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# Generation of the vorticity mode by sound in a Bingham plastic

**Research Article**

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**Abstract:** This study investigates interaction between acoustic and non-acoustic modes, such as vorticity mode, in some class of a non-newtonian fluid called Bingham plastic. The instantaneous equations describing interaction between different modes are derived. The attention is paid to the nonlinear effects in the field of intense sound. The resulting equations which describe dynamics of both sound and the vorticity mode apply to both periodic and aperiodic sound of any waveform. They use only instantaneous quantities and do not imply averaging over the sound period. The theory is illustrated by an example of acoustic force of vorticity induced in the field of a Gaussian sound beam. Some unusual peculiarities in both sound and the vorticity induced in its field as compared to a newtonian fluid, are discovered.

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## 1. Introduction

Time-independent, or purely viscous fluids, include two types of fluids, newtonian and non-newtonian. Newtonian fluids have a viscosity independent on the shear rate [1]. Fluids whose viscosity is a single valued function of the shear rate are termed non-newtonian fluids [2]. In contrast with newtonian fluids, the non-newtonian properties are caused by the viscous dissipation of energy due to collisions between large particles, or to the distortion of or collision between colloidal structures. Within this group there are three main types: Bingham plastic and plastic behavior, shear-thinning behavior and shear-thickening behavior [1–5]. An ideal Bingham plastic material is characterized by the fact that the shear rate is proportional to the shear stress after a finite value of shear stress has been attained [1, 3]. It is a newtonian fluid with a yield stress. The explanation of Bingham plastic and plastic behavior is that these fluids possess a three-dimensional structure, which can resist shear stress as high as the yield stress, but which breaks down above this shear stress. When the shear stress is reduced below the yield-value, the structure forms again. Bingham plastic flow can be readily demonstrated using a tube of toothpaste, and by considering the properties required of such a product it can be appreciated how useful the Bingham plastic property is. It is required that toothpaste from a tube should flow out easily at a controllable rate yet not flow when it is resting on the toothbrush before use. The yield stress of the toothpaste is essential to its action. It will also

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permit the marketing of a striped toothpaste. Most of drilling muds are made from Bingham plastic materials, and they are widely used in paint manufactures. Examples of Bingham plastics include fresh cement, suspensions of chalk, grain and rock, and sewage sludge [6]. Many food substances like margarine, mayonnaise, and ketchup, are good examples of Bingham plastics. Production of composite materials usually involves proceeding of fiber suspensions that often exhibit yield behavior. Concentrated suspensions of solid particles in newtonian liquids show a yield stress followed by nearly newtonian behavior after yielding and flow.

As far as the authors know, acoustics of Bingham plastics is not investigated yet, all the more so nonlinear acoustics and nonlinear phenomena of sound there. The subject of this investigation is generation of the vortex flow in the field of sound, and, in particular, acoustic streaming in a Bingham plastic. Acoustic streaming is the mean motion of a fluid caused by periodic acoustic waves. Extensive reviews on this subject exist in the literature [7–9]. More recent discussion of acoustic streaming can be found in [10–12]. The authors of [11, 12] suggest that there is an unresolved issue in the theory concerning acoustic streaming, the effect of compressibility: studies usually start with equations describing incompressible liquid, though sound itself may propagate only over compressible medium. The weak point of the former theory is also inconsistency in distinction between the vorticity and entropy modes [8, 13]. Finally, the former theory considered only averaged over the sound period quantities relating to the generation of non-acoustic modes [8, 14]. We can avoid these inconsistencies by means of immediate projecting of the initial equations into dynamic equations governing every specific mode (Sec. 2). The method was previously applied by one of the authors in some problems of nonlinear flow of newtonian fluids and in fluids where reversible or irreversible thermodynamic processes occur [15, 16]. This method is valid for both periodic and aperiodic sound. It uses instantaneous quantities and therefore does not require averaging over the sound period at any stage. Relatively to generation of the vortical mode in the field of sound, the method makes possible to derive dynamic equations considering every branch of acoustic and vortical motions individually.

In the present study, we consider a weakly nonlinear flow in an unbounded volume of a Bingham plastic. The nonlinear generation of the vorticity flow in the field of intense sound is studied. The nonlinear equations governing both sound and the secondary flow, induced by sound, are derived. Some unusual peculiarities in both sound and the vorticity induced in its field as compared to a newtonian fluid, are caused by different dependence of stress tensor on shear rates in the domains of positive and negative shear rates. The example relates to the acoustic force of vorticity in the field of a Gaussian beam (Sec. 3).

# 2. Decomposition of the vorticity and sound modes in a Bingham plastic

## 2.1. Basic equations describing motion of a Bingham plastic

We assume that a fluid is homogeneous in composition, that its unperturbed density and pressure are uniform. The momentum and energy equations and the mass conservation equation in a viscous fluid take the form:

$$\begin{aligned}
&\frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}=\frac{1}{\rho}\left(-\vec{\nabla}p+Div\,\mathbf{P}\right),\\
&\frac{\partial e}{\partial t}+(\vec{v}\cdot\vec{\nabla})e=\frac{1}{\rho}\left(-p(\vec{\nabla}\cdot\vec{v})+\mathbf{P}:Grad\,\vec{v}\right),\\
&\frac{\partial \rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{v})=0,
\end{aligned} \tag{1}$$

where $\vec{v}$ denotes velocity of a fluid, $\rho$, $p$ are its density and pressure, $e$ marks the specific internal energy per unit mass, $x_i, t$ denote spacial Cartesian coordinates ($x_1 \equiv x, x_2 \equiv y, x_3 \equiv z$) and time. Operator $Div$ denotes tensor divergency, $Grad$ is a dyad gradient, and $\mathbf{P}$ is the tensor of viscous stress. In the model of Bingham plastic, the viscous stress tensor relates to the shear rate in the following manner:

$$\mathbf{P}_{ik}=\begin{cases} P_0+\mu\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right), & \left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right)>0,\\ -P_0+\mu\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right), & \left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right)<0.\end{cases} \tag{2}$$

The important and unusual property of a Bingham plastic is to describe differently domains of positive or negative shear rate, and dependence of the shear stress on the yield stress, $P_0$. The series of excess internal energy $e(p,\rho)$ supplement the system (1):

$$e'=\frac{E_1}{\rho_0}p'+\frac{E_2 p_0}{\rho_0^2}\rho'+\frac{E_3}{p_0\rho_0}p'^2+\frac{E_4 p_0}{\rho_0^3}\rho'^2+\frac{E_5}{\rho_0^2}\rho'p'+\dots. \tag{3}$$

Primes denote perturbations, and the ambient quantities are marked by index 0. The series of excess internal energy allows to consider thermodynamic state of a fluid in the most general form in the vicinity of zero perturbations of pressure and density, where $E_1,\dots E_5$ are dimensionless coefficients. The common practice in nonlinear acoustics is to focus on the equations of the second order of acoustic Mach number $M = \frac{v_0}{c_0}$, where $v_0$ is a typical particle velocity magnitude,

$$c_0 = \sqrt{\frac{(1-E_2)p_0}{E_1\rho_0}}$$

is an infinitely small signal velocity if $P_0 = 0$ and without account for viscosity $\mu$. We use also the small dimensionless parameter responsible for viscosity,

$$\eta = \frac{\mu}{(\rho_0 c_0 \Lambda)}$$

($\Lambda$ is a characteristic scale of a flow). Our primary objective is to derive dynamic equations valid at order $\eta M^2$. It is convenient to rearrange formulae in the dimensionless quantities as follows

$$p^{nd} = \frac{p'}{c_0^2 \cdot \rho_0}, \quad \rho^{nd} = \frac{\rho'}{\rho_0}, \quad \overrightarrow{v^{nd}} = \frac{\overrightarrow{v}}{c_0}, \quad x^{nd} = \frac{x}{\Lambda},$$
$$y^{nd} = \frac{y}{\Lambda}, \quad z^{nd} = \frac{z}{\Lambda}, \quad t^{nd} = \frac{c_0}{\Lambda}t. \tag{4}$$

Everywhere below in the text, superscripts by dimensionless quantities will be omitted. In the dimensionless quantities, Eqs. (1) read:

$$\begin{aligned}
&\frac{\partial v_x}{\partial t} + \frac{\partial p}{\partial x} - \eta\triangle v_x - \eta\frac{\partial}{\partial x}(\vec{\nabla}\cdot\vec{v}) = -(\vec{v}\cdot\vec{\nabla})v_x + \rho\frac{\partial p}{\partial x} - \rho\left(\eta\triangle v_x + \eta\frac{\partial}{\partial x}(\vec{\nabla}\cdot\vec{v})\right),\\
&\frac{\partial v_y}{\partial t} + \frac{\partial p}{\partial y} - \eta\triangle v_y - \eta\frac{\partial}{\partial y}(\vec{\nabla}\cdot\vec{v}) = -(\vec{v}\cdot\vec{\nabla})v_y + \rho\frac{\partial p}{\partial y} - \rho\left(\eta\triangle v_y + \eta\frac{\partial}{\partial y}(\vec{\nabla}\cdot\vec{v})\right),\\
&\frac{\partial v_z}{\partial t} + \frac{\partial p}{\partial z} - \eta\triangle v_z - \eta\frac{\partial}{\partial z}(\vec{\nabla}\cdot\vec{v}) = -(\vec{v}\cdot\vec{\nabla})v_z + \rho\frac{\partial p}{\partial z} - \rho\left(\eta\triangle v_z + \eta\frac{\partial}{\partial z}(\vec{\nabla}\cdot\vec{v})\right),\\
&\frac{\partial p}{\partial t} + (\vec{\nabla}\cdot\vec{v}) + \sum_{i=1,2,3,j\geqslant i}\Phi_{ij}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = -(\vec{v}\cdot\vec{\nabla})p + (D_1 p + D_2\rho)(\vec{\nabla}\cdot\vec{v}) + \rho\sum_{i=1,2,3,j\geqslant i}\Phi_{ij}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\\
&\qquad + \frac{\eta}{E_1}\left(2\sum_{i=1,2,3}\left(\frac{\partial v_i}{\partial x_i}\right)^2 + \frac{1}{2}\sum_{i\neq j}\left(\frac{\partial v_i}{\partial j} + \frac{\partial v_j}{\partial x}\right)^2\right),\\
&\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{v} = -(\vec{v}\cdot\vec{\nabla})\rho - \rho(\vec{\nabla}\cdot\vec{v}),
\end{aligned} \tag{5}$$

where $\vec{\nabla} = \begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{pmatrix}$, $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. Eqs. (5) include the dimensionless quantities

$$\begin{aligned}
D_1 =& \frac{1}{E_1}\left(-1 + 2\frac{1-E_2}{E_1}E_3 + E_5\right),\\
D_2 =& \frac{1}{1-E_2}\left(1 + E_2 + 2E_4 + \frac{1-E_2}{E_1}E_5\right),\\
&\text{if } i \neq k,\\
\Phi_{ik} =& \begin{cases} -\Phi_0, \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) > 0,\\ \Phi_0, \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) < 0,\end{cases},\\
\Phi_{ii} =& \begin{cases} -\frac{\Phi_0}{2}, \frac{\partial v_i}{\partial x_i} > 0,\\ \frac{\Phi_0}{2}, \frac{\partial v_i}{\partial x_i} < 0,\end{cases},\\
\Phi_0 \equiv& \frac{P_0}{E_1\rho_0 c_0^2}.
\end{aligned} \tag{6}$$

We will consider $\Phi_0$ as a small parameter of order not higher than $\eta$. That imposes small deviation of the behavior of Bingham plastic from a newtonian liquid.

## 2.2. Dispersion relations in a flow of infinitely-small magnitude. Projecting into the vorticity mode

The equivalent form of the system (5) is

$$\frac{\partial \Psi}{\partial t} + L\Psi = \Psi_{nl}, \tag{7}$$

where $\Psi = \begin{pmatrix} v_x & v_y & v_z & p & \rho \end{pmatrix}^T$, $L$ is a linear matrix operator including spacial derivatives, $\Psi_{nl}$ denotes a nonlinear vector. The dispersion relations following from the linearized system (5) describe three independent modes. Every type of mode is determined by relations of velocity components and perturbations of two thermodynamic functions, such as pressure and density. They depend on linear relationship between stress and shear tensors and are different in various thermoviscous fluids. Studies of motions of infinitely-small amplitudes begin usually with representing of all perturbations as a sum of planar waves:

$$\begin{aligned} f(\vec{r}, t) &= \int_{R^3} \widetilde{F}(\vec{k}, t) \exp(-i\vec{k}\vec{r}) d\vec{k} \\ &= \int_{R^3} \widetilde{f}(\vec{k}) \exp(i\omega t - i\vec{k}\vec{r}) d\vec{k}, \end{aligned} \tag{8}$$

($\widetilde{F}(\vec{k}, t)$ denotes the Fourier transform of $f(\vec{r}, t)$, $\widetilde{F}(\vec{k}, t) = \frac{1}{(2\pi)^3} \int_{R^3} f(\vec{r}, t) e^{i\vec{k}\vec{r}} d\vec{r}$). Five independent solutions of the linearized version of Eqs. (5) determine wave (two branches of sound, $n = 1$ and $n = 2$) and non-wave (the entropy mode, $n = 3$, and two vorticity branches, $n = 4$ and $n = 5$) types of motion that may in general exist in a fluid. They correspond to five roots of dispersion equation $\omega_n(k_x, k_y, k_z)$, $n = 1, \ldots 5$. The unusual property of roots of dispersion equation relatively to a flow over Bingham plastic is dependence of acoustic roots on a sign of components of the shear tensor by means of $\Phi_{ik}$. Two acoustic roots take the form

$$\begin{aligned} \omega_1 &= \sqrt{k_x^2 + k_y^2 + k_z^2} + i(k_x^2 + k_y^2 + k_z^2)\eta + \frac{1}{\sqrt{k_x^2 + k_y^2 + k_z^2}} \sum_{i=1,2,3,j\geqslant i} \Phi_{ij} k_i k_j, \\ \omega_2 &= -\sqrt{k_x^2 + k_y^2 + k_z^2} + i(k_x^2 + k_y^2 + k_z^2)\eta - \frac{1}{\sqrt{k_x^2 + k_y^2 + k_z^2}} \sum_{i=1,2,3,j\geqslant i} \Phi_{ij} k_i k_j, \end{aligned} \tag{9}$$

the third root describes the entropy mode

$$\omega_3 = 0, \tag{10}$$

and the frequencies specifying two rotational modes are

$$\omega_4 = \omega_5 = i\left(k_x^2 + k_y^2 + k_z^2\right)\eta. \tag{11}$$

The roots (9) are different in the subspaces of solutions described differently by the viscous stress tensor. All roots are evaluated in the leading (first) order with respect to small quantities $\Phi_{11}, \ldots, \Phi_{33}, \eta$. The second terms in both acoustic roots are the same as in a newtonian liquid, they manifest standard attenuation of sound. The terms proportional to $\Phi_{11}, \ldots \Phi_{33}$ are responsible for variation in sound speed (its dimensional value equals 1) and depend on a sign of components of the shear rate tensor. The relations between the Fourier transforms of perturbations of both acoustic modes in the leading order are following ($n = 1, 2$):

$$\begin{aligned} \widetilde{\Psi}_n &= \begin{pmatrix} \widetilde{v}_{n,x} & \widetilde{v}_{n,y} & \widetilde{v}_{n,z} & \widetilde{p}_n & \widetilde{\rho}_n \end{pmatrix}^T \\ &= \begin{pmatrix} \frac{\omega_n k_x}{k_x^2+k_y^2+k_z^2} & \frac{\omega_n k_y}{k_x^2+k_y^2+k_z^2} & \frac{\omega_n k_z}{k_x^2+k_y^2+k_z^2} & 1 + 2\sum_{i=1,2,3,j\geqslant i} \Phi_{ij} \frac{k_i k_j}{k_x^2+k_y^2+k_z^2} & 1 \end{pmatrix}^T \widetilde{\rho}_n. \end{aligned} \tag{12}$$

Multiplication by $-ik_l$ in the space of Fourier transforms corresponds to application the partial derivative $\frac{\partial}{\partial l}$ on the relative quantity. Among other ratios, Eqs. (12) determine the important relations between components of sound velocity in the $(\vec{r}, t)$ space:

$$\vec{\nabla} \times \vec{v}_n = \vec{0}, \quad n = 1, 2. \tag{13}$$

That is the well-known property of sound velocity to be irrotational field independently on attenuation in a medium of sound propagation. The entropy type of motion in the absence of thermal conductivity specifies zero velocity [13, 17]. Velocity of the entropy mode is also a potential field, its excess pressure is zero, but its excess density differs from zero:

$$\vec{v}_3 = \vec{0}, \quad p_3 = 0. \tag{14}$$

Two branches of the "vorticity" velocity flow satisfy relations as follows:

$$\vec{\nabla}\cdot\vec{v}_n=-\eta\Delta\rho_n,\ p_n=0,\ \vec{\nabla}\cdot\left((\mathbf{I}+\mathbf{F})\vec{v}_n\right)=0,\quad n=4,5, \tag{15}$$

where

$$\mathbf{F}=\Phi+\begin{pmatrix}\Phi_{11} & 0 & 0\\ 0 & \Phi_{22} & 0\\ 0 & 0 & \Phi_{33}\end{pmatrix}, \tag{16}$$

and $\mathbf{I}$ is the unit matrix. The unusual property of the last non-acoustic branches is that they are not represented by strictly solenoidal velocity field. Density perturbation in it may differ from zero. In newtonian flows, the voricity mode possesses zero perturbations both in density and pressure. The correspondent projecting matrix operators may be determined by use of Eqs. (13)-(15) [13]. They uniquely decompose an appropriate mode from the overall vector of perturbations. For example, application of the operator $P_{vort}$ decomposes a sum of vorticity modes:

$$P_{vort}\Psi=\Psi_{vort}=\Psi_4+\Psi_5. \tag{17}$$

A sum of all projectors is the unit matrix, they are pairwise orthogonal, and every projector squared equals itself. Each projector is a matrix of spatial operators consisting of five rows and five columns. Temporal derivatives or other operators connected with time are not included in projectors. The vorticity projector in fact applies on three components of the overall velocity. Its part, which applies on the total velocity vector, $P_{vort,\vec{v}}$, consists of three rows and three columns. The projector, decomposing velocity representing a sum of forth and fifth modes in the overall velocity, takes the form

$$P_{vort,\vec{v}}=\left(\mathbf{I}-W^{-1}\begin{pmatrix}\frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x\partial y} & \frac{\partial^2}{\partial x\partial z}\\ \frac{\partial^2}{\partial x\partial y} & \frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial y\partial z}\\ \frac{\partial^2}{\partial x\partial z} & \frac{\partial^2}{\partial y\partial z} & \frac{\partial^2}{\partial z^2}\end{pmatrix}(\mathbf{I}+\mathbf{F})\right), \tag{18}$$

where $W=\vec{\nabla}\cdot(\mathbf{I}+\mathbf{F})\vec{\nabla}$, and $W^{-1}$ is the inverse operator. The solenoidal part of velocity may be decomposed by application of projector $P_{vort,\vec{v}}$,

$$P_{vort,\vec{v}}\begin{pmatrix}\sum_{n=1}^{5}v_{n,x}\\ \sum_{n=1}^{5}v_{n,y}\\ \sum_{n=1}^{5}v_{n,z}\end{pmatrix}=\begin{pmatrix}v_{vort,x}\\ v_{vort,y}\\ v_{vort,z}\end{pmatrix}. \tag{19}$$

## 2.3. Equation governing the vorticity mode caused by sound

The linearized theory does not indicate any interaction among the modes as long as the domain of interest is far from boundaries. It considers weak fluctuations. Interaction of modes can not be ignored in a nonlinear flow of finite magnitude, even over unbounded volume [8, 17, 18]. Interaction of modes yields, among other, generation of one modal field by another in the thermoviscous flow. The main idea is to use properties of linear projecting in order to decompose modes in the linear part of equations. In studies of weakly nonlinear flows, we fix relations specifying modes in a flow of infinitely small magnitude of perturbations. We suppose also, that each of the field variables contains contributions from each of modes, for example, $\vec{v}=\sum_{n=1}^{5}\vec{v}_n$. The method of projecting, proposed and applied by one of the authors in some problems of thermoviscous flow [13, 15, 16], enables decomposition the system (5) into specific dynamic equations for each mode. Among other properties of projecting, it is of importance, that it distributes the nonlinear terms between dynamic equations in a correct manner. These nonlinear terms may be considered as driving forces of specific modes conditioned by nonlinearity of a flow, including those responsible for the "self-action". Application of $P_{vort}$ on the system (5) cancels all acoustic and entropy terms in the linear part, but yields nonlinear source in its right-hand nonlinear part. The nonlinear part of the resulting equations includes mixed quadratic terms of all modes. We will consider among them only acoustic ones. Application of $P_{vort,\vec{v}}$ on the first three equations from the system (5) (they represent the momentum equation), results in the dynamic equation which governs the vorticity mode:

$$\frac{\partial\vec{v}_{vort}}{\partial t}-\eta\Delta\vec{v}_{vort}=P_{vort,\vec{v}}\begin{pmatrix}-(\vec{v}\cdot\vec{\nabla})v_x+\rho\frac{\partial p}{\partial x}-\rho\eta\left(\Delta v_x+\frac{\partial}{\partial x}(\vec{\nabla}\cdot\vec{v})\right)\\ -(\vec{v}\cdot\vec{\nabla})v_y+\rho\frac{\partial p}{\partial y}-\rho\eta\left(\Delta v_y+\frac{\partial}{\partial y}(\vec{\nabla}\cdot\vec{v})\right)\\ -(\vec{v}\cdot\vec{\nabla})v_z+\rho\frac{\partial p}{\partial z}-\rho\eta\left(\Delta v_z+\frac{\partial}{\partial z}(\vec{\nabla}\cdot\vec{v})\right)\end{pmatrix}_a. \tag{20}$$

It may be rearranged into the following equation:

$$\frac{\partial \vec{v}_{vort}}{\partial t} - \eta \Delta \vec{v}_{vort} = P_{vort,\vec{v}} \left( -\sum_{n=1}^{2} \rho_n \cdot \frac{\partial}{\partial t} \sum_{n=1}^{2} \vec{v}_n \right). \tag{21}$$

Eq. (21) does not exhibit that attenuation (or a yield stress) is a necessary condition for sound and the vorticity modes to interact. Dependence of the driving force in the right-hand side of Eq. (21) on a concrete attenuation process in a fluid manifests itself by means of dependence of $\vec{v}_n$ on $\rho_n$ different for various thermoviscous flows. One can obtain another form of Eq. (20) in terms of vorticity $\vec{\Omega} = \vec{\nabla} \times \vec{v}_{vort}$, and evaluate the leading-order nonlinear part of it in the case of a flow over a Bingham plastic accounting for Eqs. (9), (12):

$$\frac{\partial \vec{\Omega}}{\partial t} - \eta \Delta \vec{\Omega} \equiv \vec{F}_a = \vec{\nabla} \times \left( -\sum_{n=1}^{2} \rho_n \cdot \frac{\partial}{\partial t} \sum_{n=1}^{2} \vec{v}_n \right) = \left( \vec{\nabla} \sum_{n=1}^{2} \rho_n \right) \times \left( -2\eta \Delta \sum_{n=1}^{2} \vec{v}_n + \frac{\vec{\nabla}}{\Delta} \vec{\nabla} \cdot \mathsf{F} \vec{\nabla} \sum_{n=1}^{2} \rho_n \right), \tag{22}$$

where $\vec{F}_a$ denotes the acoustic force of the vorticity mode. It equals zero in newtonian non-viscous flows in view of relations linking $\rho_n$ and $\vec{v}_n$ in acoustic modes. In order to evaluate the driving force of the vorticity mode induced in the field of sound, it is sufficient to determine one of velocity components or acoustic pressure for everyone from two acoustic branches, because all other perturbations may be expressed in terms of one of them (Eqs. (12)).

## 2.4. Propagation of sound

The linear equation governing sound in terms of excess density, $\rho_a = \rho_1 + \rho_2$, agrees with Eqs. (9):

$$\frac{\partial^2 \rho_a}{\partial t^2} - \left( \Delta + 2 \sum_{i=1,2,3, j \geqslant i} \Phi_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \right) \rho_a - \eta \frac{\partial^3 \rho_a}{\partial t^3} = 0. \tag{23}$$

Applying projecting to a weakly nonlinear flow, one gets the nonlinear acoustic term which corrects the linear equation by the "self-action acoustic force" [13]:

$$\frac{\partial^2 \rho_a}{\partial t^2} - \left( \Delta + 2 \sum_{i=1,2,3, j \geqslant i} \Phi_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \right) \rho_a + \frac{1 - D_1 - D_2}{2} \frac{\partial^2 \rho_a^2}{\partial t^2} - \eta \frac{\partial^3 \rho_a}{\partial t^3} = 0. \tag{24}$$

Eq. (24) accounts for both nonlinearity and attenuation, and it describes the summary acoustic perturbation, that's why it includes the second order partial derivative with respect to time. The projection of the system (5) onto dynamic equations for every branch of sound results in two equations for each acoustic mode. There is no general analytical solution of Eq. (24) even in the case of standard viscous fluid ($\Phi_{ij} = 0$ for all $i, j$) [20]. The problem of detailed description of sound propagation in a Bingham liquid is also hardly expected to be solvable analytically. The projection of equations onto the first branch of sound yields in the leading order

$$\frac{\partial \rho_1}{\partial t} + \left( \Delta^{\frac{1}{2}} + \sum_{i=1,2,3, j \geqslant i} \Phi_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \Delta^{-\frac{1}{2}} \right) \rho_1 - \eta \Delta \rho_1 = \frac{1}{2} \left( (D_1 + D_2) \rho_1 (\vec{\nabla} \cdot \vec{v}_1) - \vec{v}_1 \cdot (\vec{\nabla} \rho_1) \right). \tag{25}$$

For a quasi-planar flow over the standard attenuating fluids, Eq. (25) rearranges into the famous Khokhlov-Zabolotskaya-Kuznetsov equation [18, 19]. There is no analytical solution of the KZK equation describing one of the sound modes, with the exception of fully nonlinear KZK, called the Khokhlov-Zabolotskaya equation [19, 21]. The analogue of the KZK equation for a Bingham plastic may be derived by introduction of one more small parameter responsible for diffraction, and considering the leading order terms. Let $y$ designate the nominal axis of the sound beam pointing in the propagation direction; $x, z$ are coordinates perpendicular to that axis. We will consider a beam propagating in the positive direction of $y$. The following assumptions will be made: a transmitter is defined at the plane $y = 0$, it has a characteristic dimensionless radius $R$, and it radiates at frequencies satisfying inequality $R \gg 1$. The last assumption ensures that the beam is reasonably directional. That allows to expand the Laplacian in a power series in the small parameter, $R^{-1}$,

$$\Delta = \frac{\partial^2}{\partial y^2} + \Delta_\perp, \quad \sqrt{\Delta} \approx \frac{\partial}{\partial y} + 0.5 \Delta_\perp \int dy, \tag{26}$$

where $\Delta_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$, and to simplify relations of perturbations specifying the rightwards progressive beam. It follows from Eqs. (9), (12) and (26):

$$\Psi_1 = \begin{pmatrix} v_{1,x} & v_{1,y} & v_{1,z} & p_1 & \rho_1 \end{pmatrix}^T = \begin{pmatrix} -\eta\frac{\partial}{\partial x} + (1+\Phi_{22})\frac{\partial}{\partial x}\int dy & -\eta\frac{\partial}{\partial y} + 1 + \Phi_{22} & -\eta\frac{\partial}{\partial z} + (1+\Phi_{22})\frac{\partial}{\partial z}\int dy & 1+2\Phi_{22} & 1 \end{pmatrix}^T \rho_1. \tag{27}$$

In the leading order, the longitudinal component of velocity in the beam is governed by the nonlinear equation

$$\frac{\partial v_{1,y}}{\partial t} + (1+\Phi_{22})\frac{\partial v_{1,y}}{\partial y} + \frac{1}{2}\int \Delta_\perp v_{1,y} dy + \frac{1-D_1-D_2}{2} v_{1,y}\frac{\partial v_{1,y}}{\partial y} - \eta\frac{\partial^2 v_{1,y}}{\partial y^2} = 0. \tag{28}$$

It is an analogue of the KZK equation in a Bingham plastic. Note that a sign of $\Phi_{22}$ depends on a sign of $\frac{\partial v_{1,y}}{\partial y}$ making the local sound speed dependent on a sign of the shear rate.

## 3. Acoustic force in the field of a Gaussian beam

Let us consider the solution of the limiting case of Eq. (28) where $\eta = 0$ and diffraction discarded. Assuming a small distortion of sound due to the yield stress, the solution depends mostly on the retarded time $t-y$ and "the slow scale" distance from a transducer, $\Phi_0 y$ [8]. It has the form of a Gaussian beam $v_{1,y}(t,r,y) = \exp\left(-(\frac{r}{R})^2\right)\phi(t-y,\Phi_0 y)$, where $r = \sqrt{x^2+z^2}$. It is a solution of the simplified dynamic equations (linear, (a), or weakly nonlinear, (b)) depending on a sign of gradient of $\phi$ by means of $\Phi_{22}$:

$$\begin{aligned} &\frac{\partial\phi}{\partial t} + \left(1-\frac{\Phi_0}{2}\right)\frac{\partial\phi}{\partial y} = 0, \text{ if } \frac{\partial\phi}{\partial y} \geqslant 0,\\ &\frac{\partial\phi}{\partial t} + \left(1+\frac{\Phi_0}{2}\right)\frac{\partial\phi}{\partial y} = 0, \text{ if } \frac{\partial\phi}{\partial y} < 0, \end{aligned} \tag{29a}$$

$$\begin{aligned} &\frac{\partial\phi}{\partial t} + \left(1-\frac{\Phi_0}{2}\right)\frac{\partial\phi}{\partial y} + \frac{1-D_1-D_2}{2}\phi\frac{\partial\phi}{\partial y} = 0, \text{ if } \frac{\partial\phi}{\partial y} \geqslant 0,\\ &\frac{\partial\phi}{\partial t} + \left(1+\frac{\Phi_0}{2}\right)\frac{\partial\phi}{\partial y} + \frac{1-D_1-D_2}{2}\phi\frac{\partial\phi}{\partial y} = 0, \text{ if } \frac{\partial\phi}{\partial y} < 0. \end{aligned} \tag{29b}$$

Accounting for $\rho_1 \approx v_{1,y}$ and other relations linking acoustic perturbations in accordance to Eq. (27), and calculating the leading order terms in view of that transversal spatial derivatives of acoustic perturbations are much smaller than longitudinal, one readily obtains the acoustic driving force of the vorticity mode which stands in the right-hand side of Eq. (22). The transversal component of it is larger than the longitudinal one. In the leading order, with account for that $\Phi_{12} = \Phi_0 sgn(x) sgn(\phi)$, $\Phi_{23} = \Phi_0 sgn(z) sgn(\phi)$, it takes the form

$$\begin{aligned} \vec{F}_{a,\perp} \approx & -\frac{4\eta}{R^2}\left(-z\vec{i} + x\vec{k}\right)\exp\left(-2\left(\frac{r}{R}\right)^2\right)\left(\left(\frac{\partial\phi(t-y,\Phi_0 y)}{\partial y}\right)^2 - \phi(t-y,\Phi_0 y)\frac{\partial^2\phi(t-y,\Phi_0 y)}{\partial y^2}\right)\\ & -\frac{8\Phi_0 sgn(\phi)}{R^4}\left(z\vec{i} - x\vec{k}\right)(|x|+|z|)\exp\left(-2\left(\frac{r}{R}\right)^2\right)\phi(t-y,\Phi_0 y)^2\\ & -\frac{4\Phi_0 sgn(z) sgn(\phi)\vec{i}}{R^4}\left(R^2 - 2z^2 - 2|xz|\right)\exp\left(-2\left(\frac{r}{R}\right)^2\right)\frac{\partial\phi(t-y,\Phi_0 y)}{\partial y}\int\phi(t-y,\Phi_0 y)dy\\ & +\frac{4\Phi_0 sgn(x) sgn(\phi)\vec{k}}{R^4}\left(R^2 - 2x^2 - 2|xz|\right)\exp\left(-2\left(\frac{r}{R}\right)^2\right)\frac{\partial\phi(t-y,\Phi_0 y)}{\partial y}\int\phi(t-y,\Phi_0 y)dy, \end{aligned} \tag{30}$$

where $\vec{i}$ and $\vec{k}$ are unit vectors along positive directions of axes $OX$ and $OZ$. For $\phi \equiv \sin(t-y)$, the acoustic driving force of the vorticity mode may be readily rearranged as

$$\vec{F}_{a,\perp} \approx -\frac{4\eta}{R^2}\left(-z\vec{i}+x\vec{k}\right)\exp\left(-2\left(\frac{r}{R}\right)^2\right) - \frac{8\Phi_0 sgn(\sin(t-y))}{R^4}\left(z\vec{i}-x\vec{k}\right)(|x|+|z|)\exp\left(-2\left(\frac{r}{R}\right)^2\right) + \frac{4\Phi_0 sgn(\sin(t-y))(sgn(z)\vec{i}-sgn(x)\vec{k})}{R^2}\exp\left(-2\left(\frac{r}{R}\right)^2\right)\cos^2(t-y)\,. \quad (31)$$

The driving force of the vorticity mode is solenoidal,

$$\vec{\nabla}\cdot\vec{F}_a = 0. \quad (32)$$

That coincides with the property that vorticity be a solenoidal field, $\vec{\nabla}\cdot\vec{\Omega}=0$. The first part of the driving force of acoustic streaming is proportional to viscosity. That reflects the origins of the phenomenon of acoustic streaming, such as nonlinearity and viscosity. These features agree with the conclusions of the theory and experiment dealing with the standard absorbing fluids [23, 24]. The second part includes terms proportional to the yield stress specific for Bingham plastics exclusively.

In this study, the "yield" parameter's order of magnitude is not higher than the "standard" one. While the terms connected with standard attenuation are proportional to $\eta$, the terms coming from the yield stress are proportional to $\Phi_0$ which is expressed in terms of $\left(\frac{\partial e}{\partial p}\right)\Big|_{\rho=const}$ (Eq. (3)). Sound of larger frequency produces larger velocities of the vortex flow (losses in momentum of high-frequency sound are stronger), but the part of acoustic force connected with the yield stress, does not depend on the sound frequency. So that it is relatively large in the domain of small sound frequencies. Estimations of $\Phi_0$ are difficult in view of absence of thermodynamic data necessary to evaluate $E_1$ in the literature not for only Bingham plastics, but for the majority of liquids. Liquid water is the only exception. For preliminary estimation, if we use $E_1$, $\rho_0$ and $c_0$ for liquid water at normal conditions ($E_1 = 9.6$, $\rho_0 = 10^3$ kg/m$^3$, $c_0 = 1500$ m/s), $\Phi_0$ equals $5\cdot 10^{-11}P_0\cdot 1$ Pa$^{-1}$. In water, $\eta = 4\cdot 10^{-11}\frac{c_0}{\Lambda}\cdot 1$ s. The ratio of dimensionless $\eta$ and $\Phi_0$ depends on the concrete yield stress $P_0$ and sound frequency. $P_0$ may take quantities of 30 Pa and 100 Pa (chocolate and stirred yogurt [6]), $10^3$ Pa (highly pigmented, flocculated ink [25]) and even $10^4$ Pa in Plasticine and some food-like materials. As for the sound frequency $\frac{c_0}{\Lambda}$, it is typically tens kilohertz. So, $\Phi_0$ is very believable of order of magnitude smaller than $\eta$ or smaller. The "yield" part in the acoustic force induced by a Gaussian beam in Eq. (30), includes the term which achieves maximum at the axis of a beam ($r=0$). It may be comparatively large in the vicinity of a beam axis.

There is obvious transition to a limit of "standard" viscous liquid if the "yield" parameter tends to zero. In this case, velocity of the vortex flow is directed according to the propagation of sound at the axis of a beam. The important peculiarity of dynamics of both sound and vortices induced in the sound field in a Bingham plastic, is its dependence on a sign of the shear rate. As a rule, for periodic in time acoustic sources, the averaged over period acoustic force of the vorticity mode is of interest. In this case, the vortex flow is associated with acoustic streaming. The part originated from the yield stress is hardly expected to exceed the first "standard" part in the acoustic force of streaming in view of its dependence on a sign of the shear rate varying over the sound period at any point. In the example described by Eq. (31), the average value of vorticity over the sound period, $2\pi$, equals zero in two last terms in the acoustic force which are responsible for effects connected with yield stress. However, contribution of this second part may be noticeable while cased by an one-polar acoustic pulse. Account for the "standard" part would produce variations of the mean velocity. The streamlines of vortices do not differ much from the "standard" ones in view of that their velocity differs from strictly solenoidal only weakly in accordance to Eq. (15).

## 4. Concluding remarks

The area of media different form the standard Newtonian, but which are important in medicine, technique and food engineering, enlarges constantly. Studies of acoustics in these media and relative phenomena of sound are of great importance in view of that they may point out a possibility to govern parameters of a medium remotely and to conclude about yield stress where direct measurements are difficult. As far as the authors know, propagation of sound waves and associated with it nonlinear phenomena in a Bingham plastic is a new subject of investigation, in spite of the fact that there exists a number of studies concerning solid media (including inhomogeneous) determined by different dependence of shear stress on the shear strain [26–28]. The conclusions, though approximate, follow from analytical formulae. They are valid only for the mobile liquid, at shear stresses small (in sense that are proportional to $M$) but enough large for experimental data to confirm non-elastic behavior of a medium. The lowest value is usually determined by accuracy of a viscometer, about $10^{-1}$ s$^{-1}$, or even smaller, $10^{-2}$ s$^{-1}$ [6]. If the liquid behavior is confirmed at shear stress larger than $10^{-1}$ s$^{-1}$, that gives for $M = 10^{-3}$ value of $\frac{c_0}{\Lambda} \geqslant 10^2$ s$^{-1}$ which supports enough large gradients of velocity in the sound wave. At practically zero shear rates, where a medium behaves as solid, the shear stress depends on strain, or, that is very likely in rheological media, both on strain and shear rate. As for the nonlinear phenomena of sound, they are still poorly analyzed even in respect to fluids with standard attenuation in view of mathematical difficulties of solving nonlinear equations. Conclusions of the present study may be considered as preliminary because the nonlinear flow of Bingham plastic is even more complex.

In this study, the authors use ideas of decomposing of linear dynamic equations for every mode in their linear parts. That may be done by means of linear combination of equations of continuity, momentum and energy and results in the nonlinear terms reflecting the interaction between modes in a weakly nonlinear flow. The main result of this study are equations governing sound and the vorticity, induced in its field, Eqs. (22), (25), (28), (30), describing general and quasi-planar geometry of sound. The method based on these ideas was applied previously in some problems of acoustic heating and streaming in the standard thermoviscous fluids [13], as well as in some others [15, 16]. The effects of thermal conductivity are not considered in the present study. They may influence on the dynamics of the vorticity mode by means of sound terms participating in the acoustic force. It is well-established, that attenuation of sound depends on the overall absorption which includes thermal conductivity [14]. The example of this study, however, does not even consider the effect of attenuation due to the first viscosity $\eta$ on the sound. The reason for that is difficulty in analytical solution of the dynamic equation governing sound.

## References


[1] A.A. Collyer, Physics Education 8, 333 (1973)
[2] W.H. Wilkinson, Non-newtonian fluids (Pergamon Press, New York, 1960)
[3] E.C. Bingham, U.S. Bureau of Standards Bulletin 13, 309 (1916)
[4] S. Benito, C.-H. Bruneau, T. Colin et al., Eur. Phys. J. E 25, 225 (2008)
[5] A.S. Lodge, Elastic liquids: an introductory vector treatment of finite-strain polymer rheology (AP, London, 1964)
[6] H.A. Barnes, J. Non-Newton. Fluid 81, 133 (1999)
[7] W.L. Nyborg, In: W.P. Manson (Ed.), Physical Acoustics, Vol. II. part B (Academic Press, New York, 1965) 265
[8] O.V. Rudenko, S.I. Soluyan, Theoretical foundations of nonlinear acoustics (Plenum, New York, 1977)
[9] M.J. Lighthill, J. Sound. Vib. 61, 391 (1978)
[10] S. Tjøtta, J.N. Tjøtta, In: H. Hobaek (Ed.), 13th International Symposium on Nonlinear Acoustics, 28 June - 2 July 1993, Bergen, Norway, (World Scientific, Singapore, 1993) 601
[11] Q. Qi, J. Acoust. Soc. Am. 94, 1090 (1993)
[12] L. Menguy, J. Gilbert, Acta Acust. 86, 249 (2000)
[13] A. Perelomova, Acta Acust. united Ac. 89, 754 (2003)
[14] S. Makarov, M. Ochmann, Acustica, 82, 579 (1996)
[15] A. Perelomova, Acta Acust. united Ac. 96, 43 (2010)
[16] A. Perelomova, Can. J. Phys. 88, 293 (2010)
[17] B.-T. Chu, L.S.G. Kovasznay, J. Fluid Mech. 3, 494 (1958)
[18] V.P. Kuznetsov, Sov. Phys. Acoust. 16, 467 (1971)
[19] M. Hamilton, V. Khokhlova, O.V. Rudenko, J. Acoust. Soc. Am. 101, 1298 (1997)
[20] M. Hamilton, In: M. Hamilton, D. Blackstock (Eds.), Nonlinear Acoustics (Academic Press, San Diego, California, USA, 1998) 233
[21] K.E. Frøysa, F. Coulouvrat, J. Acoust. Soc. Am. 99, 3319 (1996)
[22] V. Gusev, J. Acoust. Soc. Am. 107, 3047 (2000)
[23] V. Gusev, O. Rudenko, Sov. Phys. Acoust. 25, 493 (1979)
[24] T. Kamakura, T. Sudo, K. Matsuda, Yo. Kumamoto, J. Acoust. Soc. Am. 100, 132 (1996)
[25] M. Fernandez et al., J. Rheol. 42, 239 (1998)
[26] T. Szabo, J. Wu, J. Acoust. Soc. Am. 107, 2437 (2000)
[27] V.E. Nazarov, A.V. Radostin, Dokl. Phys. 48, 633 (2003)
[28] V.E. Nazarov, A.V. Radostin, Acoust. Phys. 52, 713 (2006)