal graphs  $G_1, G_2$  results in the ranking  $k$ -minimal graph  $G$  ([7] Lemma 2.1). In what follows we prove even a stronger statement concerning critical graphs.

**Theorem 11.** *Let  $G_1, G_2$  be vertex disjoint connected graphs such that  $\chi_r(G_i) = k - 1$ ,  $i \in \{1, 2\}$  and let  $G$  be a graph obtained from  $G_1 \cup G_2$  by addition of an edge  $v_1v_2$  between some vertices  $v_1 \in V(G_1)$ ,  $v_2 \in V(G_2)$ . Then*

- (a)  $\chi_r(G) = k$ ,
- (b)  $G$  is ranking  $k$ -critical if and only if  $G_1, G_2$  are ranking  $(k - 1)$ -critical.

**Proof.** (a) Since  $\chi_r(G_i) = k - 1$ ,  $i \in \{1, 2\}$  there exist vertices  $x_1 \in V(G_1)$  and  $x_2 \in V(G_2)$  coloured  $k - 1$ , which following the last part of Theorem 10 may be any vertices of the appropriate  $(k - 1)$ -critical subgraphs  $H_i \leq G_i$  and hence are assumed to be different from  $v_1$  and  $v_2$ . Therefore there exists a path  $(x_1, \dots, v_1, v_2, \dots, x_2)$ , which spoils ranking as long as some vertex coloured  $k$  appears between  $x_1$  and  $x_2$ . Hence  $\chi_r(G) \geq k$ . On the other hand to see that  $\chi_r(G) \leq k$ , let  $\varphi_i$  be a  $k$ -ranking of  $G_i$ ,  $i \in \{1, 2\}$  and let the ranking  $\varphi$  of  $G$  be defined as follows:  $\varphi(v) = \varphi_1(v)$  for  $v \in V(G_1)$ ,  $\varphi(v) = \varphi_2(v)$  for  $v \in V(G_2) - \{v_2\}$  and  $\varphi(v_2) = k$ .

(b) ( $\Leftarrow$ ) From (a) it follows that  $\chi_r(G) = k$ . Let  $v \in V(G)$ ; without loss of generality assume  $v$  to be from  $V(G_1)$ . We argue that  $\chi_r(G - v) = k - 1$ . Since  $G_1$  is  $(k - 1)$ -critical,  $G_1 - v$  has a  $(k - 2)$ -ranking  $\varphi'_1$ . By Theorem 10 graph  $G_2$  has such a  $(k - 1)$ -ranking  $\varphi_2$  that  $\varphi_2(v_2) = k - 1$ . Notice that colouring  $\varphi$  defined as  $\varphi(v) = \varphi'_1(v)$  for  $v \in V(G_1 - v)$  and  $\varphi(v) = \varphi_2(v)$  for  $v \in V(G_2)$  is a  $(k - 1)$ -ranking of  $G - v$ . Hence  $G$  is ranking  $k$ -critical.

( $\Rightarrow$ ) Let us assume that  $G_1$  is not  $(k - 1)$ -critical and let  $v \in V(G_1)$ . Since  $G$  is  $k$ -critical,  $\chi_r(G - v) \leq k - 1$  and since  $\chi_r(G_2) = k - 1$ , the only vertex coloured  $k - 1$  must belong to  $V(G_2)$ . Accordingly,  $G_1 - v$  has a  $(k - 2)$ -ranking, i.e.  $\chi_r(G_1 - v) \leq k - 2$  and it follows that  $G_1$  is  $(k - 1)$ -critical. By symmetry the same reasoning applies to  $G_2$ . ■

6. PARITY AND RANKING ON TREES

In what follows we use the *canonical trees* which can be defined recursively. A graph  $K_1$  is the first canonical tree  $T_1$  with the only vertex as its root. The canonical tree  $T_k$  is obtained by taking two disjoint copies of trees  $T_{k-1}$  and joining their roots by an edge, then taking the root of the second copy to be the root of  $T_k$ .

**Lemma 3.** *The canonical tree  $T_k$  is ranking  $k$ -critical.*

**Proof.** By Theorem 11 for any two disjoint ranking  $(k - 1)$ -critical graphs  $G_1$  and  $G_2$ , the graph  $G = (G_1 \cup G_2) + v_1v_2$  is ranking  $k$ -critical for any  $v_1 \in V(G_1)$  and  $v_2 \in V(G_2)$ . Obviously,  $K_2 \simeq T_2$  is ranking 2-critical. Hence following the above mentioned result  $T_3 \simeq P_4 = (K_2 \cup K_2) + v_1v_2$  is ranking 3-critical. Assume  $T_{k-1}$  to be ranking  $(k - 1)$ -critical. It follows by induction that  $T_k$  is ranking  $k$ -critical. ■

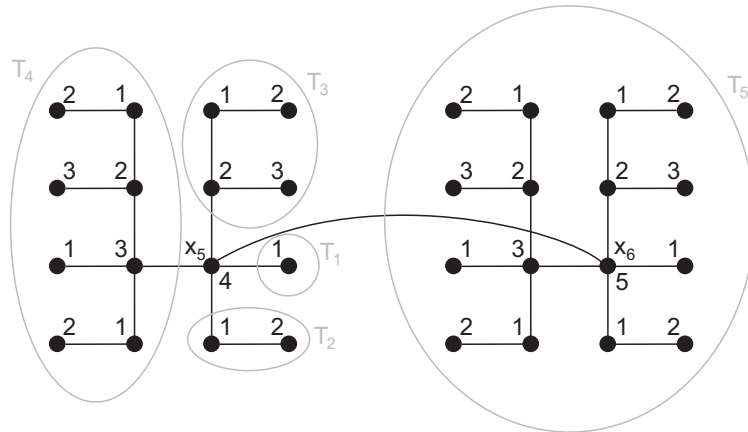


Figure 3. Parity vertex colouring of the canonical tree  $T_6$ .

In order to obtain a  $k$ -ranking of  $T_k$ , observe that removing all vertices of degree 1 results in  $T_{k-1}$ . Now, colour greedily the subsequent vertices ordered non-decreasingly with respect to their degrees, which produces a vertex  $k$ -ranking of  $T_k$ .

**Theorem 12.** *For any canonical tree  $T_k$ ,  $k \geq 4$ , we have*

$$\chi_r(T_k) - \chi_p(T_k) \geq 1.$$

**Proof.** It is not hard to see that for  $k \in \{1, 2, 3\}$  we have  $\chi_p(T_k) = k$  while  $\chi_p(T_4) = 3$  (see Figure 3). The crucial property of canonical trees is that removing the root-vertex  $x_k$  from  $T_k$  gives a forest consisting of the  $k - 1$  components  $H_i$  isomorphic to the appropriate  $T_i$ ,  $i \in \{1, \dots, k - 1\}$  respectively. Assume that  $\chi_p(T_i) \leq i - 1$  holds for all  $T_i$ ,  $4 \leq i \leq k - 1$ . It is enough to colour the root  $x_k$  of  $T_k$  using colour  $k - 1$  to obtain a  $(k - 1)$ -parity vertex colouring of  $T_k$ . Hence by induction on  $k$ , we have  $\chi_p(T_k) \leq k - 1$  for  $k \geq 4$  and since by Lemma 3,  $\chi_r(T_k) = k$  for  $k \geq 1$ , the theorem follows. ■

We strongly believe that it is possible to prove the following

**Conjecture 1.** For any tree  $T$  we have  $\chi_r(T) - \chi_p(T) \leq 1$ .

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