

## MINIMIZATION OF THE NUMBER OF PERIODIC POINTS FOR SMOOTH SELF-MAPS OF CLOSED SIMPLY-CONNECTED 4-MANIFOLDS

GRZEGORZ GRAFF

Faculty of Applied Physics and Mathematics  
Gdansk University of Technology  
Narutowicza 11/12, 80-233 Gdansk, Poland

JERZY JEZIERSKI

Institute of Applications of Mathematics  
Warsaw University of Life Sciences (SGGW)  
Nowoursynowska 159, 00-757 Warsaw, Poland

**ABSTRACT.** Let  $M$  be a smooth closed simply-connected  $m$ -dimensional manifold,  $f$  be a smooth self-map of  $M$  and  $r$  be a given natural number. The invariant  $D_r^m[f]$  defined by the authors in [Forum Math. 21 (2009)] is equal to the minimum of  $\#\text{Fix}(g^r)$  over all maps  $g$  smoothly homotopic to  $f$ . In this paper we calculate the invariant  $D_r^4[f]$  for the class of smooth self-maps of 4-manifolds with fast growth of Lefschetz numbers and for  $r$  being a product of different primes.

**1. Introduction.** One of the fundamental problems in periodic point theory is to find minimal number of periodic points in the homotopy class of a given map. Let  $f$  be a self-map of a compact manifold  $M$ . B. Jiang introduced in 1983 the invariant  $NF_r(f)$  which estimates from above  $\#\text{Fix}(g^r)$  for all  $g$  homotopic to  $f$  [14]. J. Jezierski proved in 2006 that the invariant is the best estimation if the dimension of  $M$  is at least 3 [12]. This means that  $NF_r(f)$  is equal to the minimal number of elements in  $\text{Fix}(g^r)$  over all  $g$  homotopic to  $f$ . In the last years the invariant was computed in many special cases, see for example: [10], [13], [16], [18].

In the recent papers [4], [6] the authors developed the theory for the smooth (i.e.  $C^1$ ) category, searching for the minimum in smooth homotopy class. As a result, two counterparts of  $NF_r(f)$  were found:  $D_r^m[f]$  for simply-connected manifolds [4] and its generalization  $NJD_r^m[f]$  for non simply-connected ones [6]. The crucial demanding for effective computation of the invariants is the knowledge of all sequences of local fixed point indices of iterations at a periodic  $p$ -orbit for smooth maps in the given dimension  $m$ , called  $DD^m(p)$  sequences. This information was provided in dimension 3 in the paper [9], which made it possible to compute the value of  $D_r^3[f]$  for  $S^2 \times I$  [4],  $S^3$  [5], a two-holed 3-dimensional closed ball [3] and  $NJD_r^3[f]$  for  $\mathbb{R}P^3$  [7]. Recently, in [8] we provided the list of all possible sequences of local indices of iterations in arbitrary dimension, which allows one to calculate the invariants for self-maps of higher dimensional manifolds. In this paper we partially realize this

---

2000 *Mathematics Subject Classification.* Primary: 37C25, 55M20; Secondary: 37C05.

*Key words and phrases.* Indices of iterations, smooth maps, Nielsen number.

Research supported by Polish National Research Grant No. N N201 373236.

programme for simply-connected manifolds and  $m = 4$ . We calculate  $D_r^4[f]$  under the assumption that the so-called *periodic expansion* of  $\{L(f^n)\}_{n=1}^\infty$ , the sequence of the Lefschetz numbers of iterations, has only non-zero coefficients. This property holds for example for maps with fast grow of the sequence of Lefschetz numbers, such as self-maps of  $S^4$  with degree  $d$  satisfying  $|d| > 1$ .

The paper is organized in the following way. First, in Section 2 we give the definition of  $D_r^m[f]$  which is expressed in terms of  $DD^m(p)$  sequences. Next, in Section 3 we provide the list of all  $DD^m(1)$  sequences and prove that in order to calculate  $D_r^4[f]$  it is enough to use only  $DD^4(1)$  sequences. Finally, in Section 4 we calculate  $D_r^4[f]$  for  $r$  being a product of different primes (Theorem 4.8).

**2. The invariant  $D_r^m[f]$ .** The notion of *Differential Dold* sequences ( $DD$  sequences in short) introduced in [4] is used in the definition of the invariant  $D_r^m[f]$ . A  $DD^m(p)$  sequence is a sequence of integers that can be locally realized as a sequence of indices on an isolated  $p$ -orbit for some smooth map.

**Definition 2.1.** A sequence of integers  $\{c_n\}_{n=1}^\infty$  is called a  $DD^m(p)$  sequence if there is a  $C^1$  map  $\phi : U \rightarrow \mathbb{R}^m$  ( $U \subset \mathbb{R}^m$ ) and its isolated  $p$ -orbit  $P$  such that  $c_n = \text{ind}(\phi^n, P)$ . If this equality holds for  $n|r$ , where  $r$  is fixed, then the finite sequence  $\{c_n\}_{n|r}$  will be called a  $DD^m(p|r)$  sequence.

Let  $r$  be fixed. The minimal decomposition of the sequence of Lefschetz numbers of iterations into  $DD^m(p|r)$  sequences gives the value of  $D_r^m[f]$ .

**Definition 2.2.** Let  $\{L(f^n)\}_{n|r}$  be a finite sequence of Lefschetz numbers. We decompose  $\{L(f^n)\}_{n|r}$  into the sum:

$$L(f^n) = c_1(n) + \dots + c_s(n), \quad (1)$$

where  $c_i$  is a  $DD^m(l_i|r)$  sequence for  $i = 1, \dots, s$ . Each such decomposition determines the number  $l = l_1 + \dots + l_s$ . We define the number  $D_r^m[f]$  as the smallest  $l$  which can be obtained in this way.

The invariant  $D_r^m[f]$  is equal to the minimal number of  $r$ -periodic points in smooth homotopy class of  $f$ .

**Theorem 2.3.** ([4]) *Let  $M$  be a smooth closed connected and simply-connected manifold of dimension  $m \geq 3$  and  $r \in \mathbb{N}$  a fixed number. Then,*

$$D_r^m[f] = \min\{\#\text{Fix}(g^r) : g \text{ is smoothly homotopic to } f\}.$$

*Periodic expansion* is a convenient method of storing the data connected with the sequence of indices of iterations. Each such sequence can be expanded as a combination of some basic periodic sequences  $\{\text{reg}_k\}_n$  taken with integral coefficients.

**Definition 2.4.** For a given  $k$  we define the basic sequence:

$$\text{reg}_k(n) = \begin{cases} k & \text{if } k|n, \\ 0 & \text{if } k \nmid n. \end{cases}$$

A sequence of indices of iterations (as well as a sequence of Lefschetz numbers of iterations) may be written down in the form of *periodic expansion* (cf. [15]), namely:

$$\text{ind}(f^n, x_0) = \sum_{k=1}^{\infty} a_k \text{reg}_k(n), \quad (2)$$

where  $a_n = \frac{1}{n} \sum_{k|n} \mu(k) \text{ind}(f^{(n/k)}, x_0)$ ,  $a_n$  are integers,  $\mu$  is the classical Möbius function, i.e.  $\mu : \mathbb{N} \rightarrow \mathbb{Z}$  is defined by the following three properties:  $\mu(1) = 1$ ,  $\mu(k) = (-1)^s$  if  $k$  is a product of  $s$  different primes,  $\mu(k) = 0$  otherwise.

The fact that the coefficients  $a_n$  are integers follows from the result of Dold [2].

The invariant  $D_r^m[f]$  is defined in terms of  $DD^m(p)$  sequences. On the other hand, it is enough to know only the forms of  $DD^m(1)$  sequences, because every  $DD^m(p)$  sequence can be obtained from some  $DD^m(1)$  one.

**Definition 2.5.** We will say that the  $DD^m(p)$  sequence  $\{\tilde{c}_n\}_n$  comes from the given  $DD^m(1)$  sequence  $\{c_n\}_n$  with the periodic expansion  $c_n = \sum_{d=1}^{\infty} a_d \text{reg}_d(n)$  if the periodic expansion of  $\{\tilde{c}_n\}_n$  has the form:

$$\tilde{c}_n = \sum_{d=1}^{\infty} a_d \text{reg}_{pd}(n).$$

**Theorem 2.6** ([4]). *Every  $DD^m(p)$  sequence comes from some  $DD^m(1)$  sequence.*

**3. Local indices of iterations in dimension 4.** In this section we give the complete list of all sequences of local indices of iterations of a smooth map in dimension 4 i.e. the list of all  $DD^4(1)$  sequences. Let us mention here that the forms of indices of iterations for continuous maps are known since 1991 [1], and recently indices of iterations have been found also for other important classes of maps, such as holomorphic maps [22] and planar homeomorphisms [17], [21].

**Definition 3.1.** Let  $H$  be a finite subset of natural numbers, we introduce the following notation.

By  $\text{LCM}(H)$  we mean the least common multiple of all elements in  $H$  with the convention that  $\text{LCM}(\emptyset) = 1$ . We define the set  $\bar{H}$  by:  $\bar{H} = \{\text{LCM}(Q) : Q \subset H\}$ .

For natural  $s$  we denote by  $L(s)$  any set of natural numbers of the form  $\bar{L}$  with  $\#L = s$  and  $1, 2 \notin L$ .

By  $L_2(s)$  we denote any set of natural numbers of the form  $\bar{L}$  with  $\#L = s + 1$  and  $1 \notin L, 2 \in L$ .

**Theorem 3.2 (Main Theorem I in [8]).** *Let  $f$  be a  $C^1$  self-map of  $\mathbb{R}^m$ , where  $m$  is even. Then the sequence of local indices of iterations  $\{\text{ind}(f^n, 0)\}_{n=1}^{\infty}$  has one of the following forms:*

$$(A^e) \text{ ind}(f^n, 0) = \sum_{k \in L_2(\frac{m-4}{2})} a_k \text{reg}_k(n).$$

$$(B^e) \text{ ind}(f^n, 0) = \sum_{k \in L(\frac{m-2}{2})} a_k \text{reg}_k(n).$$

$$(C^e), (D^e), (E^e) \text{ ind}(f^n, 0) = \sum_{k \in L_2(\frac{m-2}{2})} a_k \text{reg}_k(n),$$

where

$$a_1 = \begin{cases} 1 & \text{in the case } (C^e), \\ -1 & \text{in the case } (D^e), \\ 0 & \text{in the case } (E^e). \end{cases}$$

$$(F^e) \text{ ind}(f^n, 0) = \sum_{k \in L(\frac{m}{2})} a_k \text{reg}_k(n),$$

where  $a_1 = 1$ .

By  $[d, l]$  we denote the least common multiple of  $d$  and  $l$ .

**Theorem 3.3.** *The list of all  $DD^4(1)$  sequences is the following:*

$$(A) \quad c_A(n) = a_1 \text{reg}_1(n) + a_2 \text{reg}_2(n);$$

$$(B) \quad c_B(n) = a_1 \text{reg}_1(n) + a_d \text{reg}_d(n);$$

$$(C - E)_{\text{odd}}$$

$$c_X(n) = \varepsilon_X \text{reg}_1(n) + a_2 \text{reg}_2(n) + a_d \text{reg}_d(n) + a_{2d} \text{reg}_{2d}(n),$$

where  $\varepsilon_X \in \{-1, 0, 1\}$ ,  $X \in \{C, D, E\}$ ,  $d$  is odd.

$$(C - E)_{\text{even}}$$

$$c_X(n) = \varepsilon_X \text{reg}_1(n) + a_2 \text{reg}_2(n) + a_d \text{reg}_d(n),$$

where  $\varepsilon_X \in \{-1, 0, 1\}$ ,  $X \in \{C, D, E\}$ ,  $d$  is even.

$$(F) \quad c_F(n) = \text{reg}_1(n) + a_d \text{reg}_d(n) + a_l \text{reg}_l(n) + a_{[d,l]} \text{reg}_{[d,l]}(n),$$

In all cases  $d, l \geq 3$  and  $a_i \in \mathbb{Z}$ .

*Proof.* We apply Theorem 3.2 for  $m = 4$ , obtaining the corresponding parts of the thesis. For example, to obtain the case (F), we use  $(F^e)$  and get:

$$L\left(\frac{m}{2}\right) = L(2) = \overline{\{d, l\}} = \text{LCM}\{Q \subset \{d, l\}\} = \{1, d, l, [d, l]\}.$$

□

**Corollary 1.** *Let us notice that any  $DD^4(1)$  sequence has one of the following forms:*

$$1. \quad a_1 \text{reg}_1(n) + a_d \text{reg}_d(n);$$

for  $a_1, a_d \in \mathbb{Z}$ .

$$2. \quad \varepsilon \text{reg}_1(n) + a_2 \text{reg}_2(n) + a_d \text{reg}_d(n) + \gamma_d a_{2d} \text{reg}_{2d}(n);$$

for  $a_2, a_d \in \mathbb{Z}$ ,  $\varepsilon = 0, \pm 1$ ,  $\gamma_d = 0$  if  $d$  is even and  $\gamma_d = 1$  if  $d$  is odd.

$$3. \quad \text{reg}_1(n) + a_d \text{reg}_d(n) + a_l \text{reg}_l(n) + a_{[d,l]} \text{reg}_{[d,l]}(n);$$

for  $a_d, a_l \in \mathbb{Z}$ ,  $d, l \geq 3$ .

The next two lemmas show that during the calculation of  $D_r^4[f]$  we may consider only  $DD^4(1)$  sequences, which makes the computation much easier.

**Lemma 3.4** (Remark 4.6 in [4]). *For  $m \geq 3$  in Definition 2.2 of  $D_r^m[f]$  we can equivalently use only  $DD^m(p|r)$  sequences such that  $p < 2^{\lceil \frac{m+1}{2} \rceil}$ .*

**Lemma 3.5.** *To calculate  $D_r^4[f]$  it is enough to consider only  $DD^4(1)$  sequences.*

*Proof.* By Lemma 3.4 it is enough to consider only such  $DD^4(p|r)$  sequences for which  $p \leq 3$ .

We show that

(1) every  $DD^4(2|r)$  sequence is a sum of at most two  $DD^4(1|r)$  sequences.

(2) every  $DD^4(3|r)$  sequence is a sum of at most three  $DD^4(1|r)$  sequences.

*Proof of (1).* Using Theorem 2.6 we find the forms of all  $DD^4(2|r)$  sequences, each of which comes from some  $DD^4(1|r)$  sequences of one of the types (A)-(F). Next, we represent each  $DD^4(2|r)$  sequence as a sum of at most two  $DD^4(1|r)$  sequences.



(A)  $a_2\text{reg}_2(n) + a_4\text{reg}_4(n)$  is in fact the  $DD^4(1|r)$  sequence of the type  $(D)_{\text{even}}$ .

(B)  $a_2\text{reg}_2(n) + a_{2d}\text{reg}_{2d}(n)$  the same argument as above is true.

(C-E) For  $d$  odd we have that  $[4, 2d] = 4d$ , then

$$\varepsilon_X \text{reg}_2(n) + a_4 \text{reg}_4(n) + a_{2d} \text{reg}_{2d}(n) + a_{4d} \text{reg}_{4d}(n) = \text{reg}_1(n) + a_4 \text{reg}_4(n) + a_{2d} \text{reg}_{2d}(n) + a_{4d} \text{reg}_{4d}(n) \tag{F}$$

$$-\text{reg}_1(n) + \varepsilon_X \text{reg}_2(n) \tag{A}$$

where on the right-hand side of the above formula we indicated that the first sum is realized by a sequence of the type (F) and the second by (A).

In the same way we deal with the case of  $d$  even (every sequence is a sum of a sequence of the type (F) and (A)).

(F) Notice that  $[2d, 2l] = 2[d, l]$ , thus we get

$$\text{reg}_2(n) + a_{2d} \text{reg}_{2d}(n) + a_{2l} \text{reg}_{2l}(n) + a_{2[d,l]} \text{reg}_{2[d,l]}(n) = -\text{reg}_1(n) + \text{reg}_2(n) + \tag{A}$$

$$+\text{reg}_1(n) + a_{2d} \text{reg}_{2d}(n) + a_{2l} \text{reg}_{2l}(n) + a_{[2d,2l]} \text{reg}_{[2d,2l]}(n). \tag{F}$$

Proof of (2). Let us now consider a  $DD^4(3|r)$  sequence.

Notice that by Theorem 2.6 and Corollary 1 it has always the form with no more than four basic sequences  $\text{reg}_i$ , i.e.

$$a_p \text{reg}_p(n) + a_q \text{reg}_q(n) + a_r \text{reg}_r(n) + a_s \text{reg}_s(n),$$

where  $p, q, r, s \geq 3$ . Then we may represent this sequence as a sum of three  $DD^4(1|r)$  sequences in the following way:

$$-\text{reg}_1(n) + a_p \text{reg}_p(n) + \tag{B}$$

$$+ a_q \text{reg}_q(n) + \tag{D}$$

$$+\text{reg}_1(n) + a_r \text{reg}_r(n) + a_s \text{reg}_s(n) \tag{F}$$

This completes the proof. □

**4. Calculation of the invariant.** We work under the following standing assumptions

**Standing Assumptions**

1.  $f : M^4 \rightarrow M^4$  is a smooth self-map of a smooth closed connected and simply-connected 4-manifold,
2.  $r = p_1 \dots p_s$  is a product of different prime numbers,
3. in the periodic expansion of Lefschetz numbers

$$L(f^k) = \sum_{i=1}^{\infty} a_i \text{reg}_i(k)$$

$a_i \neq 0$  for all  $i \neq 1$  dividing  $r$ .

**Remark 1.** The assumption (3) is satisfied for a self-map  $f : S^4 \rightarrow S^4$  with  $|\text{deg}(f)| > 1$  [20]. In general, it often takes place if the growth of  $\{L(f^k)\}_k$  is quick.

We will find the formula for  $D_r^4[f]$ , under the above assumptions.

It turns out that first it is convenient to find the minimal decomposition of the sum

$$\sum_{i|r} a_i \text{reg}_i$$

into  $DD^4(1|r)$  sequences modulo  $\text{reg}_1$  i.e. we require that the equality holds only for all divisors  $i|r$  different than 1. In other words, we will temporarily ignore the coefficient at  $\text{reg}_1$ .

**Lemma 4.1.** *The two following numbers are equal:*

1. *the minimal number of summands in the decomposition of the sum*

$$\sum_{i|r} a_i \text{reg}_i$$

*modulo  $\text{reg}_1$  into  $DD^4(1|r)$  sequences,*

2. *the minimal number  $h(s)$  determining the family of pairs of subsets of  $I_s = \{1, \dots, s\}$ :*

$$\{A_1, B_1\}, \{A_2, B_2\}, \dots, \{A_{h(s)}, B_{h(s)}\}$$

*such that*

$$\bigcup_{i=1}^{h(s)} \{A_i, B_i, A_i \cup B_i\} = 2^{I_s} \setminus \{\emptyset\}$$

*i.e. for each nonempty subset  $X \subset I_s$  there is an  $i$  such that either  $X = A_i$  or  $X = B_i$  or  $X = A_i \cup B_i$ .*

*Proof.* Let us notice that to get the minimal decomposition of

$$\sum_{i|r} a_i \text{reg}_i \quad \text{modulo } \text{reg}_1,$$

we should use as much as possible the most “greedy”  $DD^4(1|r)$  sequences, with the greatest number of basic expressions  $\text{reg}_i$  i.e. of the type (2) or (3) of Corollary 1. In both of these cases we have the sequences of the form:

$$\varepsilon \text{reg}_1 + a_d \text{reg}_d + a_l \text{reg}_l + \gamma a_{[d,l]} \text{reg}_{[d,l]}, \quad (3)$$

where  $d, l$  are divisors of  $r$  different than 1,  $\gamma \in \{0, 1\}$ .

Since  $r = p_1 \cdots p_s$  is a product of different primes, there is a bijection  $G : 2^{I_s} \rightarrow \text{Div}(r)$  between  $\text{Div}(r)$ , the set of all divisors of  $r$ , and the family of all subsets of  $I_s = \{1, \dots, s\}$ :

$$\{1, \dots, s\} \supset A \rightarrow \prod_{i \in A} p_i \in \text{Div}(r),$$

with the convention that  $\prod_{i \in \emptyset} p_i = 1$ . Moreover

$$G(A \cup B) = [G(A), G(B)].$$

As a result, every triple of divisors  $d, l, [d, l]$  determining the sequence (3) is associated with a triple of subsets of  $I_s$ :  $A_j, B_j, A_j \cup B_j$ .

Now, a decomposition of the sum  $\sum_{1 \neq i|r} a_i \text{reg}_i(k)$  into  $h(s)$   $DD^4(1|r)$  sequences of the form (3) is equivalent to the existence of  $h(s)$  families of subsets of  $I_s$

$$\{A_1, B_1\}, \{A_2, B_2\}, \dots, \{A_{h(s)}, B_{h(s)}\}$$

such that

$$\bigcup_{i=1}^{h(s)} \{A_i, B_i, A_i \cup B_i\} = 2^{I_s} \setminus \{\emptyset\}$$

i.e. for each nonempty subset  $X \subset I_s$  there is an  $i$  such  $X = A_i$ ,  $X = B_i$  or  $X = A_i \cup B_i$ .  $\square$

Now our problem reduces to the following combinatorial question:

**Problem 4.1.** *Let  $s$  be a natural number. Find the minimal number  $h(s)$  such that there exist  $h(s)$  families of subsets  $\mathcal{A}_1, \dots, \mathcal{A}_{h(s)} \subset 2^{I_s}$  satisfying*

1.  *$\#\mathcal{A}_i \leq 2$  i.e. each family consists of at most two subsets,*

2. for each nonempty subset  $X \subset \{1, \dots, s\}$  there exists  $i \in \{1, \dots, s\}$  such that  $X$  is one of the sets  $A_i$ ,  $B_i$  or  $A_i \cup B_i$ , where  $\mathcal{A}_i = \{A_i, B_i\}$ .

**Theorem 4.2.** *The minimal number searched in Problem 4.1 is given by the formula*

$$h(s) = \frac{2^s + (-1)^{s+1}}{3}. \quad (4)$$

The proof of Theorem 4.2 is a consequence of the following three lemmas.

**Lemma 4.3.** *The formula (4) for  $h(s)$  can be given inductively as follows:*

$$h(2) = 1, \quad h(s+1) = 2 \cdot h(s) + (-1)^s.$$

*Proof.*

$$\begin{aligned} 2 \cdot h(s) + (-1)^s &= 2 \cdot \frac{2^s + (-1)^{s+1}}{3} + (-1)^s \\ &= \frac{2^{s+1} + 2 \cdot (-1)^{s+1} + 3 \cdot (-1)^s}{3} = \frac{2^{s+1} + (-1)^s}{3} = h(s+1). \end{aligned}$$

□

**Lemma 4.4.**  *$h(s)$  given by the formula (4) is less or equal to the minimal number satisfying the conditions in Problem 4.1.*

*Proof.* We notice that each family containing two subsets  $\{A, B\} \subset 2^{I_s}$  determines at most three nonempty subsets  $A, B, A \cup B \subset I_s$ . Thus, to realize all nonempty subsets in  $I_s$  we need at least  $(2^s - 1)/3$  pairs. The last means that the minimal number in Problem 4.1 is greater or equal to  $(2^s - 1)/3$ . On the other hand, the least natural number  $\geq (2^s - 1)/3$  is equal to  $(2^s - 1)/3$  when  $s$  is even and  $(2^s + 1)/3$  when  $s$  is odd. It remains to notice that in both cases we get  $h(s)$ . □

**Lemma 4.5.** (I) *For each  $s \geq 2$  there exist  $h(s) = \frac{2^s + (-1)^{s+1}}{3}$  families satisfying the conditions in Problem 4.1.*

(II) *Moreover, if  $s$  is even then each family must contain two different subsets, while if  $s$  is odd then  $h(s) - 1$  families must contain two different subsets and the last family can contain only one subset consisting of a single, arbitrarily chosen, element.*

*Proof.* We will show inductively that (for  $s \geq 2$ ): there exists a family  $\mathcal{A}_s = \{\{A_i, B_i\} : i = 1, \dots, h(s)\}$  whose elements are nonempty subsets  $A_i, B_i \subset I_s$  realizing all nonempty subsets in  $I_s$  and moreover

1.  $A_i \neq B_i$  if  $i = 1, \dots, h(s)$  and  $s$  is even,
2.  $A_i \neq B_i$  if  $i = 1, \dots, h(s) - 1$  and  $s$  is odd.
3.  $A_{h(s)} = B_{h(s)} = \{s\}$  for  $s$  odd.

For  $s = 2$  all nonempty subsets of  $I_2 = \{1, 2\}$  can be obtained from the family  $\{\{1\}, \{2\}\}$  which implies  $h(2) = 1$ .

Now we assume that for even  $s$  a family  $\mathcal{A}_s = \{\{A_i, B_i\} : i = 1, \dots, h(s)\}$  where  $A_i \neq B_i$  realizes all nonempty subsets in  $I_s = \{1, \dots, s\}$ . Then the family

$$\mathcal{A}_{s+1} = \{\{A_i, B_i\}, \{A_i \cup \{s+1\}, B_i \cup \{s+1\}\}, \{\{s+1\}\} : i = 1, \dots, h(s)\}$$

realizes all nonempty subsets in  $I_{s+1} = \{1, \dots, s, s+1\}$ . Moreover,

$$\#\mathcal{A}_{s+1} = 2 \cdot \#\mathcal{A}_s + 1 = 2 \cdot h(s) + 1 = 2 \cdot h(s) + (-1)^s = h(s+1)$$

since  $s$  is even.



Now, the family

$$\mathcal{A}_{s+2} = \{\{A'_i, B'_i\}, \{A'_i \cup \{s+2\}, B'_i \cup \{s+2\}\}, \{\{s+1\}, \{s+2\}\}\}$$

$$\text{where } \{A'_i, B'_i\} \in \mathcal{A}_{s+1} \setminus \{\{s+1\}\}$$

realizes all subsets in  $I_{s+2}$  and moreover

$$\#\mathcal{A}_{s+2} = 2 \cdot (\#\mathcal{A}_{s+1} - 1) + 1 = 2 \cdot h(s+1) - 1 = 2 \cdot h(s+1) + (-1)^{s+1} = h(s+2)$$

since  $s+1$  is odd.

This ends the proof of part (I). Part (II) of Lemma 4.5 follows from Lemma 4.4 and the observation that for  $s+1$  odd in the above inductive construction, the family  $\{\{s+1\}\}$ , i.e. the last element in  $\mathcal{A}_{s+1}$ , consists of one subset containing a single element. It is evident that after a permutation  $\{\{s+1\}\}$  can be replaced with  $\{\{i\}\}$  for an arbitrarily prescribed  $i \in I_{s+1}$ .  $\square$

### Proof of Theorem 4.2

Lemma 4.4 gives

$$h(s) \leq \text{minimal number in Problem 4.1}$$

while Lemma 4.5 proves the opposite inequality.  $\square$

By Theorem 4.2 we obtain

**Corollary 2.** *The minimal decomposition of the sum*

$$\sum_{i|r} a_i \text{reg}_i$$

*modulo  $\text{reg}_1$  into  $DD^4(1|r)$  sequences contains exactly*

$$h(s) = \frac{2^s + (-1)^{s+1}}{3}$$

*sequences.*

Moreover, by Lemma 4.5 (II) we get:

(A) *if  $s$  is even then the minimal decomposition must contain  $h(s)$  sequences of the type*

$$\varepsilon \cdot \text{reg}_1 + a_d \text{reg}_d + a_l \text{reg}_l + \gamma a_{[d,l]} \text{reg}_{[d,l]}, \quad (5)$$

*i.e. of the form (2) or (3) of Corollary 1 ( $\gamma \in \{0, 1\}$ );*

(B) *if  $s$  is odd then the minimal decomposition must contain  $h(s) - 1$  sequences of the type (5) while the remaining sequence may be  $a_1 \text{reg}_1(n) + a_d \text{reg}_d(n)$  (i.e. of the type (1) of Corollary 1), where  $d \neq 1$  is an arbitrarily prescribed divisor of  $r$ .*

**Remark 2.** Let us notice that in all sequences (5), appearing in the minimal decomposition modulo 1 described in Corollary 2, the divisors  $d, l$  must be different as they correspond to different subsets in Lemma 4.5, so both  $\text{reg}_d(n)$  and  $\text{reg}_l(n)$  appear with nonzero coefficients.

Now we are in a position to find the formula for  $D_r^4[f]$ , i.e. we take into account also the coefficient at  $\text{reg}_1$ .

Let us remark that  $D_r^4[f] \geq h(s)$ . In fact, in the minimal realization modulo  $\text{reg}_1$  we need  $h(s)$  of  $DD^4(1|r)$  sequences. The following lemmas make it precise when



these sequences are sufficient to obtain the decomposition with  $a_1 \text{reg}_1$  and when one additional sequence, to realize  $a_1 \text{reg}_1$ , is necessary.

**Lemma 4.6.** *Assume our Standing Assumptions are satisfied and  $s$  is even, then*

$$D_r^4[f] = \begin{cases} h(s) & \text{if } (r \text{ is odd and } L(f) = h(s)) \\ & \text{or } (r \text{ is even and } h(s) - 2 \leq L(f) \leq h(s)), \\ h(s) + 1 & \text{otherwise.} \end{cases}$$

*Proof.* By Corollary 2 (A) to realize

$$\sum_{1 \neq i|r} a_i \text{reg}_i$$

we need at least  $h(s) DD^4(1|r)$  sequences of the type (2) or (3) of Corollary 1.

If we assume that  $r$  is odd then they all must be of the type (3). Then the contribution of each of them to the coefficient at  $\text{reg}_1$  is 1. If moreover  $L(f) = h(s)$  then  $D_r^4[f] = h(s)$ , since  $a_1 = L(f)$ . Otherwise, we need one sequence of the type (1) more to realize the difference  $(a_1 - h(s)) \cdot \text{reg}_1(n)$ .

Now we consider the case of even  $r$ . Then exactly one sequence in the minimal decomposition must be of the type (2) and the remaining  $h(s) - 1$  sequences are of the type (3). Their contribution to the coefficient at  $\text{reg}_1$  is  $(h(s) - 1) + \varepsilon$  where  $\varepsilon = 0, +1, -1$ . Now, if  $h(s) - 2 \leq L(f) \leq h(s)$ , then  $a_1$  can be realized by these sequences. Otherwise we need one more sequence of the type (1).  $\square$

**Lemma 4.7.** *Assume our Standing Assumptions are satisfied and  $s$  is odd, then*

$$D_r^4[f] = h(s).$$

*Proof.* It is enough to show that  $\sum_{i|r} a_i \text{reg}_i(n)$  is the sum of exactly  $h(s) DD^4(1|r)$  sequences.

Since  $s$  is odd, by Corollary 2 (B),  $h(s) - 1$  sequences of the types (2) or (3) of Corollary 1 realize

$$\sum_i a_i \text{reg}_i,$$

where the summation runs through the set  $\text{Div}(r) \setminus \{1, d\}$ , for some  $d|r$ . Again by Corollary 2 (B), it remains to add one expression of the type (1) realizing the sum  $a_1 \text{reg}_1 + a_d \text{reg}_d$ .  $\square$

We sum up our considerations in the following

**Theorem 4.8.** *Let  $f : M^4 \rightarrow M^4$  be a smooth self-map of a smooth closed connected and simply-connected 4-manifold,  $r = p_1 \dots p_s$  be a product of different prime numbers. We assume that the coefficients  $a_i$  in the periodic expansion of  $L(f^k) = \sum_{i=1}^{\infty} a_i \text{reg}_i(k)$ , are nonzero for all  $i|r$ ,  $i \neq 1$ . Then*

$$D_r^4[f] = \begin{cases} h(s) & \text{if } (s \text{ is odd}) \text{ or } (r \text{ is odd and } L(f) = h(s)) \\ & \text{or } (r \text{ is even and } h(s) - 2 \leq L(f) \leq h(s)), \\ h(s) + 1 & \text{otherwise.} \end{cases}$$

where  $h(s) = (2^s + (-1)^{s+1})/3$ .

**Remark 3.** If in Theorem 4.8 we drop the part (3) of the Standing Assumption according which  $a_i \neq 0$  for all  $i \neq 1$  dividing  $r$  then the equality becomes the inequality and we get the estimation for  $D_r^4[f]$  from above.

## REFERENCES

- [1] I. K. Babenko and S. A. Bogatyĭ, *The behavior of the index of periodic points under iterations of a mapping*, Math. USSR Izv. **38** (1992), 1–26.
- [2] A. Dold, *Fixed point indices of iterated maps*, Invent. Math. **74** (1983), 419–435.
- [3] G. Graff, *Minimal number of periodic points for smooth self-maps of two-holed 3-dimensional closed ball*, Topol. Methods Nonlinear Anal. **33** (2009), no. 1, 121–130.
- [4] G. Graff and J. Jezierski, *Minimal number of periodic points for  $C^1$  self-maps of compact simply-connected manifolds*, Forum Math. **21** (2009), no. 3, 491–509.
- [5] G. Graff and J. Jezierski, *Minimal number of periodic points for smooth self-maps of  $S^3$* , Fund. Math. **204** (2009), 127–144.
- [6] G. Graff and J. Jezierski, *Minimizing the number of periodic points for smooth maps. Non-simply connected case*, Topology Appl. **158** (2011), no. 3, 276–290.
- [7] G. Graff, J. Jezierski and M. Nowak-Przygodzki, *Minimal number of periodic points for smooth self-maps of  $\mathbb{R}P^3$* , Topology Appl. **157** (2010), no. 10–11, 1784–1803.
- [8] G. Graff, J. Jezierski and P. Nowak-Przygodzki, *Fixed point indices of iterated smooth maps in arbitrary dimension*, J. Differential Equations (2011), doi:10.1016/j.jde.2011.05.024
- [9] G. Graff and P. Nowak-Przygodzki, *Fixed point indices of iterations of  $C^1$  maps in  $\mathbb{R}^3$* , Discrete Cont. Dyn. Systems **16** (2006), no. 4, 843–856.
- [10] E. Hart, P. Heath and E. Keppelmann, *Algorithms for Nielsen type periodic numbers of maps with remnant on surfaces with boundary and on bouquets of circles. I.*, Fund. Math. **200** (2008), no. 2, 101–132.
- [11] E. Hart and E. Keppelmann, *Nielsen periodic point theory for periodic maps on orientable surfaces*, Topology Appl. **153** (2006), no. 9, 1399–1420.
- [12] J. Jezierski, *Wecken’s theorem for periodic points in dimension at least 3*, Topology Appl. **153** (2006), no. 11, 1825–1837.
- [13] J. Jezierski, *Homotopy periodic sets of selfmaps of real projective spaces*, Bol. Soc. Mat. Mexicana (3) **11** (2005), no. 2, 294–302.
- [14] B. J. Jiang, “*Lectures on the Nielsen Fixed Point Theory*,” Contemp. Math. **14**, Amer. Math. Soc., Providence 1983.
- [15] J. Jezierski and W. Marzantowicz, “*Homotopy Methods in Topological Fixed and Periodic Points Theory*,” Topological Fixed Point Theory and Its Applications, 3. Springer, Dordrecht, 2005.
- [16] H. J. Kim, J. B. Lee and W. S. Yoo, *Computation of the Nielsen type numbers for maps on the Klein bottle*, J. Korean Math. Soc. **45** (2008), no. 5, 1483–1503.
- [17] P. Le Calvez, *Dynamique des homéomorphismes du plan au voisinage d’un point fixe*, Ann. Sci. École Norm. Sup. (4) **36**, (2003), no. 1, 139–171.
- [18] J. B. Lee and X. Zhao, *Nielsen type numbers and homotopy minimal periods for maps on the 3-nilmanifolds*, Sci. China Ser. A **51** (2008), no. 3, 351–360.
- [19] J. B. Lee and X. Zhao, *Nielsen type numbers and homotopy minimal periods for maps on 3-solvmanifolds*, Algebr. Geom. Topol. **8** (2008), no. 1, 563–580.
- [20] J. Llibre, J. Paranõs and J. A. Rodriguez, *Periods for transversal maps on compact manifolds with a given homology*, Houston J. Math. **24** (1998), no. 3, 397–407.
- [21] F. Ruiz del Portal and J.M. Salazar, *A Poincaré formula for the fixed point indices of the iterates of arbitrary planar homeomorphisms*, Fixed Point Theory Appl., Volume 2010 (2010), Article ID 323069, 31 pages.
- [22] G.Y. Zhang, *Fixed point indices and periodic points of holomorphic mappings*, Math. Ann. **337** (2007), no. 2, 401–433.

Received July 2010; revised February 2011.

*E-mail address:* graff@mif.pg.gda.pl

*E-mail address:* jezierski@acn.waw.pl

