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# Fixed point indices of iterated smooth maps in arbitrary dimension ☆

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ABSTRACT

Let $f$ be a smooth self-map of $\mathbb{R}^m$, when $m$ is an arbitrary natural number. We give a complete description of possible sequences of indices of iterations of $f$ at an isolated fixed point, answering in affirmative the Chow, Mallet-Paret and Yorke conjecture posed in [S.N. Chow, J. Mallet-Parret, J.A. Yorke, A periodic point index which is a bifurcation invariant, in: Geometric Dynamics, Rio de Janeiro, 1981, in: Lecture Notes in Math., vol. 1007, Springer, Berlin, 1983, pp. 109–131].



## 1. Introduction

Let $x_0$ be a fixed point of a map $f$. The local fixed point index $\operatorname{ind}(f, x_0) \in \mathbb{Z}$ is a topological invariant that plays important role in fixed point theory. It can also be used to study a structure of periodic points and dynamical properties of a map. In such a case the sequence of fixed point indices of iterations $\{\operatorname{ind}(f^n, x_0)\}_{n=1}^{\infty}$ is applied.

We will consider local self-maps $f$ of $\mathbb{R}^m$ in a neighborhood of the fixed point 0. Let $f : U \to \mathbb{R}^m$, where $U$ is an open subset of $\mathbb{R}^m$, be a map such that 0 is an isolated fixed point for each iteration. Then, the local fixed point index at 0, $\operatorname{ind}(f^n, 0)$, is well defined for each $f^n$ taken on a small enough neighborhood of 0.

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The knowledge about the form of the sequence $\{\operatorname{ind}(f^n, x_0)\}_{n=1}^{\infty}$ allows one to deduce valuable information about the existence of periodic points, number of orbits and dynamical behavior of $f$ in a neighborhood of a fixed point (cf. for example [1,13,21]).

Thus, the important task is to find all possible restrictions for the sequence of fixed point indices. In 1971 Krasnosel'skii and Zabreiko noticed that (for continuous maps) $\{\operatorname{ind}(f^n, x_0)\}_{n=1}^{\infty}$ cannot take arbitrary values but its elements must satisfy some congruences [26]. Namely, they observed that for any prime number $p$ holds:

$$\operatorname{ind}(f, x_0) \equiv \operatorname{ind}\bigl(f^p, x_0\bigr) \pmod p.$$

These restrictions, known as *mod p property*, were proved by Steinlein in 1972 [25]. In 1984 Albrecht Dold found much more general congruences for indices [4], called *Dold relations*:

$$\sum_{k|n} \mu(n/k) \operatorname{ind}\bigl(f^k, x_0\bigr) \equiv 0 \pmod n, \tag{1.1}$$

where $\mu$ is the arithmetic Möbius function.

The natural question is whether there are any further restrictions for local indices of a continuous map for $m > 1$. (In dimension one there are just few sequences of indices of a very special form.) Graff and Nowak-Przygodzki showed in [11] that the answer is negative, even in dimension 2 every sequence of integers which satisfies Dold relations can be realized as a sequence of fixed point indices of an iterated map (cf. also [1]).

Nowadays, the problem of determining the form of indices for different classes of maps is studied very intensively, the main attention is focused on planar homeomorphisms [10,14,15,19]; $\mathbb{R}^3$-homeomorphisms [20]; smooth maps [3,17,18,23]; simplicial maps [6,24] and holomorphic maps [2, 5,27,28].

The theory for smooth maps was initiated in 1974 by a very elegant result of Shub and Sullivan [23], who proved that, unlike in the continuous case, the sequence of indices is always bounded.

In 1983 Chow, Mallet-Paret and Yorke gave, in terms of the derivative of $f$ at 0, further restrictions for indices of a smooth map, which we call *CMPY conditions* [3] (cf. Theorem 2.6). They also formulated the conjecture, which may be presented in the following form: *the only restrictions for indices of iterations of a smooth map are CMPY conditions*. The proof of this hypothesis was given in 1990 for dimension 2 [1] and in 2006 for dimension 3 [9]. The aim of this paper is to prove the conjecture for arbitrary dimension. In this way we obtain the complete description of the forms of indices in $\mathbb{R}^m$ for each $m$. This knowledge may be successfully used in many branches of dynamical systems, differential equations and periodic point theory. Let us mention here an application in determining minimal number of periodic points in a smooth homotopy class of a given map. This problem is a classical one in continuous category (cf. [12]), establishing the complete list of possible indices of iterations for smooth maps makes it possible to deal with it also in the smooth category [7,8]. Our results enable us also to confirm old Babenko and Bogatyi conjecture (cf. [1]), which states that any bounded sequence satisfying Dold relations is a sequence of indices of an isolated fixed point of a smooth map of an Euclidean space.

The paper is organized in the following way: in Section 2 we introduce the notation and recall Chow, Mallet-Paret and Yorke theorem. In the third section we formulate our main result: we obtain the list of sequences that are admissible by CMPY conditions in $\mathbb{R}^m$ (Theorem 3.1), and next we state that every sequence from this list can be realized (Theorem 3.2). The second part is much more difficult, because we must simultaneously control topological and differential properties of constructed maps to obtain the given values of indices, preserving smoothness. Sections 4–8 are devoted to the detailed construction of one of the classes of realizations, and in Section 9 we discuss how to realize the other forms of indices. In the final section we consider Babenko and Bogatyi problem and the question of the realization of indices in some narrower classes of smooth maps. In particular we discuss the recent result obtained by Ruiz del Portal and Salazar by Conley index methods in dimension 3 [22].

## 2. Chow, Mallet-Paret and Yorke theorem in $\mathbb{R}^n$

Due to Dold relations (cf. formula (1.1)) one may write down indices of iterations in the convenient form of a combination of some basic sequences. This representation is called a periodic expansion.

**Definition 2.1.** For a given $k \in \mathbb{N}$ we define

$$\mathrm{reg}_k(n) = \begin{cases} k & \text{if } k \mid n, \\ 0 & \text{if } k \nmid n. \end{cases}$$

In other words, $\mathrm{reg}_k$ is the periodic sequence:

$$(0, \ldots, 0, k, 0, \ldots, 0, k, \ldots),$$

where the non-zero entries appear for indices divisible by $k$.

By $\mu$ we will denote the classical Möbius function, i.e., $\mu : \mathbb{N} \to \mathbb{Z}$ is defined by the following three properties: $\mu(1) = 1$, $\mu(k) = (-1)^s$ if $k$ is a product of $s$ different primes, $\mu(k) = 0$ otherwise.

**Theorem 2.2.** *(Cf. [13].) A sequence $\{\mathrm{ind}(f^n, 0)\}_{n=1}^{\infty}$ can be written down uniquely in the following form of a periodic expansion*:

$$\mathrm{ind}\big(f^n, 0\big) = \sum_{k=1}^{\infty} a_k \, \mathrm{reg}_k(n),$$

*where*

$$a_n = \frac{1}{n} \sum_{k \mid n} \mu\left(\frac{n}{k}\right) \mathrm{ind}\big(f^k, 0\big).$$

Notice that by Dold relations (1.1) coefficients $a_n$ are always integers.

**Definition 2.3.** Assume that a periodic expansion for a sequence $\{\mathrm{ind}(f^n, 0)\}_{n=1}^{\infty}$ is given. Let $B = \{n \in \mathbb{N}\colon\ a_n \neq 0\}$. The set $B$ is called *the set of local algebraic periods*.

**Definition 2.4.** Let $H$ be a finite subset of natural numbers, we introduce the following notation.

By $\mathrm{LCM}(H)$ we mean the least common multiple of all elements in $H$ with the convention that $\mathrm{LCM}(\emptyset) = 1$. We define the sets $\overline{H}$, $m \cdot H$, $H_{odd}$ by:

$$\overline{H} = \big\{\mathrm{LCM}(Q)\colon\ Q \subset H\big\},$$

$$m \cdot H = \{mh\colon\ h \in H\}, \qquad H_{odd} = \{h \in H\colon\ h \text{ is odd}\}.$$

Now we start to consider the smooth case. Let $f : U \to \mathbb{R}^m$, where $U$ is an open subset of $\mathbb{R}^m$ containing 0, be a $C^1$ map with 0 an isolated fixed point for each iteration.

**Definition 2.5.** By $\Delta$ we will denote the set of degrees of all primitive roots of unity which are contained in $\sigma(Df(0))$, the spectrum of the derivative at 0.

Chow, Mallet-Paret and Yorke showed that the set of local algebraic periods $B$ for $\{\mathrm{ind}(f^n, 0)\}_{n=1}^{\infty}$, where $f$ is a $C^1$ map, is finite and depends only on the set $\Delta$.

We denote by $\sigma_+$ the number of real eigenvalues of $Df(0)$ greater than 1, $\sigma_-$ the number of real eigenvalues of $Df(0)$ less than $-1$, in both cases counting with multiplicity.

Below we state the theorem of Chow, Mallet-Paret and Yorke [3], expressed in the language of periodic expansions (cf. [13]).

**Theorem 2.6.** *Let $U \subset \mathbb{R}^m$ be an open neighborhood of 0, $f: U \to \mathbb{R}^m$ be a $C^1$ map having 0 as an isolated fixed point for each iteration. Then*:

$$\operatorname{ind}(f^n, 0) = \sum_{k\in O} a_k \operatorname{reg}_k(n),$$

*where*

(∗) $O = \begin{cases} \overline{\Delta} & \text{if } \sigma_- \text{ is even}, \\ \overline{\Delta} \cup 2\overline{\Delta}_{odd} & \text{if } \sigma_- \text{ is odd}. \end{cases}$

(∗∗) *If $\sigma_-$ is odd and $k \in 2\overline{\Delta}_{odd} \setminus \overline{\Delta}$, then $a_k = -a_{k/2}$.*

(∗∗∗) *The bounds for the coefficients $a_1$ and $a_2$ are the following*:

(1) $a_1 = (-1)^{\sigma_+}$ *if* $1 \notin \sigma(Df(0))$.

(2) $a_1 \in \{-1, 0, 1\}$ *if* 1 *is an eigenvalue of* $Df(0)$ *with multiplicity* 1.

(3) $a_2 \in \{0, (-1)^{\sigma_+ +1}\}$ *if* $1 \notin \sigma(Df(0))$ *and* $-1$ *is the eigenvalue of* $Df(0)$ *with multiplicity* 1.

## 3. Indices of iterations in $\mathbb{R}^m$

In this section we formulate the main result of the paper, i.e. we give the complete list of all forms of indices in arbitrary dimension.

We use the following notation: for natural $s$ we denote by $L(s)$ any set of natural numbers of the form $\bar{L}$ with $\#L = s$ and $1, 2 \notin L$.

By $L_2(s)$ we denote any set of natural numbers of the form $\bar{L}$ with $\#L = s + 1$ and $1 \notin L$, $2 \in L$.

Let us remind that $\mathrm{LCM}(\emptyset) = 1$.

**Theorem 3.1** *(Main Theorem I). Let $f$ be a $C^1$ self-map of $\mathbb{R}^m$, $m > 1$. Then the sequence of local indices of iterations $\{\operatorname{ind}(f^n, 0)\}_{n=1}^{\infty}$ has one of the following forms.*

(I) *For m odd*

$$(A^o) \quad \operatorname{ind}(f^n, 0) = \sum_{k\in L_2(\frac{m-3}{2})} a_k \operatorname{reg}_k(n),$$

$$(B^o), (C^o), (D^o) \quad \operatorname{ind}(f^n, 0) = \sum_{k\in L(\frac{m-1}{2})} a_k \operatorname{reg}_k(n),$$

*where*

$$a_1 = \begin{cases} 1 & \text{in the case } (B^o), \\ -1 & \text{in the case } (C^o), \\ 0 & \text{in the case } (D^o), \end{cases}$$

$$(E^o), (F^o) \quad \operatorname{ind}(f^n, 0) = \sum_{k\in L_2(\frac{m-1}{2})} a_k \operatorname{reg}_k(n),$$

*where* $a_1 = 1$ *and*

$$a_2 = \begin{cases} 0 & \textit{in the case } (E^o), \\ -1 & \textit{in the case } (F^o). \end{cases}$$

(II) *For m even*:

$$\big(A^e\big) \quad \operatorname{ind}\big(f^n, 0\big) = \sum_{k \in L_2(\frac{m-4}{2})} a_k \operatorname{reg}_k(n),$$

$$\big(B^e\big) \quad \operatorname{ind}\big(f^n, 0\big) = \sum_{k \in L(\frac{m-2}{2})} a_k \operatorname{reg}_k(n),$$

$$\big(C^e\big), \big(D^e\big), \big(E^e\big) \quad \operatorname{ind}\big(f^n, 0\big) = \sum_{k \in L_2(\frac{m-2}{2})} a_k \operatorname{reg}_k(n),$$

*where*

$$a_1 = \begin{cases} 1 & \textit{in the case } (C^e), \\ -1 & \textit{in the case } (D^e), \\ 0 & \textit{in the case } (E^e), \end{cases}$$

$$\big(F^e\big) \quad \operatorname{ind}\big(f^n, 0\big) = \sum_{k \in L(\frac{m}{2})} a_k \operatorname{reg}_k(n),$$

*where* $a_1 = 1$.

**Theorem 3.2** *(Main Theorem II). Every sequence of integers which is of one of the forms* (*A*)–(*F*) *can be realized as a sequence of local indices of iterations of a* $C^1$ *self-map of* $\mathbb{R}^m$.

**Example 3.3.** Let us consider the following sequence of integers:

$$(2, 2, 5, 2, 2, 5, \ldots). \tag{3.1}$$

Can this sequence be a sequence of local indices of iterations of a smooth map in the plane?

We see that the above sequence may be written down as $2\operatorname{reg}_1(n) + \operatorname{reg}_3(n)$. On the other hand, by Theorem 3.1 in dimension $m = 2$ the only admissible sequences are ($m$ even, and $L_2(-1) = L(0) = \overline{\emptyset} = \operatorname{LCM}(\emptyset) = 1$):

$$\big(A^e\big) = \big(B^e\big) \quad \operatorname{ind}\big(f^n, 0\big) = a_1 \operatorname{reg}_1(n),$$

$$\big(C^e\big), \big(D^e\big), \big(E^e\big) \quad \operatorname{ind}\big(f^n, 0\big) = a_1 \operatorname{reg}_1(n) + a_2 \operatorname{reg}_2(n),$$

where

$$a_1 = \begin{cases} 1 & \text{in the case } (C^e), \\ -1 & \text{in the case } (D^e), \\ 0 & \text{in the case } (E^e), \end{cases}$$

$$\big(F^e\big) \quad \operatorname{ind}\big(f^n, 0\big) = \operatorname{reg}_1(n) + a_k \operatorname{reg}_k(n),$$

where $k > 2$.

As the sequence (3.1) is not of any of the forms listed above, it cannot be obtained as a sequence of local indices of some iterated smooth map in dimension 2. □

### 3.1. Proof of Theorem 3.1

Now we prove Theorem 3.1, the proof of Theorem 3.2, which is in fact Chow, Mallet-Paret and Yorke conjecture is much more difficult and will be given in the forthcoming sections.

Let $G = \{d_1, \ldots, d_s\} = \Delta \setminus \{1, 2\}$ (see Definition 2.5) and define $G_2 = G \cup \{2\}$. Now we classify possible forms of indices of iterations in dependence on $s$:

**Lemma 3.4.** *Let $f$ be a smooth local self-map of $\mathbb{R}^m$ having 0 as an isolated fixed point for each iteration.*

(I) *If $m = 2s$ then*

$$\operatorname{ind}(f^n, 0) = \sum_{k \in \overline{G}} a_k \operatorname{reg}_k(n) \quad \text{and} \quad a_1 = 1. \tag{3.2}$$

(II) *If $m = 2s + 1$ then*

$$\operatorname{ind}(f^n, 0) = \sum_{k \in \overline{G}} a_k \operatorname{reg}_k(n) \quad \textit{and} \quad a_1 \in \{-1, 0, +1\} \tag{3.3}$$

*or*

$$\operatorname{ind}(f^n, 0) = \sum_{k \in \overline{G}_2} a_k \operatorname{reg}_k(n) \quad \textit{and} \quad a_1 = 1,\ a_2 \in \{0, -1\}. \tag{3.4}$$

(III) *If $m = 2s + 2$ then*

$$\operatorname{ind}(f^n, 0) = \sum_{k \in \overline{G}_2} a_k \operatorname{reg}_k(n) \quad \textit{and} \quad a_1 \in \{-1, 0, +1\} \tag{3.5}$$

*or*

$$\operatorname{ind}(f^n, 0) = \sum_{k \in \overline{G}} a_k \operatorname{reg}_k(n). \tag{3.6}$$

(IV) *If $m \geqslant 2s + 3$ then*

$$\operatorname{ind}(f^n, 0) = \sum_{k \in \overline{G}_2} a_k \operatorname{reg}_k(n). \tag{3.7}$$

**Proof.** We make use of Theorem 2.6. Notice that in any case $O \subset \overline{G}_2$ (for the definition of $O$ see Theorem 2.6), thus we start with a general remark that for any map

$$\operatorname{ind}(f^n, 0) = \sum_{k \in \overline{G}_2} a_k \operatorname{reg}_k(n)$$

for some integers $a_k$ (not necessarily non-zero). This gives the case (IV). Now we will show that in the first three cases some restrictions on the numbers $a_k$ must be satisfied.

(I) By dimension arguments $Df(0)$ has no real eigenvalues. In particular neither 1 nor 2 belongs to $\Delta$ and moreover $\sigma_+ = \sigma_- = 0$. As a result, by Theorem 2.6(1), $a_1 = (-1)^{\sigma_+} = 1$ and by $(*)$ of Theorem 2.6 $O = \overline{\Delta} = \overline{G}$. These restrictions give the formula (3.2).

(II) In this case there is a single real eigenvalue. Let us consider two cases in dependence whether $\sigma_-$ is even or odd.

(A) $\sigma_-$ is even (hence $\sigma_- = 0$).
   (a) If $1 \in \sigma(Df(0))$ then its multiplicity is 1, hence Theorem 2.6(2) implies $a_1 \in \{-1, 0, +1\}$. Here $-1$ is not an eigenvalue ($2 \notin \Delta$), hence there are no restrictions for $a_2$. By $(*)$ of Theorem 2.6 the fact that $\sigma_-$ is even implies $O = \overline{\Delta} = \overline{G}$. This leads to the formula (3.3).
   (b) If $-1 \in \sigma(Df(0))$ then its multiplicity is one and 1 is not an eigenvalue. Now Theorem 2.6 parts (1) and (3) imply $a_1 = (-1)^{\sigma_+} = 1$ and $a_2 \in \{-1, 0\}$. This gives the formula (3.4).
   (c) If neither $+1$ nor $-1$ is an eigenvalue of $Df(0)$ then by the same argument as in item (a) $O = \overline{G}$. By Theorem 2.6(1) $a_1 = (-1)^{\sigma_+} = \pm 1$, and so this case is covered by the formula (3.3).

(B) $\sigma_-$ is odd. Then the only real eigenvalue $\nu$ satisfies $\nu < -1$, thus $\Delta = G$. Item $(*)$ of Theorem 2.6 implies that $O = \overline{G} \cup 2\overline{G}_{odd}$. Additionally by $(**)$ $a_k = -a_{k/2}$ for $k \in 2\overline{G}_{odd} \setminus \overline{G}$ and by (1) $a_1 = 1$. Thus, item $(**)$ implies $a_2 = -a_1 = -1$ (since $2 \notin \sigma(Df(0))$).
   Let us notice however, that $\overline{G} \cup 2\overline{G}_{odd} \subset \overline{G}_2$. As a consequence, this case is covered by the formula (3.4).

(III) Now there are at most two real eigenvalues (counting multiplicities). Let $\mu$ denote the multiplicity of the eigenvalue $+1$.

(A) Let $\mu \leqslant 1$. Then by Theorem 2.6 parts (1) and (2) $a_1 \in \{-1, 0, +1\}$ which gives the formula (3.5).

(B) Let $\mu = 2$. Then $\sigma_- = 0$ and $-1$ is not an eigenvalue, so by the same argument like in item (IIAa) $O = \overline{G}$ which gives the formula (3.6). □

**Proof of Theorem 3.1.** The case of $m = 2$ was proved in [1], $m = 3$ in [9]. For $m > 3$ the proof is a consequence of Lemma 3.4. Namely, let $m$ (and thus $s$) be fixed. We rearrange the formulation of Lemma 3.4, considering separately even and odd $m$ and taking $\overline{G} = L(s)$ and $\overline{G}_2 = L_2(s)$. This ends the proof. □

## 4. Realization of the case $(F^e)$

In the forthcoming part of the paper we prove Theorem 3.2. In Sections 4–9 we concentrate on the case $(F^e)$, which is the most fundamental one. In Section 9 we will give the descriptions of realizations in the other cases. The important part of Section 4 consists of the description of the scheme of the construction (Section 4.3). The proof of Lemma 4.1 (which is formulated below) is given in Section 8.

Let $d_1, \dots, d_s$, where $s = \frac{m}{2}$, be given natural numbers greater than 2. Without loss of generality we may assume that $d_i \neq d_j$ for $i \neq j$. For each non-empty set $J \subset \{d_1, \dots, d_s\}$ we define $d_J = \mathrm{LCM}(J)$. Moreover, we assume that a collection of integers $a_{d_J}$ is given. Then, we claim the existence of the needed realization $(F^e)$ in the following lemma.

**Lemma 4.1.** *There exists a smooth map $F : \mathbb{R}^{2s} \to \mathbb{R}^{2s}$ such that 0 is the only periodic point of $F$ and*

$$\mathrm{ind}\big(F^n, 0\big) = \mathrm{reg}_1(n) + \sum_J a_{d_J} \mathrm{reg}_{d_J}(n) \tag{4.1}$$

*for every $n \in \mathbb{N}$, where the summation extends over the family of all non-empty subsets of $\{d_1, \dots, d_s\}$.*

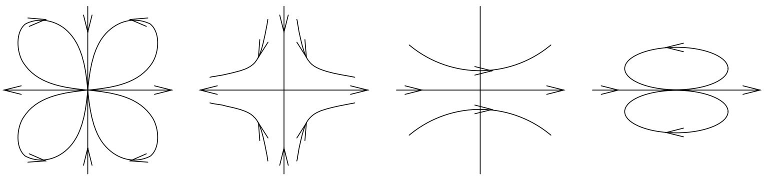


**Fig. 1.** Examples of the flows $h_p$ for different $p$.

The most difficult task is to realize (all at the same time for one map) expressions of the forms $a_{d_J} \operatorname{reg}_{d_J}(n)$, when $J$ has more than one element. In the next sections we show how to do it on some surfaces in $\mathbb{R}^{2s}$.

### 4.1. *Planar realization of* $\operatorname{reg}_1(n) + a_d \operatorname{reg}_d(n)$

Let $a_d$ be a given integer. In this section we define a two-dimensional flow, the discretization of which has indices equal to $\operatorname{reg}_1(n) + a_d \operatorname{reg}_d(n)$ (cf. [9]).

Let us consider planar flows $h_p : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ with phase portraits consisting of $2|p-1|$ hyperbolic regions for $p < 1$, $2(p-1)$ elliptic regions for $p > 1$ (the examples of such flows are given in Fig. 1); or with 0 as a source for $p = 1$.

This kind of flows may be described in the polar coordinates by the following equations:

$$\begin{aligned} \dot{r} &= ar^{k+1}\cos(p-1)\alpha, \\ \dot{\alpha} &= br^k \sin(p-1)\alpha, \end{aligned} \tag{4.2}$$

where $k \geqslant 0$ is an integer, $a$ and $b$ are positive real numbers.

The classical Poincaré–Bendixson formula for index of the discretization of such flows states that each double elliptic region gives the contribution to the index equal to $+1$; each double hyperbolic $-1$; and the fixed point itself gives the contribution equal to 1.

Thus, if we define $H_p = h_p(\cdot,\cdot,1) : \mathbb{R}^2 \to \mathbb{R}^2$, we get that $\operatorname{ind}(H_p, 0) = 1 + \frac{2(p-1)}{2} = p$.

Now, for a given natural number $d$ and an integer $a_d$ we define the map $\hat{O}$ as the $\frac{2\pi}{d}$ rotation around 0 and put $p = a_d d + 1$. Then $\hat{O}$ commutes with $H_p$ and

$$\operatorname{ind}\big((\hat{O} \circ H_p)^n, 0\big) = \begin{cases} 1 & \text{if } d \nmid n, \\ a_d d + 1 & \text{if } d \mid n \end{cases} = \operatorname{reg}_1(n) + a_d \operatorname{reg}_d(n). \tag{4.3}$$

To sum up the above considerations: in order to find a planar map that realizes the expression $\operatorname{reg}_1(n) + a_d \operatorname{reg}_d(n)$ as indices of iterations, we take the flow $h_{a_d d+1}$:

$$\begin{aligned} \dot{r} &= ar^{k+1}\cos(a_d d\alpha), \\ \dot{\alpha} &= br^k \sin(a_d d\alpha), \end{aligned} \tag{4.4}$$

where $\alpha \in [0, 2\pi]$, $k > 0$ is a fixed arbitrarily chosen integer and $a, b > 0$ are fixed arbitrarily chosen real numbers. Then the needed map is the discretization of the flow (4.4) composed with $\hat{O}$, i.e. $\frac{2\pi}{d}$ rotation around 0, and produce the indices according to the formula (4.3).

### 4.2. Rotation-like system in $\mathbb{R}^m$, $m$ is even

Assume $m$ is even, $\frac{m}{2} = s \in \mathbb{N}$. We will use the following polar coordinates:

$$\begin{aligned} x_1 &= r_1 \cos\phi_1, & y_1 &= r_1 \sin\phi_1, \\ x_2 &= r_2 \cos\phi_2, & y_2 &= r_2 \sin\phi_2, \\ &\dots & &\dots \\ x_s &= r_s \cos\phi_s, & y_s &= r_s \sin\phi_s. \end{aligned} \tag{4.5}$$

We define the flow $R : \mathbb{R}^{2s} \times \mathbb{R} \to \mathbb{R}^{2s}$ in the polar coordinates by the formula

$$R(r_1, \phi_1, \dots, r_s, \phi_s; t) = \left(r_1, \phi_1 + \frac{t}{d_1}, \dots, r_s, \phi_s + \frac{t}{d_s}\right). \tag{4.6}$$

Let us notice that:

- the only stationary point of $R$ is $0 \in \mathbb{R}^{2s}$,
- the period of the trajectory of the point $(r_1, \phi_1, \dots, r_s, \phi_s) \neq (0, \dots, 0)$ is equal to $2\pi \operatorname{LCM}\{d_i\colon r_i \neq 0\}$.

We denote by $S_{(r,\phi)}$ the trajectory of the point $(r, \phi) = (r_1, \phi_1, \dots, r_s, \phi_s)$ under the flow $R$. Now, for a given non-empty set $K \subset \{1, \dots, s\}$ we define below some special trajectory $S_K$ of $R$.

For each $1 \leqslant i \leqslant s$ we define:

$$r_i^K = \begin{cases} 1/\sqrt{\#K} & \text{if } i \in K, \\ 0 & \text{if } i \notin K. \end{cases} \tag{4.7}$$

Thus we obtain:

$$\left(r_1^K\right)^2 + \dots + \left(r_s^K\right)^2 = 1.$$

Notice that $r_i^K = r_{i'}^K$ for $i, i' \in K$ and $r_i = 0$ for $i \notin K$.

We will denote the trajectory of $(r_K, 0) = (r_1^K, 0, \dots, r_s^K, 0)$ under $R$ as $S_K$. We have:

$$S_K = \left\{\left(r_1^K, \frac{t}{d_1}, \dots, r_s^K, \frac{t}{d_s}\right)\colon t \in \mathbb{R}\right\}. \tag{4.8}$$

The trajectory $S_K$ is contained in $S^{2s-1} \subset \mathbb{R}^{2s}$ and has the period equal to $2\pi \operatorname{LCM}(K)$.

### 4.3. Scheme of the proof of Lemma 4.1

For a non-empty subset $K \subset \{1, \dots, s\}$ we denote $J_K = \{d_i\colon i \in K\}$. When $K$ is fixed we will simply write $J$ instead of $J_K$. Let us recall that $d_J = \operatorname{LCM}(J)$. For a given $K$ we will define a 2-dimensional surface in $\mathbb{R}^{2s}$, a cone over the trajectory $S_{(r,\phi)}$ by:

$$\operatorname{cone} S_{(r,\phi)} = \{\lambda l\colon l \in S_{(r,\phi)},\ \lambda \geqslant 0\}. \tag{4.9}$$

By $\operatorname{cone} S_K$ we will denote the cone over $S_K$.

**Lemma 4.2.** *For each* $(r,\alpha)$ *the surface* cone $S_{(r,\alpha)}$ *is invariant under the flow* $R$.

**Proof.** It is an immediate consequence of the definition of $R$, see the formula (4.6). □

Now we are in a position to give the general scheme of the construction. First, we notice that the space $\mathbb{R}^{2s}$ splits into cones over orbits of $R$. The cones are homeomorphic to $\mathbb{R}^2$ and mutually disjoint in the following meaning: $\operatorname{cone} S' \cap \operatorname{cone} S'' = \{0\}$ or $S' = S''$. Then the proof is divided into two parts.

(1) We fix a non-empty subset $K \subset \{1,\dots,s\}$ and we find a flow $P_{a_{d_{J_K}}}$ on the surface $\operatorname{cone} S_K$ whose 1-time map composed with 1-time map of $R$ realizes the sum

$$\operatorname{reg}_1 + a_{d_{J_K}} \operatorname{reg}_{d_{J_K}}$$

as indices of iterations. Such maps define a self-map of $\bigcup_{\emptyset \neq K \subset \{1,\dots,s\}} \operatorname{cone} S_K$ realizing the demanding sum $\sum_{\emptyset \neq K \subset \{1,\dots,s\}} a_{d_{J_K}} \operatorname{reg}_{d_{J_K}}$ with $a_1 = 1$.

(2) It remains to extend this map to a smooth self-map of $\mathbb{R}^{2s}$ with 0 as the only periodic point in such a way that the fixed point indices of all iterations are the same as those of the self-map of $\bigcup_{\emptyset \neq K \subset \{1,\dots,s\}} \operatorname{cone} S_K$.

The first part will be done in Section 5. Since the second part is a bit technical one we present its more detailed scheme.

(I) The desired extension will be the composition $F = S_1^{\phi} \circ S_1^{R} \circ R_1 \circ P_1$ of time-one maps of some flows ($R$ is the rotation-like system defined in Section 4.2).

(II) The flow $P$ is given by the formula (5.1) in $\operatorname{cone} S_K$. In the cone-neighborhood $V_K(\varepsilon)$ of $\operatorname{cone} S_K$ it is combined with a sink, and outside $V_K(\varepsilon)$ it is just a sink (Section 7.1).

(III) The flows $R$ and $P$ preserve the cones of the rotation-like system $R$. The restriction of the composition $R_1 \circ P_1$ to $\operatorname{cone} S_K$ realizes $\operatorname{reg}_1 + a_{d_J} \operatorname{reg}_{d_J}$ (Corollary 5.1). Moreover $R \circ P$ is a sink (to 0) outside $V_K(\varepsilon)$.

(IV) The composition $S_1^{\phi} \circ S_1^{R}$ is identity on $\operatorname{cone} S_K$, hence the restriction of $F = S_1^{\phi} \circ S_1^{R} \circ R_1 \circ P_1$ to this cone still realizes $\operatorname{reg}_1 + a_{d_J} \operatorname{reg}_{d_J}$. On the other hand, $S_1^{\phi} \circ S_1^{R}$ acts in such a way that $F$ moves the points towards $\operatorname{cone} S_K$ (Section 7.2). This implies (by Lemma 8.5)

$$\operatorname{ind}\bigl(F^n_{|V_K(\varepsilon)}, 0\bigr) = \operatorname{ind}\bigl(F^n_{|\operatorname{cone} S_K}, 0\bigr) = \operatorname{reg}_1(n) + a_{d_J} \operatorname{reg}_{d_J}(n).$$

(V) To realize the sum $\operatorname{reg}_1 + \sum_{\emptyset \neq K \subset \{1,\dots,s\}} a_{d_J} \operatorname{reg}_{d_J}$ we apply the above construction to all $K \subset \{1,\dots,s\}$ simultaneously (compare Corollary 5.2).

(VI) Finally, we prove that $F$ is smooth. This follows from the smoothness of the involved flows. The last is evident outside 0. It remains to check that the partial derivatives of discretizations are continuous at 0. This is a consequence of the homogeneity of the considered flows (Section 8.1).

## 5. Realization of the expression $a_{d_J} \operatorname{reg}_{d_J}$ on a surface in $\mathbb{R}^{2s}$

Let us fix a non-empty subset $K \subset \{1,\dots,s\}$ and let $J = \{d_i\colon i \in K\}$. Without loss of generality we may assume that $1 \in K$. Then we define the flow $P_{a_{d_J}} : \mathbb{R}^{2s} \times \mathbb{R} \to \mathbb{R}^{2s}$ in the generalized polar coordinates by:

$$\begin{cases} \dot{r}_1 = r_1|r|^k \cos(a_{d_J} d_1 \phi_1), \\ \dot{\phi}_1 = \dfrac{1}{d_1}|r|^k \sin(a_{d_J} d_1 \phi_1), \\ \dot{r}_2 = r_2|r|^k \cos(a_{d_J} d_1 \phi_1), \\ \dot{\phi}_2 = \dfrac{1}{d_2}|r|^k \sin(a_{d_J} d_1 \phi_1), \\ \dots \\ \dot{r}_s = r_s|r|^k \cos(a_{d_J} d_1 \phi_1), \\ \dot{\phi}_s = \dfrac{1}{d_s}|r|^k \sin(a_{d_J} d_1 \phi_1). \end{cases} \tag{5.1}$$

Let us investigate what is the form of the flow $P_{a_{d_J}}$ on the surface cone $S_K$.

First of all let us notice that cone $S_K$ is a two-dimensional surface, thus there are relations among parameters $r_i$ and $\phi_i$, for different $i$. Indeed:

- by the formula (4.9): for each $i \in K$, $r_i$ has the same form $r_i = \lambda \cdot \frac{1}{\sqrt{\#K}}$, we will denote this common value by $\rho$,
- by the form of $S_K$ cf. (4.8) there is a relation among $\phi_i$ for different $i$. Namely, taking $\phi_1 = t/d_1, \dots, \phi_n = t/d_n$ we get:

$$d_1\phi_1 = d_2\phi_2 = \dots = d_n\phi_n. \tag{5.2}$$

Let us denote the above common value by $\psi$.

The cone $S_K$ is composed of periodic orbits and each of them has the period $2\pi d_J$. In particular, $\psi$ defined in (5.2) may be chosen (for each point) in $[0, 2\pi d_J]$.

The restriction of the flow (5.1) to the cone $S_K$ takes the following form:

$$\begin{cases} \dot{\rho} = (\sqrt{\#K})^k \rho^{k+1} \cos(a_{d_J} \psi), \\ \dot{\psi} = (\sqrt{\#K})^k \rho^k \sin(a_{d_J} \psi). \end{cases} \tag{5.3}$$

We substitute

$$\frac{\psi}{d_J} = \alpha$$

and then we obtain:

$$\begin{cases} \dot{\rho} = (\sqrt{\#K})^k \rho^{k+1} \cos(a_{d_J} d_J \alpha), \\ \dot{\alpha} = \dfrac{1}{d_J}(\sqrt{\#K})^k \rho^k \sin(a_{d_J} d_J \alpha), \end{cases} \tag{5.4}$$

where now $\alpha \in [0, 2\pi]$.

On the other hand, this is the same formula as (4.4). Consequently, taking into account that the time-one map of the flow $R$ is $\frac{2\pi}{d_J}$ rotation around 0 on the cone, by the formula (4.3) we obtain the following

**Corollary 5.1.** *The composition of the time-one map of the flow $P_{a_{d_J}}$ with the time-one map of $R$ restricted to the surface* cone $S_K$, *realizes the sum* $\mathrm{reg}_1 + a_{d_J} \mathrm{reg}_{d_J}$.

**Corollary 5.2.** *The above compositions define a self-map $F$ of $\bigcup_{\emptyset\neq K\subset\{1,\ldots,s\}}\operatorname{cone} S_K$. The point 0 is the only periodic point and the map realizes the sum*

$$\mathrm{reg}_1+\sum_{\emptyset\neq K\subset\{1,\ldots,s\}} a_{d_{J_K}}\,\mathrm{reg}_{d_{J_K}}.$$

**Proof.** We apply $2^s-2$ times Lemma 5.3 (see below)

$$\begin{aligned}
&\operatorname{ind}\bigl(F^n_{|\bigcup_{\emptyset\neq K\subset\{1,\ldots,s\}}\operatorname{cone} S_K},0\bigr)\\
&\quad=\sum_{\emptyset\neq K\subset\{1,\ldots,s\}}\operatorname{ind}\bigl(F^n_{|\operatorname{cone} S_K},0\bigr)-\bigl(2^s-2\bigr)\,\mathrm{reg}_1(n)\\
&\quad=\sum_{\emptyset\neq K\subset\{1,\ldots,s\}}\bigl(\mathrm{reg}_1(n)+a_{d_{J_K}}\,\mathrm{reg}_{d_{J_K}}(n)\bigr)-\bigl(2^s-2\bigr)\,\mathrm{reg}_1(n)\\
&\quad=\mathrm{reg}_1(n)+\sum_{\emptyset\neq K\subset\{1,\ldots,s\}} a_{d_{J_K}}\,\mathrm{reg}_{d_{J_K}}(n),
\end{aligned}$$

and we get the desired formula. □

**Lemma 5.3.** *(Cf. [16].) Let $Y=A\cup B$ be a topological space, $x_0\in A\cap B$, $U$ be an open neighborhood of $x_0$ in $Y$. Let $F:U\to Y$, $F(U\cap A)\subset A$ and $F(U\cap B)\subset B$. If $x_0$ is an isolated fixed point of $F$ and $U$, $U\cap A$, $U\cap B$ and $U\cap A\cap B$ are ENRs, then*:

$$\operatorname{ind}(F,x_0)+\operatorname{ind}(F_{|A\cap B},x_0)=\operatorname{ind}(F_{|A},x_0)+\operatorname{ind}(F_{|B},x_0).$$

To end the proof of Lemma 4.1 it is enough to extend the map $F$ to a smooth self-map of $\mathbb{R}^{2s}$ without changing the fixed point indices of any iteration.

## 6. Deviations and neighborhoods of the cones

Now we start to extend onto $\mathbb{R}^{2s}$ the self-map of $\bigcup_{\emptyset\neq K\subset\{1,\ldots,s\}}\operatorname{cone} S_K$, realizing $\mathrm{reg}_1+\sum_{\emptyset\neq K\subset\{1,\ldots,s\}} a_{d_{J_K}}\,\mathrm{reg}_{d_{J_K}}$. We fix a non-empty subset $K\subset\{1,\ldots,s\}$ and at first we will extend the map on a cone-neighborhood of the surface $\operatorname{cone} S_K$. In order to do that we will define functions called deviations and some "regular" neighborhoods of $\operatorname{cone} S_K$ and describe its properties.

First, we prove a useful lemma which justifies the way we defined the flow $P_{a_{d_J}}$.

**Lemma 6.1.** *For each $(r,\alpha)$ the surface $\operatorname{cone} S_{(r,\alpha)}$ is invariant under the flow $P_{a_{d_J}}$.*

**Proof.** For a given point $(r,\phi)=(r_1,\phi_1,\ldots,r_n,\phi_n)$, $\operatorname{cone} S_{(r,\phi)}$ is a surface generated by two parameters $(\lambda,t)$, namely: $\operatorname{cone} S_{(r,\phi)}=\{(\lambda r_1,\frac{t}{d_1}+\phi_1,\ldots,\lambda r_n,\frac{t}{d_n}+\phi_n):\ \lambda\geqslant 0,\ t\in[0,2\pi]\}$.

Thus $V_s$ and $V_t$, the two tangent vectors to $\operatorname{cone} S_{(r,\phi)}$, have the forms:

$$V_s=(r_1,0,r_2,0,\ldots,r_n,0)$$

and

$$V_t=\left(0,\frac{1}{d_1},0,\frac{1}{d_2},\ldots,0,\frac{1}{d_n}\right).$$

Now, it suffices to notice that the vector field that generates $P_{a_{d_J}}$ (given by left-hand side of (5.1)) $(\dot{r}, \dot{\phi}) = (\dot{r}_1, \dot{\phi}_1, \ldots, \dot{r}_n, \dot{\phi}_n)$ at each point (except for 0 which is fixed) is a linear combination of $V_s$ and $V_t$:

$$(\dot{r}, \dot{\phi}) = \left[|r|^k \cos(a_{d_J} d_1 \phi_1)\right] V_s + \left[|r|^k \sin(a_{d_J} d_1 \phi_1)\right] V_t. \tag{6.1}$$

As a result, $(\dot{r}, \dot{\phi})$ is always tangent to $\operatorname{cone} S_{(r,\phi)}$, so $\operatorname{cone} S_{(r,\phi)}$ is invariant under the flow $P_{a_{d_J}}$. $\square$

Let us define a cone-neighborhood of $\operatorname{cone} S_K$:

$$V_K = \left\{(r_1, \phi_1, \ldots, r_s, \phi_s)\colon\ r_i > 0,\ |d_i\phi_i - d_j\phi_j| < \pi \text{ for } i, j \in K\right\}.$$

Now, we define the real-valued maps, called deviations, $D_\Phi$, $D_G$ on $V_K$.

Consider an element in $V_K$ represented by $(r_1, \phi_1, \ldots, r_s, \phi_s) \in \mathbb{R}^{2s}$. We denote $\Phi = \sum_{i\in K} \frac{d_i\phi_i}{\#K}$ and then define $D_\Phi(r, \phi)$:

$$D_\Phi = \sum_{i\in K} (d_i\phi_i - \Phi)^2. \tag{6.2}$$

Let us notice that the value of $\Phi$ depends on the $\phi_i$'s, however if $(r_1, \phi_1, \ldots, r_s, \phi_s)$ and $(r_1, \phi_1', \ldots, r_s, \phi_s')$ represent the same element and both satisfy the condition used in the definition of $V_K$, then the value of $D_\Phi$ is the same, hence we get a correctly defined function on $V_K$.

Furthermore, we put $G = \sum_{i\in K} \frac{r_i}{\#K}$. Then we define

$$D_G = \sum_{i\in K} \frac{(r_i - G)^2}{G^2} + \sum_{i\notin K} \frac{r_i^2}{G^2}. \tag{6.3}$$

Let us notice that the maps $D_\Phi, D_G$ are homogeneous:

$$D_\Phi(\lambda \cdot z) = D_\Phi(z), \qquad D_G(\lambda \cdot z) = D_G(z) \tag{6.4}$$

for all $0 \neq z \in V_K$ and $\lambda > 0$.

Finally, we define

$$V_K(\varepsilon) = \left\{(r, \phi)\colon\ D_\Phi(r, \phi) \leqslant \varepsilon \text{ and } D_G(r, \phi) \leqslant \varepsilon\right\}.$$

We choose $\varepsilon$ so small that the sets $V_K(\varepsilon)$ are disjoint for different $K$. Notice that for small enough $\varepsilon$ the deviations are well defined on $V_K(\varepsilon)$.

We end this section with the following observation:

**Lemma 6.2.** *The deviation maps $D_\Phi$ and $D_G$ are constant on each surface* $\operatorname{cone} S_{(r,\phi)} \subset V_K(\varepsilon)$.

**Proof.** Consider first $D_\Phi$. Let us take two points $(r, \phi)$ and $(r, \phi')$ in $\operatorname{cone} S_{(r,\phi)}$. Then $\phi = (\mu_1, \ldots, \mu_n)$ and $\phi' = (\mu_1 + \frac{t}{d_1}, \ldots, \mu_n + \frac{t}{d_n})$. We obtain:

$$\Phi(r, \phi') = \frac{\sum_{i\in K} (\mu_i + t/d_i) d_i}{\#K} = \frac{\sum_{i\in K} \mu_i d_i}{\#K} + t = \Phi(r, \phi) + t. \tag{6.5}$$

By the definition of $D_\Phi(r, \phi')$ and the equality (6.5) we get:

$$D_\Phi(r,\phi') = \sum_{i\in K}\big[(\mu_i + t/d_i)d_i - \Phi(r,\phi')\big]^2$$
$$= \sum_{i\in K}\big[d_i\mu_i + t - \Phi(r,\phi) - t\big]^2 = D_\Phi(r,\phi). \tag{6.6}$$

The corresponding equality for $D_G$ is evident, since no $\phi_i$ is used in its definition. □

Notice that by the above lemma $V_K(\varepsilon)$ consists of cones, i.e. if $(r,\phi)\in V_K(\varepsilon)$ then $\operatorname{cone} S_{(r,\phi)} \subset V_K(\varepsilon)$.

**Corollary 6.3.** *Lemma* 6.2 *together with Lemmas* 4.2 *and* 6.1 (*invariance of* $\operatorname{cone} S_{(r,\alpha)}$) *implies that the deviations for $D_G$ and $D_\Phi$ do not change their values under the action of $R$ and $P_{a_{d_J}}$.*

## 7. Flows

In this section we define three flows: $P$, $S^\phi$ and $S^r$ on $\mathbb{R}^{2s}$ which will be used directly to construct the needed realization $(F^e)$. The flow $P$ will be the extension of $P_{a_{d_J}}$ onto $\mathbb{R}^{2s}$ while $S^\phi$, $S^r$ make the values of $\Phi$ and $R$ smaller, which guarantees that no new periodic points appear.

We take a $C^\infty$ map $g_\varepsilon : [0,\infty) \to [0,1]$ such that

- $g_\varepsilon(0) = 1$,
- $g_\varepsilon(x) = 0$ iff $x \in [\varepsilon,\infty)$,
- $g_\varepsilon$ is decreasing on $[0,\varepsilon]$ and the derivatives at 0 and $\varepsilon$ are equal to 0.

Let

$$\gamma_J(r,\phi) = g_\varepsilon\big(D_\Phi(\phi)\big)\cdot g_\varepsilon\big(D_G(r)\big). \tag{7.1}$$

**Remark 7.1.** Notice that outside 0 the following equalities hold:

$$\gamma_{J\,|\partial V_K(\varepsilon)} \equiv 0, \tag{7.2}$$
$$\gamma_{J\,|\operatorname{cone} S_K} \equiv 1. \tag{7.3}$$

### 7.1. Definition of $P$

First we define the flow $\overline{P}_{a_{d_J}}$ by the formula

$$\overline{P}_{a_{d_J}} = \gamma_J P_{a_{d_J}} + (1-\gamma_J)z, \tag{7.4}$$

where $z$ is the following flow for which 0 is a sink:

$$\begin{cases} \dot r_1 = -r_1^{k+1}, \\ \dot r_2 = -r_2^{k+1}, \\ \dots \\ \dot r_n = -r_n^{k+1}. \end{cases} \tag{7.5}$$

Now $P$ is defined by:

$$P = \sum_J \overline{P}_{a_{d_J}}. \tag{7.6}$$

Thus, on each $V_K(\varepsilon)$, $P$ is a combination of $P_{a_{d_J}}$ and a sink, while $P$ is just a sink outside each $V_K(\varepsilon)$.

### 7.2. Definition of pushing flows $S^\phi$ and $S^r$

Again, a subset $K \subset \{1, \dots, s\}$ is fixed. We will define two flows on $V_K(\varepsilon)$ which:

- disappear on cone $S_K$ and on the boundary of $V_K(\varepsilon)$,
- move the cones in $V_K(\varepsilon)$ towards cone $S_K$.

These maps will be used to remove periodic points of some retractions during the calculation of indices (cf. Section 8.2).

The flow $S^\phi$ will be given by

$$\begin{gathered}\dot{\phi}_i = \frac{1}{d_i} \cdot r^{k+1} \gamma_J(r, \phi) \cdot (\Phi - d_i \phi_i) \quad \text{if } i \in K, \\ \dot{\phi}_i = 0 \quad \text{if } i \notin K.\end{gathered} \tag{7.7}$$

In the similar way we define the flow $S^r$

$$\begin{gathered}\dot{r}_i = (G - r_i) r^k \cdot \gamma_J(r, \phi) \quad \text{if } i \in K, \\ \dot{r}_i = -r_i r^k \cdot \gamma_J(r, \phi) \quad \text{if } i \notin K.\end{gathered} \tag{7.8}$$

The next lemma provides the main property of the flows $S^\phi, S^r$: their composition pushes each cone $S(r, \phi) \subset V_K(\varepsilon)$ towards cone $S_K$.

**Lemma 7.2.** *Let us consider the flow $S^\phi \circ S^r$ and a point $(r, \phi) \in \operatorname{Int} V_K(\varepsilon) \setminus \operatorname{cone} S_K$. Then either $D_G$ or $D_\Phi$ decreases when $t$ grows.*

**Proof.** First notice that $G$ is constant for the flow $S^r$:

$$\begin{aligned}\frac{d}{dt} G = \frac{d}{dt} \sum_{i \in K} \frac{r_i}{\#K} &= \sum_{i \in K} \frac{1}{\#K} \frac{dr_i}{dt} \\ = r^k \cdot \gamma_J \sum_{i \in K} \frac{1}{\#K} (G - r_i) &= r^k \cdot \gamma_J \left( G - \sum_{i \in K} \frac{r_i}{\#K} \right) = 0.\end{aligned} \tag{7.9}$$

Assume that $r_i \neq G$ for $i \in K$ or $r_i \neq 0$ for $i \notin K$. (Recall that $r_i > 0$ for all $i \in K$.) Basing on the observation that $G$ in the formula (7.8) is a constant, it is easy to notice that $D_G$ decreases as $t$ grows, because each $r_i$ given in (7.8) tends to $G$.

If $r_i = G$ for $i \in K$ and $r_i = 0$ for $i \notin K$, we get that $D_G(r, \phi) = 0$, but for these points we may repeat the same reasoning for $D_\Phi$ and obtain that $D_\Phi$ decreases if $d_i \phi_i \neq \Phi$. This ends the proof. □

## 8. Proof of Lemma 4.1

Now we are in a position to give the proof of Lemma 4.1.

We define an extension $F$ of a self-map of $\bigcup_K \operatorname{cone} S_K$ (given in Section 5) to a self-map of $\mathbb{R}^{2s}$ by the formula:

$$F := S_1^{\phi} \circ S_1^{r} \circ R_1 \circ P_1, \tag{8.1}$$

where $S_1^{\phi}$, $S_1^{r}$, $R_1$ and $P_1$ denote time-one maps of the respective flows.

### *8.1. Smoothness of F*

First, we prove some lemmas which are needed to show that $F$ is smooth.

**Definition 8.1.** We call $\Gamma : \mathbb{R}^d \to \mathbb{R}$ a homogeneous function of degree $p$ if for every real $\lambda > 0$ there is: $\Gamma(\lambda x_1, \ldots, \lambda x_d) = \lambda^p \Gamma(x_1, \ldots, x_d)$.

**Lemma 8.2.** *Let $G : \mathbb{R}^{2s} \to \mathbb{R}$ have the following form expressed in the polar coordinates* (4.5): *$G(x) = h_1(r)h_2(\phi)$, where $h_1 : \mathbb{R}^s_+ \to \mathbb{R}$ is a homogeneous function of degree $p$. Then $G$ is also a homogeneous function of degree $p$.*

**Proof.**

$$G(\lambda x) = h_1(\lambda r)h_2(\phi) = \lambda^p h_1(r)h_2(\phi) = \lambda^p G(x). \qquad \square$$

**Lemma 8.3.** *Let us assume that a flow $W$ on $\mathbb{R}^{2s}$ is given in the polar coordinates by the formulas*

$$\begin{aligned} \dot{r}_i &= r_i L_i(r_1, \ldots, r_s) \cdot H_i(\phi_1, \ldots, \phi_s), \\ \dot{\phi}_i &= L_i(r_1, \ldots, r_s) \cdot T_i(\phi_1, \ldots, \phi_s), \end{aligned} \tag{8.2}$$

*where $i = 1, \ldots, s$; $H_i$, $T_i$ are $C^1$, real-valued and $2\pi$-periodic; $L_i : \mathbb{R}^s_+ \to \mathbb{R}$ is a homogeneous $C^1$ function of degree $k \geqslant 1$ for each $i$.*

*Then $W_1$, the time-one map of the flow $W$, is a $C^1$ map.*

**Proof.** After differentiating the formulas (4.5) we get:

$$\begin{aligned} \dot{x}_i &= \dot{r}_i \cos\phi_i - \dot{\phi}_i r_i \sin\phi_i, \\ \dot{y}_i &= \dot{r}_i \sin\phi_i + \dot{\phi}_i r_i \cos\phi_i. \end{aligned} \tag{8.3}$$

Using the formulas (8.3) we may rewrite the equations for the flow (8.2) in such a form that the left-hand sides are expressed in Cartesian coordinates and right-hand sides in the polar coordinates:

$$\begin{cases} \dot{x}_i = r_i L_i(r_1, \ldots, r_s) \cdot H_i(\phi_1, \ldots, \phi_s) \cos\phi_i - r_i L_i(r_1, \ldots, r_s) \cdot T_i(\phi_i, \ldots, \phi_s) \sin\phi_i, \\ \dot{y}_i = r_i L_i(r_1, \ldots, r_s) \cdot H_i(\phi_1, \ldots, \phi_s) \sin\phi_i + r_i L_i(r_1, \ldots, r_s) \cdot T_i(\phi_i, \ldots, \phi_s) \cos\phi_i. \end{cases} \tag{8.4}$$

Let us denote the right-hand sides of the formulas (8.4) by $Q_i$, $i = 1, \ldots, 2s$. In order to show that the discretization of the flow (8.4) is $C^1$ it is enough to state that all $Q_i$ (as the functions on $\mathbb{R}^{2s}$ in the Cartesian coordinates) are $C^1$. It is immediate that each such map is $C^1$ in $\mathbb{R}^{2s} \setminus \{0\}$. We show that they are $C^1$ near the origin. Let $z = (z_1, \ldots, z_{2s}) = (x_1, y_1, \ldots, x_s, y_s)$.

We get that each $Q_i$ has the form $h_1(r)h_2(\phi)$ where $h_1$ is a homogeneous function of degree $k+1$, thus by Lemma 8.2 $Q_i$ is also a homogeneous function of degree $k+1$.

Next, by the Euler theorem, partial derivatives of $Q_i$ are homogeneous function of degree $k$. We obtain finally:

$$\lim_{z\to 0}\frac{\partial Q_i}{\partial z_j}(z)=\lim_{z\to 0}\frac{\partial Q_i}{\partial z_j}\left(\|z\|\frac{z}{\|z\|}\right)$$
$$=\lim_{z\to 0}\|z\|^k\frac{\partial Q_i}{\partial z_j}\left(\frac{z}{\|z\|}\right)=0. \tag{8.5}$$

The last equality results from the fact that $\frac{\partial Q_i}{\partial z_j}$ are bounded on the unit sphere. Consequently, each partial derivative of $Q_i$ at zero exists and is continuous. $\square$

**Remark 8.4.** Notice that all partial derivatives of $Q_i$ converge to zero, as $z\to 0$, thus the map $Q=(Q_1,\dots,Q_{2s})$ has the derivative at 0 equal to 0. This implies that the derivative of the time-one map of the flow (8.4) at 0 is equal to identity.

**Proof of the smoothness of *F*.** The map $F$ is the composition of four maps given in (8.1). The smoothness of $R_1$ is obvious. The flow $P$ has the form (8.2), thus by Lemma 8.3 $P_1$ is smooth. The same argument as for $P$ (with only small modifications) applies to $S^\phi$ and $S^r$. This completes the proof. $\square$

*8.2. Calculation of indices*

Now we show that the fixed point indices of iterations of $F$ are given by the formula (4.1). By Corollary 5.2 it suffices to prove the following lemma.

**Lemma 8.5.**

$$\operatorname{ind}\left(F^n_{|V_K(\varepsilon)},0\right)=\operatorname{ind}\left(F^n_{|\operatorname{cone}S_K},0\right). \tag{8.6}$$

**Proof.** Let $r$ be a retraction $r:V_K(\varepsilon)\to\operatorname{cone}S_K$. We show that there exists a homotopy $H:V_K(\varepsilon)\times I\to V_K(\varepsilon)$, such that $H(\cdot,0)=\mathrm{id}$ and $H(\cdot,1)=r$. Then, for each fixed $n$, the homotopy $h^{(n)}=F^n\circ H$ joins $F^n$ with $F^n\circ r$. As a result, if for each $t$ the composition $F^n\circ H_t:V_K(\varepsilon)\to V_K(\varepsilon)$ has no fixed points (except for 0) we obtain the formula (8.6) by homotopy invariance of fixed point index.

We define $H$ by:

$$H(r,\phi,t)=\left(H'(r,t),H''(\phi,t)\right), \tag{8.7}$$

where

$$H'(r,t)=\begin{cases}(1-t)r_i+tG & \text{for } i\in K,\\ (1-t)r_i & \text{for } i\notin K,\end{cases} \tag{8.8}$$

$$H''(\phi,t)=\begin{cases}(1-t)\phi_i+t\frac{\Phi}{d_i} & \text{for } i\in K,\\ \phi_i & \text{for } i\notin K.\end{cases} \tag{8.9}$$

To complete the proof we must show that $F^n\circ H_t$ has the unique periodic point 0. Let us recall that the map $F$ changes the cones in $V_K(\varepsilon)$ (by Lemma 7.2 $F$ diminishes $D_G$ or $D_\Phi$). On the other hand, each cone in $V_K(\varepsilon)\setminus\operatorname{cone}S_K$ is pushed in each moment of the homotopy $H_t$ (for $t\geqslant 0$) to a cone whose (at least one) deviation $D_G$ or $D_\Phi$ is smaller. The last follows from the fact that the homotopies $H'$ and $H''$ are convex combinations between the coordinates and their mean values, and thus the deviations are decreasing. This completes the proof. $\square$

## 9. Realizations of other cases

In this section we describe the constructions of realizations of all other cases. We follow the steps of Lemma 3.4, i.e. we start with the case when the dimension $m$ is equal to $2s$, then we get the case $(F^e)$, realized in the previous section. Next, we gradually increase the dimension, which enables us to construct the realizations with less and less number of restrictions. However, we cannot obtain the realizations just as the products of $(F^e)$ on $\mathbb{R}^{2s}$ and some map on complementing space (Remark 9.1). We need in particular two additional cones (cf. $\tilde{S}_{\pm K}$ below) interchanged by symmetry, in order to obtain not only $a_d \operatorname{reg}_d$ appearing in $(F^e)$ but also $a_{2d} \operatorname{reg}_{2d}$ with arbitrary $a_{2d}$.

The smoothness in all cases follows again from the homogeneity of the considered vector fields.

### 9.1. *Realizations for $m = 2s + 1$*

We represent $\mathbb{R}^{2s+1}$ as $\mathbb{R}^{2s} \times \mathbb{R}$, and will denote $(2s+1)$-axis by $M_1$.

We take $r^K \in \mathbb{R}^{2s}$ defined by the formula (4.7) and consider

$$\tilde{S}_K = \left\{ \left( r_1^K, \frac{t}{d_1}, \ldots, r_s^K, \frac{t}{d_s}, 0 \right) : t \in \mathbb{R} \right\} \tag{9.1}$$

and

$$\tilde{S}_{\pm K} = \left\{ \left( r_1^K, \frac{t}{d_1}, \ldots, r_s^K, \frac{t}{d_s}, \pm 1 \right) : t \in \mathbb{R} \right\}. \tag{9.2}$$

Next, we will consider the cones $\operatorname{cone} \tilde{S}_K \subset \mathbb{R}^{2s} \times \{0\}$ and $\operatorname{cone} \tilde{S}_{\pm K} \subset \mathbb{R}^{2s+1}$.

We define the rotation flow in $\mathbb{R}^{2s+1}$ by

$$\begin{aligned} \tilde{R}(r_1, \phi_1, \ldots, r_s, \phi_s, x_{2s+1}; t) &= \left( r_1, \phi_1 + \frac{t}{d_1}, \ldots, r_s, \phi_s + \frac{t}{d_s}, x_{2s+1} \right) \\ &= \big( R(r_1, \phi_1, \ldots, r_s, \phi_s; t), x_{2s+1} \big). \end{aligned} \tag{9.3}$$

We use the symmetry map $s : \mathbb{R}^{2s+1} \to \mathbb{R}^{2s+1}$:

$$s(x_1, x_2, \ldots, x_{2s}, x_{2s+1}) = (x_1, x_2, \ldots, x_{2s}, -x_{2s+1}).$$

Notice that

$$s(\operatorname{cone} \tilde{S}_{\pm K}) = \operatorname{cone} \tilde{S}_{\mp K}. \tag{9.4}$$

We take on the additional axis $M_1$ one of the three flows, a sink or a source or straight (straight flow is a flow with the fixed point 0 removable by any small perturbation) and denote it as $\Gamma_i$, where $i \in \{\mathit{sink}, \mathit{straight}, \mathit{source}\}$.

Each of them generates time-one map $\Gamma_{i,1}$, which realizes fixed point indices on $M_1$ of the form

$$\begin{aligned} \operatorname{ind}\big(\Gamma^n_{\mathit{sink},1}, 0\big) &= \operatorname{reg}_1(n), \\ \operatorname{ind}\big(\Gamma^n_{\mathit{straight},1}, 0\big) &= 0, \\ \operatorname{ind}\big(\Gamma^n_{\mathit{source},1}, 0\big) &= -\operatorname{reg}_1(n). \end{aligned} \tag{9.5}$$

We extend each of the flows $\Gamma_i$ onto the small cone-neighborhood of $M_1$, $V_{M_1}$, and denote the extended flow as $\tilde{\Gamma}_i$.

Let us remind that by $J$ we denote any subset of $\{d_1, \dots, d_s\}$ (determined by a set of indices $K \subset \{1, \dots, s\}$), $d_J = \mathrm{LCM}(J)$, $\mathrm{LCM}(\emptyset) = 1$.

**Remark 9.1.** The realizations which can be obtained as a product of $F$ (realizing $(F^e)$) and a one-dimensional map are of very special forms. Namely, by the multiplicativity property of fixed point index for the self-maps $F \times \Gamma_{i,1}$ of $\mathbb{R}^{2s} \times \mathbb{R}$, where $i \in \{sink, straight, source\}$, we get

$$\mathrm{ind}\big(F^n \times \Gamma_{i,1}^n, 0 \times 0\big) = \mathrm{ind}\big(F^n, 0\big) \cdot \mathrm{ind}\big(\Gamma_{i,1}^n, 0\big).$$

On the other hand, the multiplication of two expansions is expressed by the formula

$$\mathrm{reg}_k(n) \cdot \mathrm{reg}_l(n) = \mathrm{GCD}\{k, l\}\, \mathrm{reg}_{\mathrm{LCM}\{k,l\}}(n), \tag{9.6}$$

where $\mathrm{GCD}\{k, l\}$ denotes the greatest common divisor of $k$ and $l$.

As a consequence, by the formula (9.6) we obtain:

$$F \times \Gamma_{sink,1} \text{ gives a subcase of } \big(E^o\big).$$

$$F \times \Gamma_{source,1} \text{ gives the case } \big(C^o\big).$$

Except for the sequences $\mathrm{reg}_1(n), -\mathrm{reg}_1(n), 0$, in the one-dimensional space we can realize one more: $\mathrm{reg}_1(n) - \mathrm{reg}_2(n)$ by $s \circ \Gamma_{source,1}$. Thus, by the product $F \times (s \circ \Gamma_{source,1})$ we can obtain also the case $(F^o)$ but with the additional restrictions $a_d = -a_{2d}$ for $d$ odd.

Summing it up, by the products we cannot realize all possible sequences of indices.

*Realization of the sequence defined by* (3.3), i.e.

$$\sum_J a_{d_J}\, \mathrm{reg}_{d_J}, \quad a_1 \in \{-1, 0, +1\}.$$

We consider the following counterpart of the formula (8.1):

$$\tilde{F} := \tilde{S}_1^{\phi} \circ \tilde{S}_1^{r} \circ \tilde{R}_1 \circ \tilde{P}_1, \tag{9.7}$$

where $\tilde{R}_1$ is the time-one map of the flow defined in (9.3). The flow $\tilde{P}$ is constructed from $P$ (see Section 7.1) by the following modifications:

- we add a $(2s+1)$ component of the flow $P_{a_{d_J}}$, cf. (5.1):

  $$\dot{x}_{2s+1} = x_{2s+1} |r|^k \cos(a_{d_J} d_1 \phi_1),$$

- on the cone-neighborhoods of cone $\tilde{S}_{\pm K}$ and cone $\tilde{S}_K$ the flow $\tilde{P}$ is the combination of $P_{a_{d_J}}$ and a sink on $\mathbb{R}^{2s+1}$, in the same way as in the formula (7.4),
- on $V_{M_1}$ $\tilde{P}$ is the combination of $\tilde{\Gamma}_i$ and a sink,
- outside the above cone-neighborhoods we define $\tilde{P}$ as a sink. (9.8)

The maps $\tilde{S}_1^{\phi}$ and $\tilde{S}_1^{r}$ are the counterpart of $S_1^{\phi}$ and $S_1^{r}$ and move the points towards the cones (or to $M_1$ in the case of $V_{M_1}$) so that the fixed point indices will be equal to those calculated on the cones (or on $M_1$, respectively). As a result, we obtain by Lemma 5.3 and the formula (9.5):

$$\begin{aligned} \operatorname{ind}(\tilde{F}^n, 0) &= \Bigl(\operatorname{reg}_1(n) + \sum_{J\neq\emptyset} a_{d_J} \operatorname{reg}_{d_J}(n)\Bigr) + \bigl(\operatorname{ind}(\Gamma_{i,1}^n, 0)\bigr) - \operatorname{reg}_1(n) \\ &= a_1 \operatorname{reg}_1(n) + \sum_{J\neq\emptyset} a_{d_J} \operatorname{reg}_{d_J}(n), \end{aligned} \tag{9.9}$$

where:

$$a_1 = \begin{cases} 1 & \text{for } i = \mathit{sink}, \\ 0 & \text{for } i = \mathit{straight}, \\ -1 & \text{for } i = \mathit{source}. \end{cases}$$

*Realization of the sequence defined by* (3.4), i.e.

$$\sum_J a_{d_J} \operatorname{reg}_{d_J} + \sum_{J:\ d_J \text{ is odd}} a_{2d_J} \operatorname{reg}_{2d_J}, \quad a_1 = 1,\ a_2 \in \{0, -1\}.$$

Now we use additionally the cones $\tilde{S}_{\pm K}$, and in their neighborhoods we define $\tilde{P}$ in the same way as described in (9.8). We define the realization as

$$\tilde{F} := s \circ \tilde{S}_1^{\phi} \circ \tilde{S}_1^{r} \circ \tilde{R}_1 \circ \tilde{P}_1. \tag{9.10}$$

Observe that $\tilde{F}_{|M_1} = s \circ \Gamma_{i,1}$, and

$$\begin{aligned} \operatorname{ind}(s^n \circ \Gamma_{sink,1}^n, 0) &= \operatorname{reg}_1(n), \\ \operatorname{ind}(s^n \circ \Gamma_{source,1}^n, 0) &= \operatorname{reg}_1(n) - \operatorname{reg}_2(n). \end{aligned} \tag{9.11}$$

Finally, $\tilde{F}$ realizes:

- $a_{d_J} \operatorname{reg}_{d_J}(n)$ on $\tilde{S}_K$,
- $a_{2d_J} \operatorname{reg}_{2d_J}$ on $\tilde{S}_{\pm K}$,
- $a_1 \operatorname{reg}_1(n) + a_2 \operatorname{reg}_2(n)$ on $M_1$, where the sum has one of two forms of the formula (9.11). (9.12)

*9.2. Realizations for $m = 2s + 2$*

We represent $\mathbb{R}^{2s+2}$ as $\mathbb{R}^{2s} \times \mathbb{R}^2$, and will denote $(x_{2s+1}, x_{2s+2})$-plane by $M_2$.
We repeat the similar construction as in the previous case, namely we define analogously

$$\tilde{S}_K = \left\{ \left( r_1^K, \frac{t}{d_1}, \ldots, r_s^K, \frac{t}{d_s}, 0, 0 \right):\ t \in \mathbb{R} \right\} \tag{9.13}$$

and

$$\tilde{S}_{\pm K} = \left\{ \left( r_1^K, \frac{t}{d_1}, \ldots, r_s^K, \frac{t}{d_s}, 1, \pm 1 \right) : t \in \mathbb{R} \right\}. \tag{9.14}$$

Next, we build the cones $\operatorname{cone} \tilde{S}_K \subset \mathbb{R}^{2s} \times \{(0,0)\}$ and $\operatorname{cone} \tilde{S}_{\pm K} \subset \mathbb{R}^{2s+2}$.
We define the rotation flow in $\mathbb{R}^{2s+2}$ by

$$\begin{aligned} &\tilde{R}(r_1, \phi_1, \ldots, r_s, \phi_s, x_{2s+1}, x_{2s+2}; t) \\ &\quad = \big(R(r_1, \phi_1, \ldots, r_s, \phi_s; t), x_{2s+1}, x_{2s+2}\big). \end{aligned} \tag{9.15}$$

We use the symmetry map

$$s(x_1, x_2, \ldots, x_{2s}, x_{2s+1}) = (x_1, x_2, \ldots, x_{2s}, x_{2s+1}, -x_{2s+1}).$$

*Realization of the sequence defined by* (3.5), i.e.

$$\sum_J a_{d_J} \operatorname{reg}_{d_J} + \sum_{J:\, d_J \text{ is odd}} a_{2d_J} \operatorname{reg}_{2d_J}, \quad a_1 \in \{-1, 0, +1\}.$$

In this case we define the flow $P$ by the formulas (5.1) with the additional pair of the polar coordinates. This formula is valid also in the neighborhoods of each cone $\tilde{S}_{\pm K}$.

On the space $M_2$ we take one of the planar flows $h_p$ described in Section 4.1, such that it is symmetric along the axis $x_{2s+2}$. Considering indices of iterations of its discretization, which have the form $a_1 \operatorname{reg}_1 + a_2 \operatorname{reg}_2$, it is easy to observe that the coefficient $a_1$ depends on whether the flow is sink, straight, or source on the $(x_{2s+1})$-axis, and is equal to $1, 0, -1$ respectively. On the other hand, we may obtain arbitrary $a_2$ by taking appropriate number of symmetric sectors on $M_2$.

We define $\tilde{F} = s \circ \tilde{S}_1^{\phi} \circ \tilde{S}_1^{r} \circ \tilde{R}_1 \circ \tilde{P}_1$ which realizes:

- $a_{d_J} \operatorname{reg}_{d_J}(n)$ on $\operatorname{cone} \tilde{S}_K$,
- $a_{2d_J} \operatorname{reg}_{2d_J}$ on $\operatorname{cone} \tilde{S}_{\pm K}$,
- $a_1 \operatorname{reg}_1(n) + a_2 \operatorname{reg}_2(n)$ on $M_2$ with $a_1 \in \{-1, 0, 1\}$ and arbitrary $a_2$. (9.16)

*Realization of the sequence defined by* (3.6), i.e.

$$\sum_J a_{d_J} \operatorname{reg}_{d_J}, \quad a_1 \text{ arbitrary}.$$

Here we define $P$ as a sink in the neighborhood of each cone $\tilde{S}_{\pm K}$. On the space $M_2$ we take one of the planar flows $h_p$ described in Section 4.1. The discretization of such flow enables us to realize $a_1 \operatorname{reg}_1$ with arbitrary coefficient $a_1$.

Next, we define $\tilde{F} = \tilde{S}_1^{\phi} \circ \tilde{S}_1^{r} \circ \tilde{R}_1 \circ \tilde{P}_1$ which realizes:

- $a_{d_J} \operatorname{reg}_{d_J}(n)$ on $\operatorname{cone} \tilde{S}_K$,
- $a_1 \operatorname{reg}_1(n)$ on $M_2$ with arbitrary $a_1$. (9.17)

*9.3. Realizations for $m \geqslant 2s+3$*

We represent $\mathbb{R}^{2s+3}$ as $\mathbb{R}^{2s} \times \mathbb{R}^3$, and denote $(x_{2s+1}, x_{2s+2}, x_{2s+3})$-space by $M_3$. In this case we may find a flow on $M_3$, symmetric along the $(x_{2s+1}, x_{2s+2})$-plane, such that its discretization realizes on $M_3$ every sequence of the form $a_1 \operatorname{reg}_1(n) + a_2 \operatorname{reg}_2(n)$ with no restrictions for the coefficients $a_1$ and $a_2$ (see [9] for details). Repeating then exactly the same scheme as in (9.17), we realize the sequence given in the formula (3.5), i.e.

$$\sum_J a_{d_J} \operatorname{reg}_{d_J} + \sum_{J:\, d_J \text{ is odd}} a_{2d_J} \operatorname{reg}_{2d_J}, \quad a_1, a_2 \text{ arbitrary.}$$

## 10. Final remarks

*10.1. Problem of Babenko and Bogatyi*

In [1] Babenko and Bogatyi asked whether *any bounded sequence satisfying Dold relations is a sequence of indices of an isolated fixed point of a smooth map of an Euclidean space* (Problem 4, p. 22). Theorem 3.2 enables us to answer this question positively. Namely, any bounded sequence satisfying Dold relations can be written down as a finite combination of basic sequences $\operatorname{reg}_k$ (cf. [13]). As a consequence, if we take $s$ big enough in Lemma 3.4 we are able to realize any sequence $\{b(n)\}_n$ of the form $b(n) = \sum_{k\in A} a_k \operatorname{reg}_k(n)$, where $A$ is finite. Furthermore, for a given sequence $\{b(n)\}_n$ we may establish the lowest dimension $m$ such that $\{b(n)\}_n$ can be realized by a smooth map in $\mathbb{R}^m$. In any case, if $\#A = p$ we are able to realize $\{b(n)\}_n$ for $m = 2p+3$ (see (IV), formula (3.7) of the proof of Lemma 3.4).

*10.2. Realizations in the narrower classes of maps*

At the end of the article we discuss the problem of finding the realizations in some narrower classes of maps. We study the case of $C^1$ maps, but it is an interesting question whether the sequences enlisted in Theorem 3.1 may be obtained as indices of some more regular maps, which satisfy additional restrictions.

Let us recall that the map $F$ realizing expression in the formula (4.1) of Lemma 4.1 is the composition of a non-singular linear map $R$ and some flows. On the other hand, the flows may be taken arbitrarily small. The same is true for all realizations of the remaining cases. As a result, we get the following

**Corollary 10.1.** *The map realizing any of admissible expressions given in Theorem* 3.1 *can be chosen a diffeomorphism being an arbitrarily small deformation of a non-singular linear map.*

A class which is very important in dynamical systems consists of maps $f$ for which $\{0\}$ is an isolated invariant set, i.e. there is $U$, an open neighborhood of 0, such that

$$\bigcap_{k\in\mathbb{Z}} f^k(U) = \{0\}. \tag{10.1}$$

An open problem is whether there are always $C^1$ realizations for which $\{0\}$ is an isolated invariant set. Notice that in some of our realizations elliptic sectors are used, so they do not satisfy the assumption (10.1) in such cases. Ruiz del Portal and Salazar considered in [22] that problem in dimension 3 by a use of Conley index methods. These authors gave the list (Proposition 1 in [22]) of examples of $\mathbb{R}^3$-diffeomorphisms satisfying (10.1) which do not cover all sequences admissible by Theorem 3.1 in dimension 3. They also conjectured that the list is optimal in the following sense. Let $f$ be an $\mathbb{R}^3$-diffeomorphism such that $\operatorname{Fix}(f) = \operatorname{Per}(f) = \{0\}$, if the sequence of indices of $f$ at an isolated fixed point 0 does not follow any of the patterns from their list then $\{0\}$ is not an isolated invariant set of $f$.

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