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A Nearly Optimal Fractional Delay Filter Design Using an Asymmetric Window

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***Abstract*—In this paper a numerically efficient filter design method suitable for variable fractional delay (VFD) filter implementation is investigated. We propose to use a well known window method with an asymmetric window extracted from optimal filter designed beforehand. As we will demonstrate, such an approach, if additional gain correction is applied, allows for nearly optimal VFD filter design. Thus, the proposed approach combines window method simplicity with performance comparable to that of optimal filters. Efficiency of the presented technique makes it suitable for designing filters with varying delay in real time.**

***Keywords*—Asymmetric window, extracted window, fractional delay filter, nearly optimal filter, window method.**

I. Introduction

In many digital signal processing applications, such as synchronization in digital modems [1], incommensurate sampling rate conversion [2] and speech coding [3], there is a necessity for delaying signals. This problem can be readily solved when a signal needs to be delayed by an integer multiple of sampling period. In such situation signal samples are simply stored in a register for several sampling periods. However, in many cases, like in modeling of musical instruments sounds [4] and time delay estimation (TDE) [5], required delay is a fraction of sampling period and fractional delay (FD) filters [6], [7] must be utilized. Moreover, in many of these applications variable fractional delay (VFD) is required. This involves continuous changes of filter impulse response, often for each processed sample, which creates demand for simple and efficient design algorithms.

The ideal fractional delay filter has infinite impulse response, which is described by only one parameter – total delay [6], [7]. In order to implement FD filter in real time, its impulse response is usually approximated with causal finite impulse response (FIR), which leads to inevitable approximation errors [8]. From all FIR fractional delay filters the most efficient are optimal FD filters which offer the best performance for given filter length. One of the most commonly utilized types of optimal filters are fractional delay filters optimal in the least squares sense and filters optimal in the Chebyshev sense (called also minimax filters) [6]. These types of filters guarantee the best performance, in the sense of given optimality criteria, from all FIR filters of the same length and approximation band. However, these filters are very hard to design in real time because of complicated design algorithms which makes them unsuitable for VFD applications.

On the other hand, very simple filter design method is the commonly known window method [8], [9]. The design procedure consists only of ideal impulse response multiplication by window function. Simplicity of window method makes it ideal for variable fractional delay filter design. However, selection of window satisfying given specifications is usually difficult. That is why filter design with window method requires iterative approach with results often significantly worse than optimal.

It is worth noting that additional advantage in case of VFD filter design an additional advantage of using window method is that a single symmetric window can be used to design a VFD filter [7], [10]–[13]. However, gain of the designed filter must be corrected with different value for each desired fractional delay [7], [10]–[13]. As only a single window is required it can be carefully selected beforehand and used to design the whole family of FD filters differing only in fractional delay. Moreover, instead of searching for best window formula, a window can be extracted from optimal filter with some arbitrarily selected fixed fractional delay [10]–[13]. Next, this reference window can be used to compute proper gain correction factor which changes with fractional delay. The gain correction factor can be sampled and stored in a look-up table (LUT) or approximated with low order polynomial. Now, the design process at runtime is simple and consists only of ideal filter impulse response $h_{id,\epsilon}[n]$ multiplication by the reference window $w_{ref}[n]$ (like in traditional window method) and additional filter gain correction $\alpha(\epsilon)$ (Fig. 1).

The idea of FD filters designed with symmetric reference window has been investigated in several papers [10]–[13] and it has been demonstrated that nearly optimal performance can be achieved that way. In this paper we will investigate another

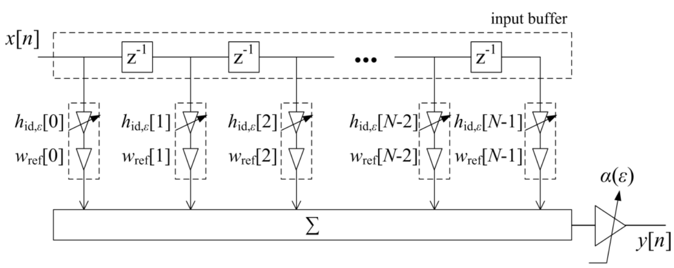

Fig. 1. The VFD filter implementation based on window method. $x[n]$ and $y[n]$ represent input and output signals, "z^{-1}" blocks represent the unit delay elements storing input samples for one sampling period.

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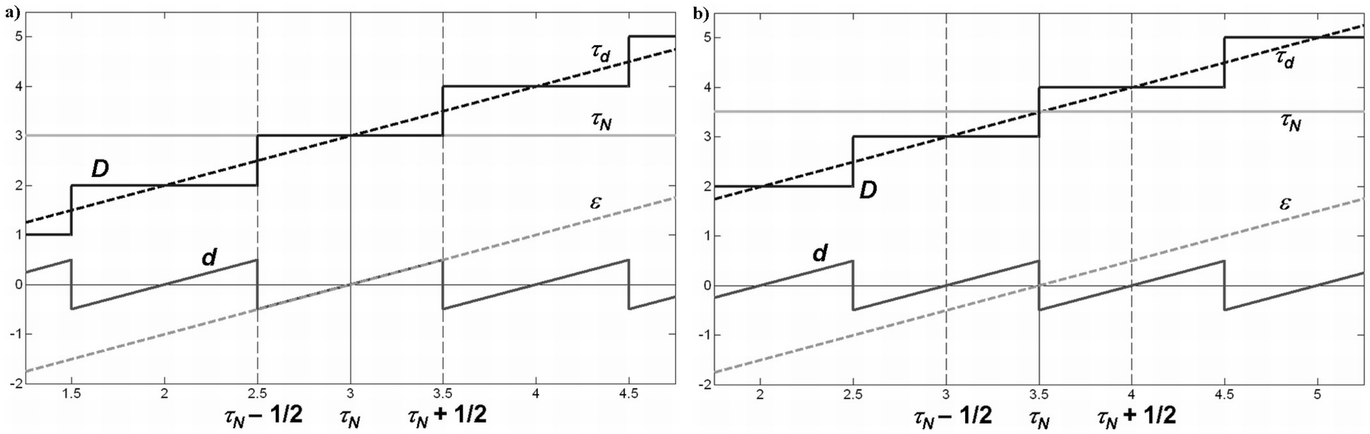

Fig. 2. Bulk delay τ_N, net delay ϵ, integer delay D and fractional delay d versus total delay τ_d for fractional delay filters with lengths $N = 7$ (a) and $N = 8$ (b).

concept. Windows extracted from optimal minimax or least squares filters are in general asymmetric therefore we propose to use single asymmetric window extracted for some arbitrarily selected reference delay as the reference window. That way filter designed for the delay for which reference window was extracted is optimal. For other delays, as we will demonstrate, with proper gain correction nearly optimal solutions can be achieved.

II. Variable Fractional Delay Filter

The ideal fractional delay filter [6], [7], [13] with total delay τ_d is characterized by the following frequency response

$$H_{id}(f) = \exp(-j2\pi f \tau_d), \qquad |f| < 1/2 \tag{1}$$

and the corresponding impulse response

$$h_{id}[n] = \mathrm{sinc}(n - \tau_d), \qquad n = 0, \pm 1, \dots \tag{2}$$

where f is normalized frequency and n is the discrete-time index. Total delay τ_d of any FD filter can be split into integer delay D =round (τ_d) and fractional delay $d \in [-1/2, 1/2)$ or, assuming N-point FIR approximation of ideal fractional delay filter, into bulk delay $\tau_N = (N-1)/2$ and net delay ϵ

$$\tau_d = \tau_N + \epsilon = D + d \tag{3}$$

Relationships between the above mentioned definitions of delay are presented in Fig. 2. The best FD filter performance can be achieved when the total delay $\tau_d \in [\tau_N - 1/2, \tau_N + 1/2]$, which means that in practical applications net delay ϵ is usually limited to the range $\epsilon \in [-1/2, +1/2]$. Taking this into account it is worth noting that for odd filter lengths (Fig. 2a) fractional delay d and net delay ϵ are the same, whereas for even N values (Fig. 2b) these delays are different but can be easily transformed into each other.

One should notice that for non-integer total delay impulse response (2) is infinite as well as non-causal and filter with such impulse response cannot be implemented. This leads us to the problem of the ideal frequency response H_{id} (1) approximation using causal filter with finite impulse response (FIR) $h_N[n]$. Approximation errors are inevitable and must be taken into account during design process. The most general measure of approximation errors is complex approximation error

$$E(f) = H_N(f) - H_{id}(f) \tag{4}$$

where

$$H_N(f) = \sum_{n=0}^{N-1} h_N[n] \exp(-j2\pi f n), \qquad |f| \le 1/2 \tag{5}$$

is the frequency response of FIR filter approximating the ideal FD filter frequency response H_{id} (1).

In practice, instead of complex approximation error (4), scalar parameters are used, which allows for simpler evaluation of filter performance. Examples are peak error (PE)

$$PE(f_a) = \max_{f \in [-f_a, f_a]} |E(f)| \tag{6}$$

and least squares error (LSE)

$$LSE(f_a) = \int_{-f_a}^{f_a} |E(f)|^2 df \tag{7}$$

evaluated in desired approximation band limited by its upper frequency f_a.

Depending on design process, requirements on PE or LSE and approximation band may be satisfied by FIR filters of different lengths. As we have mentioned before, there are methods for optimal fractional delay filter design [6], [14]–[16]. Filters designed using these methods demonstrate minimal approximation errors for given impulse response length. Using them we can readily find filters satisfying our quality requirements with minimal impulse response length. In practice, most frequently utilized optimal FD filters are optimal in the Chebyshev sense (minimax filters) minimizing PE (6) and in the least squares (LS) sense minimizing LSE (7). Due to the fact that the design of these types of filters is complicated, they are not appropriate for variable delay applications.

III. An Extracted Window or a Nearly Optimal VFD Filter Design

Filter design method investigated in this paper uses reference window extracted from single optimal FD filter to design

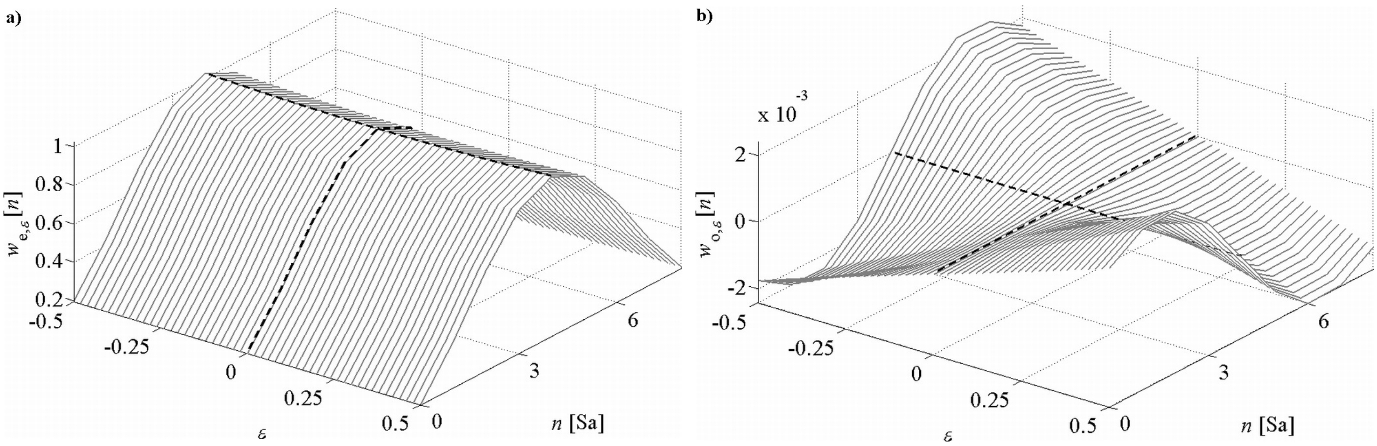

Fig. 3. Even (a) and odd (b) parts of windows extracted from filters with $N = 9$ and $f_a = 0.35$ optimal in the Chebyshev sense for different net delays ϵ. Dashed lines show values for $\epsilon = 0$ and $n = (N-1)/2$.

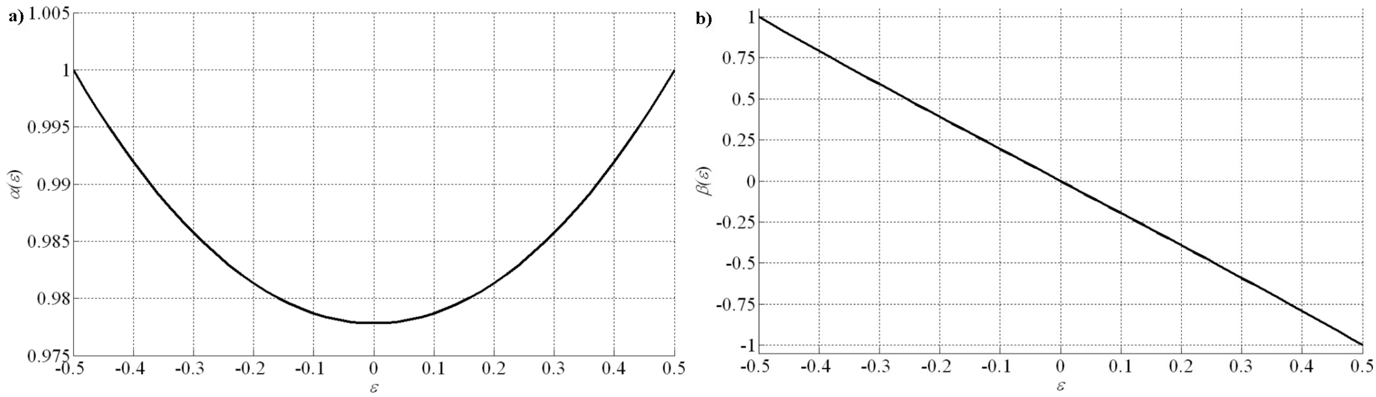

Fig. 4. Curves of $\alpha(\epsilon)$ (a) and $\beta(\epsilon)$ (b) factors computed for windows extracted from FD filters from Fig. 3 and reference window selected at net delay $\epsilon_{ref} = -0.5$. Values of these factors reflect changes in scale of even (a) and odd (b) parts of windows extracted from optimal filters with different net delays ϵ.

a family of filters with varying fractional delay. In this work fractional delay filters optimal in the Chebyshev sense (minimax) and filters optimal in least squares sense are considered, as they give designer explicit control of approximation band. Unfortunately, there are no formulas for windows that can be used to design these kinds of optimal filters. Nevertheless, we can assume that their impulse responses $h_{opt}[n]$ might be computed by multiplying ideal FD filter impulse response $h_{id}[n]$ (2) by some window $w_{opt}[n]$

$$h_{opt}[n] = h_{id}[n] w_{opt}[n] \tag{8}$$

Therefore, basing on the aforementioned assumption, for filter of any delay we can extract window using the following formula

$$w_{opt}[n] = h_{opt}[n]/h_{id}[n] \tag{9}$$

Let us now investigate properties of windows extracted from filters with different delays. As fractional delay filter impulse response and, thus, extracted window are in general asymmetric, to verify properties of $w_{opt}[n]$ we split it into even

$$w_e[n] = (w_{opt}[n] + w_{opt}[N-1-n])/2 \tag{10}$$

and odd part

$$w_o[n] = (w_{opt}[n] - w_{opt}[N-1-n])/2 \tag{11}$$

As we can see in Fig. 3a the even part of the extracted window seems to be independent of filter delay regardless of optimal filter type. Additionally, the odd part of the extracted window is significantly smaller but non-zero, which indicates asymmetry of this window (Fig. 3b). More thorough investigations, described in more detail later, confirmed that change in the shape of both even and odd part of reference window is very slight. Therefore we can in most cases assume that only scale of even and odd part of the extracted window changes with filter delay (Fig. 4). This means that single asymmetric reference window extracted for net delay ϵ_{ref} with proper scaling of its even and odd part can be used to reconstruct a window $w_{rec,\epsilon}[n]$ for filter with different net delay $\epsilon \neq \epsilon_{ref}$

$$w_{opt,\epsilon}[n] \cong w_{rec,\epsilon}[n] = \alpha(\epsilon) w_{e,ref}[n] + \beta(\epsilon) w_{o,ref}[n] \tag{12}$$

In particular, window extracted for $\epsilon_2 = -\epsilon_1$ is the same as time reversed window extracted for ϵ_1

$$w_{opt,\epsilon}[n] = w_{opt,-\epsilon}[N-1-n] \cong w_{rec,\epsilon}[n] = w_{rec,-\epsilon}[N-1-n] \tag{13}$$

Therefore high quality VFD filters with negative net delays can be designed using only single asymmetric reference window extracted for an arbitrary selected positive net delay, and vice versa

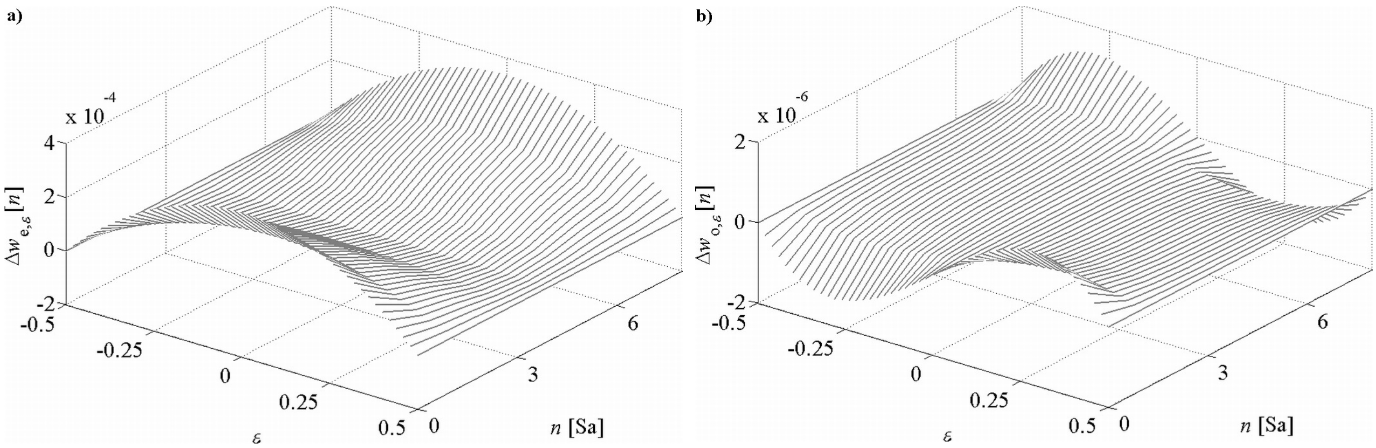

Fig. 5. Errors in reconstruction of even (a) and odd (b) parts of windows from Fig. 3 for the reference window extracted for $\epsilon_{ref} = -0.5$ and reconstruction factors from Fig. 4.

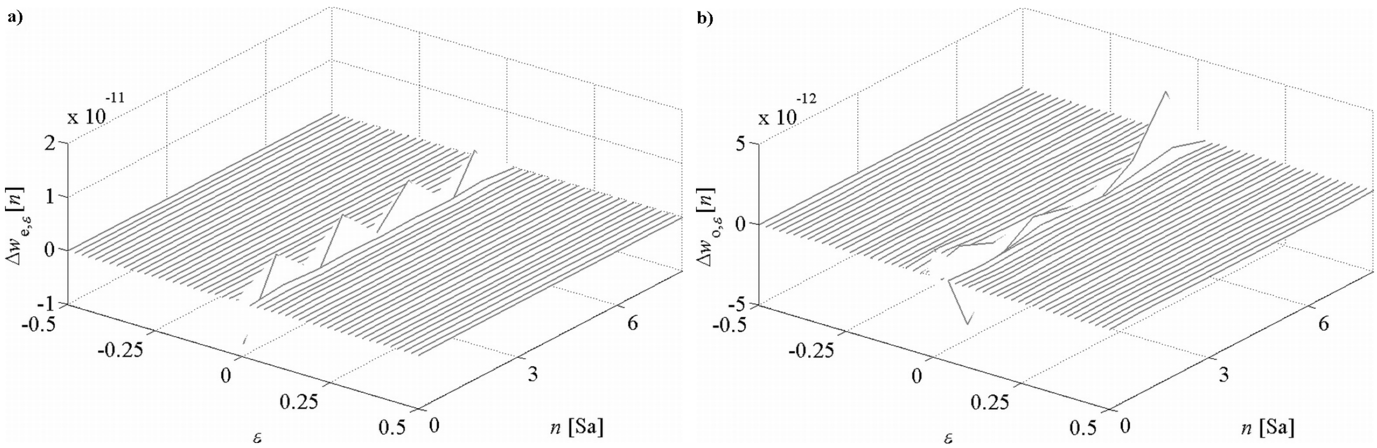

Fig. 6. Errors in reconstruction of even (a) and odd (b) parts of windows extracted from filters optimal in the least square sense. Reference window extracted at $\epsilon_{ref} = -0.5$. Filter with $N = 9$ and $f_a = 0.35$.

$$h_{opt,\epsilon}[n] \cong h_{rec,\epsilon}[n] = h_{id,\epsilon}[n]w_{rec,\epsilon}[n] = \\ = h_{id,\epsilon}[n]w_{rec,-\epsilon}[N-1-n] \quad (14)$$

This leads to general formula for VFD filter design

$$h_{opt,\epsilon}[n] \cong h_{rec,\epsilon}[n] = \\ = \begin{cases} h_{id,\epsilon}[n]w_{rec,\epsilon}[n], & \text{if } \mathrm{sgn}(\epsilon) = \mathrm{sgn}(\epsilon_{ref}) \text{ or } \epsilon = 0 \\ h_{id,\epsilon}[n]w_{rec,-\epsilon}[N-1-n], & \text{otherwise} \end{cases} \quad (15)$$

It is worth noting that window reversing operation in (15) does not consume many signal processor resources and significantly improves designed filter quality. We need, however, to compute factors $\alpha(\epsilon)$ and $\beta(\epsilon)$ from (12), which can be done with the following formulas

$$\alpha(\epsilon) \cong \frac{1}{f_a}\int_0^{f_a} |H_e(f)|/|H_{e,ref}(f)|df \quad (16)$$

and

$$\beta(\epsilon) \cong \frac{\mathrm{sgn}(\epsilon_{ref})\mathrm{sgn}(\epsilon)}{f_a}\int_0^{f_a} |H_o(f)|/|H_{o,ref}(f)|df \quad (17)$$
$$\epsilon_{ref} \neq 0$$

where $H_e(f)$ is a frequency response of FD filter with net delay ϵ designed using even part of window $w_{opt,\epsilon}[n]$ extracted for net delay ϵ while $H_{e,ref}(f)$ is a frequency response of FD filter designed using even part of reference window $w_{ref}[n]$. Impulse responses of these filters are described by the following formulas

$$h_e[n] = h_{id,\epsilon}[n]w_{e,\epsilon}[n], \quad h_{e,ref}[n] = h_{id,\epsilon}[n]w_{e,ref}[n] \quad (18)$$

By analogy, $H_o(f)$ and $H_{o,ref}(f)$ are frequency responses of FD filters designed using odd parts of windows $w_{opt,\epsilon}[n]$ and $w_{ref}[n]$. Impulse responses of these filters can be described by the following equations

$$h_o[n] = h_{id,\epsilon}[n]w_{o,\epsilon}[n], \quad h_{o,ref}[n] = h_{id,\epsilon}[n]w_{o,ref}[n] \quad (19)$$

Typical curves of α and β parameters are presented in Fig. 4. The scale $\alpha(\epsilon)$ of the even part of extracted window varies very little (Fig. 4a), while the odd part is approximately proportional to the net delay (Fig. 4b).

Let us consider now reconstruction errors of even and odd parts of extracted window $w_{opt,\epsilon}[n] = w_{e,\epsilon}[n] + w_{o,\epsilon}[n]$ ((9), Fig. 3) based on reference window $w_{ref}[n] = w_{e,ref}[n] + w_{o,ref}[n]$ with gain correction factors $\alpha(\epsilon)$ and $\beta(\epsilon)$ ((16),

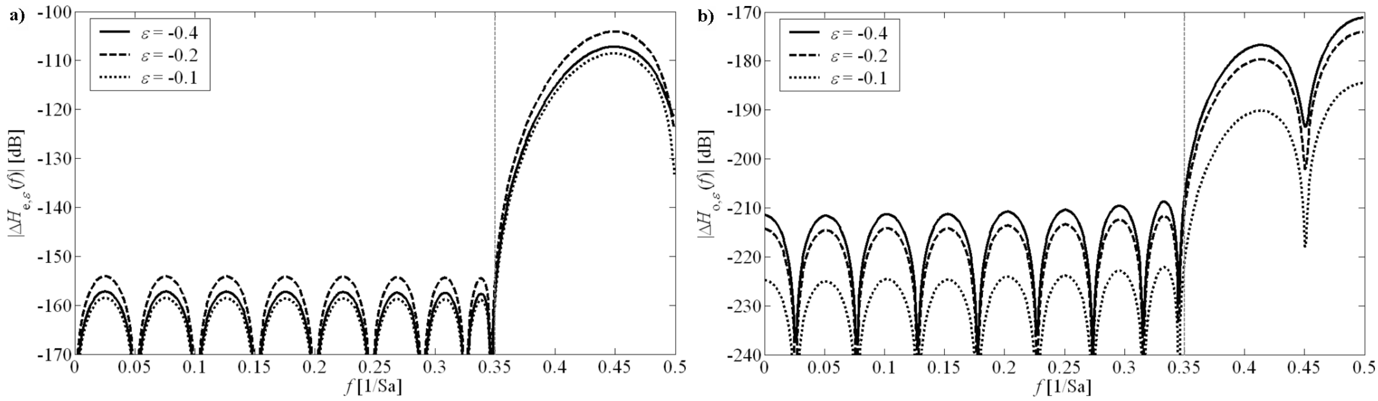

Fig. 7. Magnitude responses of reconstruction errors of optimal filter related to even (a) and odd (b) parts of reference window for different net delays. Reference window extracted from filter with $N = 19$ and $f_a = 0.35$ optimal in the Chebyshev sense at $\epsilon_{ref} = -0.5$.

(17), Fig. 4)

$$\Delta w_{e,\epsilon}[n] = w_{e,\epsilon}[n] - \alpha(\epsilon) w_{e,ref}[n] \tag{20}$$

and

$$\Delta w_{o,\epsilon}[n] = w_{o,\epsilon}[n] - \beta(\epsilon) w_{o,ref}[n] \tag{21}$$

As we can see in Fig. 5 and 6 we have achieved good reconstruction of both parts of reference windows. We can also see that there are no noticeable variations of window shape extracted from LS filter. Therefore perfect reconstruction of extracted window for all net delays can be achieved (Fig. 6). On the other hand, in case of minimax filter, some small changes in the shape of the extracted window can be observed (Fig. 5), which results in small but visible reconstruction errors. That is why, as we will observe in the next section, VFD filter design with the extracted window method is characterized with slightly worse performance in comparison to the LS case.

Using introduced above definitions of reconstruction errors of extracted windows parts (20) and (21), even and odd part of reconstruction error ($\Delta h_{e,\epsilon}[n]$ and $\Delta h_{o,\epsilon}[n]$) of the optimal FD filter impulse response

$$h_{rec,\epsilon}[n] = h_{id,\epsilon}[n] w_{rec,\epsilon}[n] = h_{opt,\epsilon}[n] + \Delta h_{e,\epsilon}[n] + \Delta h_{o,\epsilon}[n] \tag{22}$$

can be defined as follows

$$\Delta h_{e,\epsilon}[n] = h_{id,\epsilon}[n] \Delta w_{e,\epsilon}[n] \tag{23}$$

$$\Delta h_{o,\epsilon}[n] = h_{id,\epsilon}[n] \Delta w_{o,\epsilon}[n] \tag{24}$$

Based on magnitude responses of errors (23) and (24) for minimax filters presented in Fig. 7 we can say that reconstructed filters performance is virtually optimal. It is worth noting that the energy of those errors is very low with most of it located mainly outside of the approximation band (above f_a). That is why calculations in (16) and (17) are limited to approximation band, which is especially important for minimax filters. Since shape of windows extracted from filters optimal in the least square sense do not change, errors (23) and (24) for such filters are equal to zero.

IV. Performance

In the previous section we have investigated properties of windows extracted from optimal FD filters. We have also discussed how single reference window can be used to reconstruct window for any delay required in VFD filter implementation. We have demonstrated that in case of LS filters using both even and odd parts of reference window perfect extracted window reconstruction can be achieved (Fig. 6). With minimax filters some reconstruction errors can be observed (Fig. 5) but nonetheless filters designed with reconstructed windows are virtually optimal (Fig. 7). The problem is that such window reconstruction (12) needs independent scaling of even and odd part of reference window, while, as it was stated in section I, we are interested in simpler VFD filter implementation (Fig. 1). The proposed VFD filter design process at runtime (Fig. 1) consists only of ideal filter impulse response $h_{id,\epsilon}[n]$ multiplication by generally asymmetric reference window $w_{ref}[n]$ and filter gain correction $\alpha(\epsilon)$

$$h_{opt,\epsilon}[n] \cong h_{rec,\epsilon}[n] = h_{id,\epsilon}[n] w_{rec,\epsilon}[n] = \\ = h_{id,\epsilon}[n] \alpha(\epsilon) w_{ref}[n] \tag{25}$$

Additional improvement can be achieved by reversing of reconstructed window $w_{rec}[n]$ (15), as described in section III.

Since the odd part of the extracted window is small in comparison to its even part (Fig. 3) it can be simply ignored for low performance FD filters (e.g. filter with $f_a = 0.43$ in Fig. 8). Fig. 8 and 9 present design effects for assumption that $\beta(\epsilon)$ in (12) is equal to zero. This leads to the following formula for reconstructing impulse response of FD filter using only even part of the extracted window

$$h_{opt,\epsilon}[n] \cong h_{rec,\epsilon}[n] = h_{id,\epsilon}[n] w_{rec,\epsilon}[n] = \\ = h_{id,\epsilon}[n] \alpha(\epsilon) w_{e,ref}[n] \tag{26}$$

Performed research indicates that for high performance FD filters, with PE (6) or LSE (7) below –50 dB, the odd part of the extracted window although small is still vital. In such case discarded odd extracted window part results in significant degradation of designed filter performance. In papers [10]–[13] only even (symmetric) part of the reference window has been

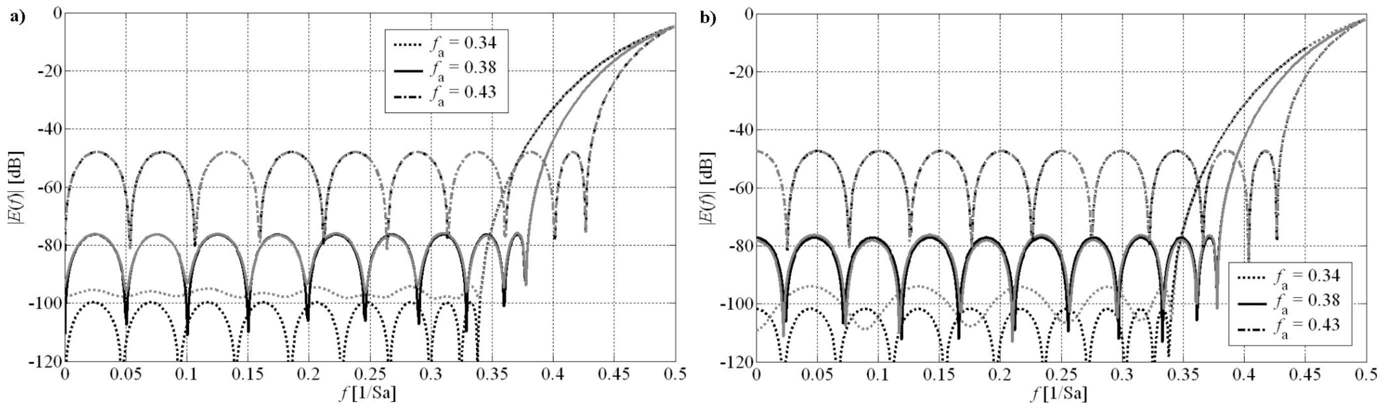

Fig. 8. Magnitudes of complex approximation errors for minimax fractional delay filters (black lines) with $N = 19$, $\epsilon = -0.2$ (a) as well as with $N = 20$, $\epsilon = -0.2$ (b) and their equivalents with discarded odd part of extracted windows (gray lines). Filter gain correction is not applied.

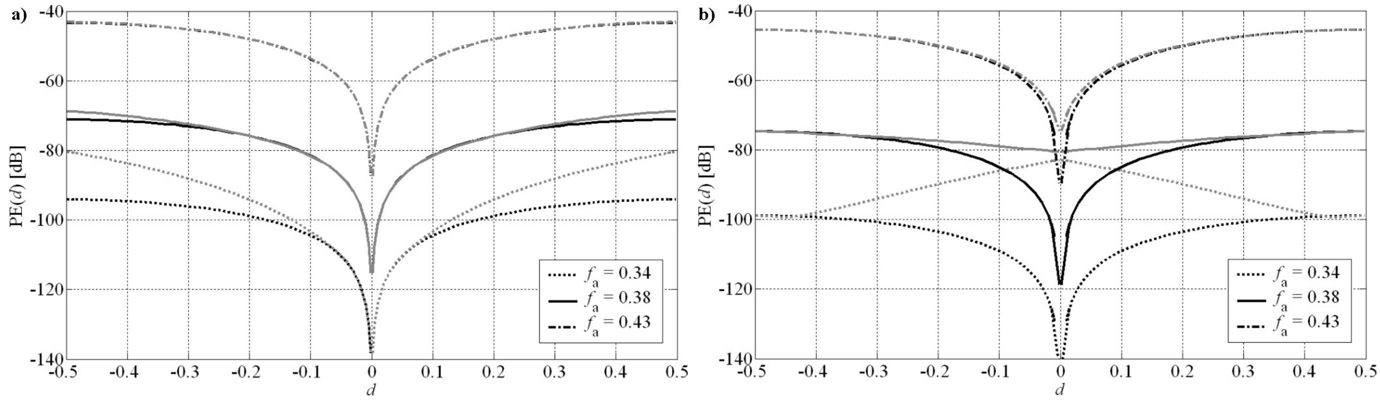

Fig. 9. PE for minimax variable fractional delay filters (black lines) with $N = 19$ (a) as well as $N = 20$ and their equivalents with discarded odd part of extracted windows (gray lines). Filter gain correction is not applied.

used with additional optimization of gain correction factor. As the actual extracted windows are asymmetric, we propose to use asymmetric window extracted for some arbitrary selected net delay ϵ_{ref}. It is worth noting that during the design process any reference delay can be assumed, however, due to FD filter impulse response properties as well as numerical limitations, it is advisable not to use $\epsilon = 0$ for odd filter lengths and $\epsilon = \pm 0.5$ for even filter lengths.

As we can see in Fig. 10 filter design (14) using asymmetric window extracted for ϵ_{ref}, when no gain correction is applied, results in optimal solution for ϵ_{ref} and $-\epsilon_{ref}$. For delays different than $\pm\epsilon_{ref}$ filter performance quickly deteriorates with distance from reference ϵ_{ref}. Still for low performance filters a feasible solution is to store several asymmetric reference windows, extracted for different delays, in a look-up-table (LUT) and select them based on the desired delay (Fig. 11). However, for high performance filters such approach would require too many reference windows, as a single window can be used only in extremely narrow delay range around ϵ_{ref}.

Although LUT approach can be sensible in some applications, in most cases it is better to use single reference window with additional gain correction factor dependent on net delay ϵ (15), (25). For each desired net delay we take $\alpha(\epsilon)$ (16) as initial point and optimize this correction factor value for best designed filter performance. The same $\alpha(\epsilon)$ optimization algorithms as in case of symmetric reference windows [7], [10] can be used. It comes as a surprise how little we gain from using asymmetric window instead of symmetric one. Especially if we remember how large performance loss we observe when odd part of the extracted window is discarded (Fig. 8 and 9). Fig. 13 presents differences in performance between optimal filters and filters designed using several asymmetric reference windows and their symmetric even parts with optimal gain correction (Fig. 12). In both cases, designed filters achieve almost optimal performance. As we can see, asymmetric window can be used to determine the delay for which the performance is truly optimal. However, in practical applications the performance difference between filters designed with symmetric and asymmetric windows can be neglected. Generally we cannot gain no more than tenths of dB at ϵ_{ref} with better results for longer filters.

An interesting difference between minimax and LS cases can be noticed though. As it has been mentioned in section III, for LS filters shape of even part of extracted window is independent of net delay, and thus with symmetric window selection of the reference delay ϵ_{ref} does not matter. On the other hand, shape of even part of windows extracted from minimax filter changes slightly, but in such way that design results are similar to cases with asymmetric window (Fig. 13). It is also worth noting that in every investigated case filter gain correction curves are continuous and even functions of delay and can be readily approximated with polynomials and

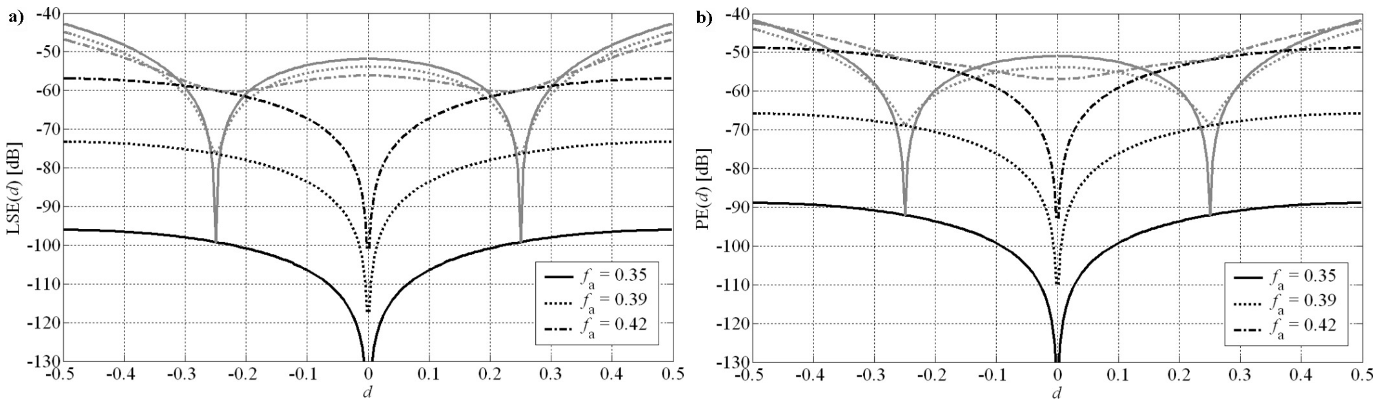

Fig. 10. Errors for VFD LS (a) and minimax (b) filters with $N = 19$ (black lines) and their equivalents designed using asymmetric windows extracted for $\epsilon_{ref} = 0.25$ without gain correction (gray lines).

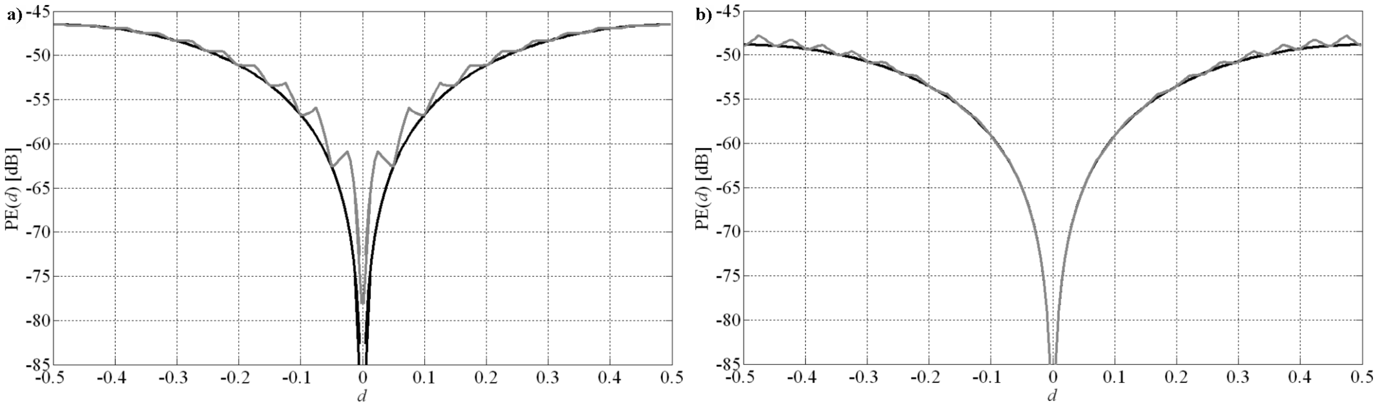

Fig. 11. PE for minimax variable fractional delay filters with $f_a = 0.42$ (black lines) and their equivalents designed using set of 10 asymmetric reference windows stored in a look-up-table (gray lines). Filters lengths $N = 18$ (a) and $N = 19$ (b).

computed at runtime [10], which makes the proposed design method as simple as traditional window method and suitable for VFD applications.

V. Conclusions

The paper is dedicated to investigation of extracted window method with particular focus on asymmetric reference window. The investigated filter design approach seems ideal for VFD filter design, as single window can be used with only additional gain correction. In order to broaden understanding of investigated design method, thorough analysis of properties of windows extracted from optimal minimax and LS filters has been presented in this paper. We have demonstrated that proper separate scaling of even and odd part of single extracted window selected as a reference window leads to almost perfect reconstruction of optimal filter for any fractional delay. Further reduction of numerical costs in VFD filter implementation with extracted window can be attained if just one scaling factor is used for asymmetric window. We have confirmed the thesis that asymmetric reference window can be used to design nearly optimal VFD filters. The use of asymmetric window allows for selection of net delay for which optimality is reached, however, the performance is almost identical as that for symmetric window (even part of asymmetric reference window). In both cases errors in reconstruction of odd part of extracted window can be readily compensated with additional gain correction of the reference window. Also another approach has been discussed here. A LUT of asymmetric windows extracted for different reference delay can be used, which do not require gain correction. The problem is that such approach is only sensible for low performance filters, as the number of required reference windows rapidly increases with desired performance of VFD filter.

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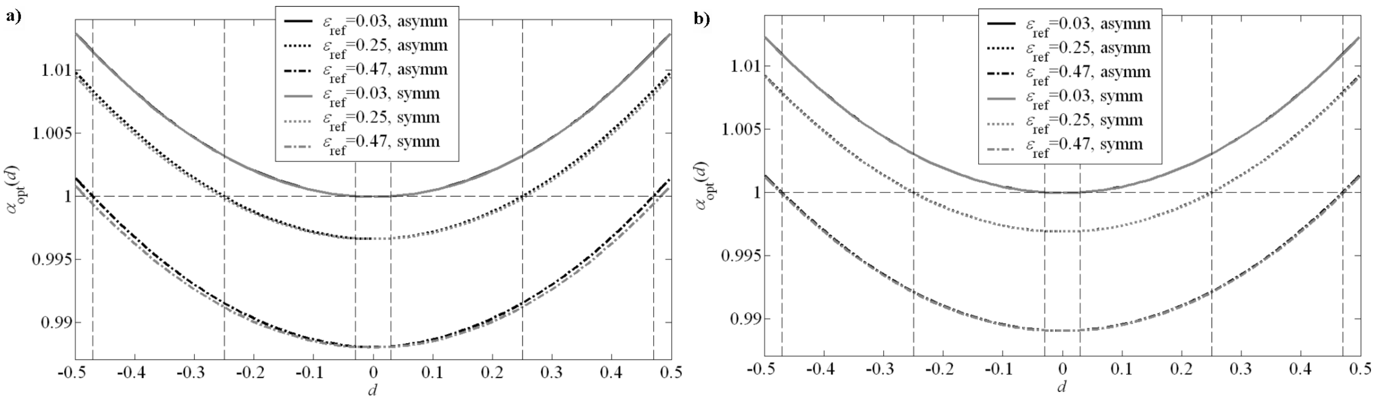

Fig. 12. Optimal gain correction factor curves for variable fractional delay filters with $N = 17$ and $f_a = 0.35$ designed using various reference windows extracted from LS (a) and minimax (b) filters.

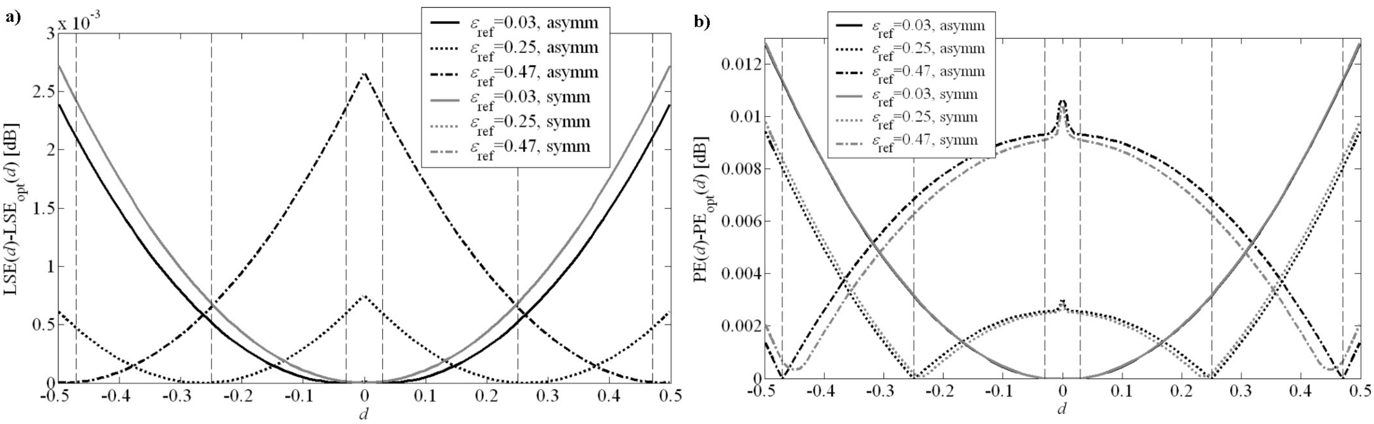

Fig. 13. Differences between errors of filters from Fig. 12 and their optimal equivalents.

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