

# On homotopy Conley index for multivalued flows in Hilbert spaces

Zdzisław Dzedzej <sup>1</sup>

Faculty of Technical Physics and Applied Mathematics  
Gdańsk University of Technology  
ul. Narutowicza 11/12, 80-952 Gdańsk, Poland  
e-mail: [zdzedzej@mif.pg.gda.pl](mailto:zdzedzej@mif.pg.gda.pl)

Grzegorz Gabor

Faculty of Mathematics and Computer Science  
Nicolaus Copernicus University  
Chopina 12/18, 87-100 Toruń, Poland  
e-mail: [ggabor@mat.uni.torun.pl](mailto:ggabor@mat.uni.torun.pl)

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## Abstract

An approximation approach is applied to obtain a homotopy version of the Conley type index in Hilbert spaces considered in [6]. The definition given in the paper is more elementary and, as a by-product, gives a natural connection between indices from [13] and [16] in a finite-dimensional case. Some geometric properties from [7] are discussed in an infinite dimensional situation.

## 1 Preliminaries on set-valued maps

Let  $X, Y$  be metric spaces. By a *set-valued map*  $\varphi$  from  $X$  into  $Y$  (written  $\varphi : X \multimap Y$ ) we mean a map that assigns to each  $x \in X$  a *closed nonempty* subset  $\varphi(x)$  of  $Y$ . If, for any closed (resp. open) set  $U \subset Y$ , the *preimage*  $\varphi^{-1}(U) := \{x \in X \mid \varphi(x) \cap U \neq \emptyset\}$  is closed (resp. open), then we say that  $\varphi$  is *upper* (resp. *lower*) *semicontinuous* (written *usc* (resp. *lsc*)); a map  $\varphi$  is *continuous* if it is upper and lower semicontinuous simultaneously. The *graph*  $\text{Gr}(\varphi) := \{(x, y) \in X \times Y \mid y \in \varphi(x)\}$  of an upper semicontinuous map  $\varphi$  is closed. A map  $\varphi$  is upper semicontinuous and has compact values (i.e., for each  $x \in X$ , the set  $\varphi(x)$  is compact) if and only if, for any sequence  $(x_n, y_n) \in \text{Gr}(\varphi)$  such that  $x_n \rightarrow x \in X$ , there is a subsequence

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$(y_{n_k})$  such that  $y_{n_k} \rightarrow y \in \varphi(x)$  (in other words the projection  $p_\varphi : \text{Gr}(\varphi) \rightarrow X$  is proper<sup>(2)</sup>); in this case the *image*  $\varphi(K) := \{y \in Y \mid y \in \varphi(x) \text{ for some } x \in K\}$  of any compact  $K \subset X$  is compact. We say that a map  $\varphi$  is *compact* if it is upper semicontinuous and  $\text{cl} \varphi(X)$  is compact;  $\varphi$  is *completely continuous* if the restriction  $\varphi|_B$  of  $\varphi$  to any bounded subset  $B \subset X$  is compact.

A proper surjection  $p : X \rightarrow Y$  is a *Vietoris map* if, for each  $y \in Y$ , the fibre  $p^{-1}(y)$  is acyclic in the sense of the Alexander-Spanier cohomology. A map  $p : (X, X') \rightarrow (Y, Y')$  of pairs  $(X, X'), (Y, Y')$  (i.e.  $p : X \rightarrow Y$  and  $p(X') \subset Y'$ ) is a *Vietoris map*, if  $p$  is a Vietoris map and  $p^{-1}(Y') = X'$  (observe that the restriction  $p' : X' \rightarrow Y'$  of  $p$  is a Vietoris map, too). A map  $\varphi : X \multimap Y$  is *admissible* (in the sense of Górniewicz) if there exist a space  $\Gamma$ , a Vietoris map  $p : \Gamma \rightarrow X$  and a continuous map  $q : \Gamma \rightarrow Y$  such that, for every  $x \in X$ ,  $\varphi(x) = q(p^{-1}(x))$ . It is clear that admissible maps are upper semicontinuous with nonempty compact values.

The class of admissible maps is rich: for example any acyclic map  $\varphi : X \multimap Y$  is admissible ( $\varphi$  is *acyclic* if it is upper semicontinuous and, for any  $x \in X$ ,  $\varphi(x)$  is acyclic); it is determined by the pair  $(p_\varphi, q_\varphi)$  where  $p_\varphi : \text{Gr}(\varphi) \rightarrow X$  and  $q_\varphi : \text{Gr}(\varphi) \rightarrow Y$  are the restrictions of the projections  $X \times Y \rightarrow X$  and  $X \times Y \rightarrow Y$ , respectively. Moreover a superposition of acyclic maps is admissible. For more details concerning admissible maps - see [9]. Let us prove the following elementary:

**Proposition 1.1** *Let  $X$  be a metric space,  $E$  - a linear normed space, and  $F : X \multimap E$  a map with nonempty values. Then for each  $\varepsilon > 0$  there exists a continuous map  $f : X \rightarrow E$  such that  $f(x) \in \text{conv}(F(B_\varepsilon(x)))$ .*

*Proof.* For each  $x \in X$  we can choose a point  $v_x \in F(x)$ . Let  $\{\lambda_s\}_{s \in S}$  be a continuous partition of unity subordinated to the covering  $\{V_s\}$ , which is a locally finite refinement of the covering  $\{B_\varepsilon(x)\}_{x \in X}$ . For  $s \in S$  we fix a point  $x_s$  such that  $\text{supp } \lambda_s \subset V_s \subset B_\varepsilon(x_s)$ . Define a continuous map

$$f(x) := \sum_{s \in S} \lambda_s(x) v_s,$$

where  $v_s = v_{x_s}$ . If  $s \in S_x = \{s \mid \lambda_s(x) \neq 0\}$ , then  $x \in B_\varepsilon(x_s)$ . Thus  $x_s \in B_\varepsilon(x)$ , and hence  $v_s \in F(B_\varepsilon(x))$ . Therefore  $f(x) \in \text{conv} F(B_\varepsilon(x))$ .  $\square$

**Remark 1.2** *Observe that  $E$  could be a topological vector space in the previous proposition. The map  $f$  is locally Lipschitz if we take a locally Lipschitz partition of unity.*

We say that a continuous map  $f : X \rightarrow Y$  is a *graph  $\varepsilon$ -approximation* of  $\varphi : X \multimap Y$  if  $f(x) \in B_\varepsilon(\varphi(B_\varepsilon(x)))$  for every  $x \in X$ . The following is a version of a classical result of A. Cellina [2] combined with the previous observation.

<sup>2</sup>Recall that a continuous map  $f : X \rightarrow Y$  is *proper* if, for each compact  $K \subset Y$ , the preimage  $f^{-1}(K)$  is compact; it is worth reminding that  $f$  is proper if and only if it is *perfect*, i.e. continuous, closed and such that, for any  $y \in Y$ ,  $f^{-1}(y)$  is compact. Observe that a continuous surjection  $f : X \rightarrow Y$  is perfect if and only if the multivalued map  $Y \ni y \multimap f^{-1}(y) \subset X$  is upper semicontinuous and has compact values.

**Theorem 1.3** *Let  $\varphi : X \multimap E$  be usc with convex values, where  $X$  is a metric space and  $E$  is a Banach space. Then, for every  $\varepsilon > 0$ , there exists a locally Lipschitz graph  $\varepsilon$ -approximation  $f$  of  $\varphi$  such that  $f(x) \in \text{conv}\varphi(B_\varepsilon(x))$  for every  $x \in X$ .*

*Proof.* Let  $\varepsilon > 0$ . From upper semicontinuity of  $\varphi$  it follows that for every  $x \in X$  there exists  $0 < \delta(x) < \frac{\varepsilon}{2}$  such that  $\varphi(B_{\delta(x)}(x)) \subset B_\varepsilon(\varphi(x))$ . Consider a locally finite covering  $\{V_s\}_{s \in S}$  of  $X$  which is a star-refinement of the covering  $\{B_{\delta(x)}(x)\}_{x \in X}$ , i.e., stars  $\text{st}(V_t) = \bigcup\{V_s : V_s \cap V_t \neq \emptyset\}$  refine the covering  $\{B_{\delta(x)}(x)\}_{x \in X}$ . Let  $\{\lambda_s\}_{s \in S}$  be a locally Lipschitz partition of unity subordinated to the covering  $\{V_s\}$ . For each  $s \in S$  we choose a point  $x_s \in V_s$  and some  $y_s \in \varphi(x_s)$ .

Define

$$f(x) := \sum_{s \in S} \lambda_s(x) y_s.$$

Let  $S_x = \{s \in S \mid \lambda_s(x) \neq 0\}$  and let  $s \in S_x$ . Then  $x \in V_s$ . It implies that  $d(x_s, x) < \delta(x_s) < \varepsilon$  and hence  $x_s \in B_\varepsilon(x)$ . Therefore

$$f(x) = \sum_{s \in S_x} \lambda_s(x) y_s \in \text{conv}\varphi(B_\varepsilon(x)).$$

Moreover, since  $x \in \bigcap_{s \in S_x} V_s$ , there exists  $x'$  such that  $\bigcup_{s \in S_x} V_s \subset B_{\delta(x')}(x')$ . Thus both  $x, x_s \in V_s$ , and thus  $d(x, x_s) < 2\delta(x') < \varepsilon$ . By our choice of  $\delta(x')$  we have  $y_s \in B_\varepsilon(\varphi(x'))$ . But the latter set is convex, thus  $f(x) = \sum \lambda_s(x) y_s \in B_\varepsilon(\varphi(x')) \subset B_\varepsilon(\varphi(B_\varepsilon(x)))$  and the proof is complete.  $\square$

**Corollary 1.4** *If  $\varphi$  is completely continuous, then the approximation  $f$  in Theorem 1.3 is also completely continuous.*

*Proof.* For every bounded set  $U \subset X$  we have  $f(U) \subset \overline{\text{conv}}\varphi(B_\varepsilon(U))$ , and the latter set is relatively compact.  $\square$

## 2 Multivalued flows

Let  $X$  be a metric space.

**Definition 2.1** By a *multivalued flow* on  $X$  we mean an upper semicontinuous mapping  $\varphi : X \times \mathbb{R} \multimap X$  with nonempty and compact values such that, for every  $s, t \in \mathbb{R}$  and  $x, y \in X$ ,

(i)  $\varphi(x, 0) = \{x\}$ ;

- (ii) if  $s \cdot t \geq 0$ , then  $\varphi(x, t + s) = \varphi(\varphi(x, t) \times \{s\})$ ;
- (iii)  $y \in \varphi(x, t)$  if and only if  $x \in \varphi(y, -t)$ .

Let  $\Delta \subseteq \mathbb{R}$ . A map  $\sigma : \Delta \rightarrow X$  is a  $\Delta$ -trajectory of  $\varphi$  if, for every  $t, s \in \Delta$ ,  $\sigma(t) \in \varphi(\sigma(s), t - s)$ . It is an easy exercise to prove that every trajectory is continuous. Indeed, let us consider a sequence  $t_n$  converging to  $t_0$ . Let  $U \ni \sigma(t_0)$  be open. Since  $\varphi$  is upper semicontinuous,  $\varphi^{-1}(U)$  is open and  $(\sigma(t_0), 0) \in \varphi^{-1}(U)$ , because  $\sigma(t_0) \in \varphi(\sigma(t_0), 0)$ . There exist  $\delta > 0$  and an open set  $V \subset X$  such that  $(\sigma(t_0), 0) \in V \times (-\delta, \delta) \subset \varphi^{-1}(U)$ . Therefore, for a large  $n$ ,  $|t_n - t_0| < \delta$  and then  $\sigma(t_n) \in \varphi(\sigma(t_0), t_n - t_0) \subset U$ .

Let  $x \in N \subseteq X$ . The set of all  $\Delta$ -trajectories in  $N$  originating in  $x$  (i.e., such that  $0 \in \Delta$ ,  $\sigma(0) = x$  and  $\sigma(t) \in N$  for  $t \in \Delta$ ) is denoted by  $Tr_N(\varphi; \Delta, x)$ .

Define the *invariant*, *right-invariant*, *left-invariant* (with respect to  $\varphi$ ) part of  $N$  by:

$$\text{Inv}(N, \varphi) := \{x \in N \mid Tr_N(\varphi; \mathbb{R}, x) \neq \emptyset\},$$

$$\text{Inv}^+(N, \varphi) := \{x \in N \mid Tr_N(\varphi; \mathbb{R}_+, x) \neq \emptyset\},$$

$$\text{Inv}^-(N, \varphi) := \{x \in A \mid Tr_N(\varphi; \mathbb{R}_-, x) \neq \emptyset\},$$

respectively.

**Definition 2.2** A subset  $K \subset X$  is *invariant* (resp. *positively* (*negatively*) *invariant*) with respect to  $\varphi$  if  $\text{Inv}(K, \varphi) = K$  (resp.  $\text{Inv}^+(K, \varphi) = K$  ( $\text{Inv}^-(K, \varphi) = K$ )).

Note that, given  $N \subset X$ , the set  $K := \text{Inv}(N, \varphi)$  is invariant with respect to  $\varphi$  and it is the maximal invariant subset of  $N$ .

**Proposition 2.3** ([6], Proposition 3.9) *Let  $\Lambda$  be a metric space,  $N \subset X$  be closed and let  $\eta : X \times \mathbb{R} \times \Lambda \multimap X$  be a family of multivalued flows (i.e.,  $\eta$  is upper semicontinuous and, for each  $\lambda \in \Lambda$ ,  $\eta(\cdot, \lambda) : X \times \mathbb{R} \multimap X$  is a multivalued flow). Then the graph of the set-valued map*

$$\Lambda \ni \lambda \mapsto \text{Inv}(N, \eta(\cdot, \lambda))$$

*is closed, i.e. for any sequence  $(x_n, \lambda_n) \in N \times \Lambda$  such that  $x_n \in \text{Inv}(N, \eta(\cdot, \lambda_n))$ , if  $(x, \lambda) = \lim_{n \rightarrow \infty} (x_n, \lambda_n)$ , then  $x \in \text{Inv}(N, \eta(\cdot, \lambda))$ .*

**Definition 2.4** A closed and bounded set  $N \subset X$  is an *isolating neighborhood* for  $\varphi$  if  $\text{Inv}(N, \varphi) \subset \text{int } N$ . We say that a set  $K$  invariant with respect to  $\varphi$  is *isolated* if there is an isolating neighborhood  $N$  for  $\varphi$  such that  $K = \text{Inv}(N, \varphi)$ .

In particular, if  $X = \mathbb{R}^n$ , then each isolating neighborhood  $N$  for  $\varphi$  is compact and, by Proposition 2.3,  $K = \text{Inv}(N, \varphi)$  is closed in  $N$ , hence compact.

### 3 Conley index in Hilbert spaces

We shall assume the following:

Let  $\mathbb{H} = (\mathbb{H}, \langle \cdot, \cdot \rangle)$  be a real Hilbert space and  $L : \mathbb{H} \rightarrow \mathbb{H}$  a linear bounded operator with spectrum  $\sigma(L)$ . We assume the following

- $\mathbb{H} = \bigoplus_{k=0}^{\infty} \mathbb{H}_k$  with all subspaces  $\mathbb{H}_k$  being mutually orthogonal and of finite dimension;
- $L(\mathbb{H}_0) \subset \mathbb{H}_0$  where  $\mathbb{H}_0$  is the invariant subspace of  $L$  corresponding to the part of spectrum  $\sigma_0(L) = i\mathbb{R} \cap \sigma(L)$  lying on the imaginary axis,
- $L(\mathbb{H}_k) = \mathbb{H}_k$  for all  $k > 0$ ,
- $\sigma_0(L)$  is isolated in  $\sigma(L)$ , i.e.  $\sigma_0(L) \cap \text{cl}(\sigma(L) \setminus \sigma_0(L)) = \emptyset$ .

**Definition 3.1** A multivalued flow  $\varphi : \mathbb{H} \times \mathbb{R} \multimap \mathbb{H}$  is called an *L-flow*, if it has the form

$$\varphi(x, t) = e^{tL}x + U(t, x),$$

where  $U : \mathbb{H} \times \mathbb{R} \multimap \mathbb{H}$  is an admissible map which is completely continuous.

Let  $\Lambda$  be a metric space. By a *family of L-flows* we understand a set-valued map  $\eta : \mathbb{H} \times \mathbb{R} \times \Lambda \multimap \mathbb{H}$  of the form  $\eta(x, t, \lambda) = e^{tL}x + U(x, t, \lambda)$ , where  $U : \mathbb{H} \times \mathbb{R} \times \Lambda \multimap \mathbb{H}$  is an admissible completely continuous mapping, such that, for each  $\lambda \in \Lambda$ ,  $\eta(\cdot, \lambda) : \mathbb{H} \times \mathbb{R} \multimap \mathbb{H}$  is a multivalued flow.

It is clear that, if  $\eta : \mathbb{H} \times \mathbb{R} \times \Lambda \multimap \mathbb{H}$  is a family of *L-flows*, then, for each  $\lambda \in \Lambda$ ,  $\eta(\cdot, \lambda) : \mathbb{H} \times \mathbb{R} \multimap \mathbb{H}$  is an *L-flow*. Moreover, each *L-flow* is an admissible flow.

**Proposition 3.2** ([6], Prop.3.15) *If  $X \subset \mathbb{H}$  is closed and bounded, then the set-valued map  $\Lambda \ni \lambda \mapsto \text{Inv}(X, \eta(\cdot, \lambda)) \subset X$  is usc and it has compact (possibly empty) values.*

**Definition 3.3** An usc mapping  $f : \mathbb{H} \multimap \mathbb{H}$  is an *L-vector field* if it is of the form  $f(x) = Lx + K(x)$ , where  $K : \mathbb{H} \multimap \mathbb{H}$  is completely continuous with compact convex values, and if  $f$  induces an *L-flow*  $\pi$  on  $H$ .

Given an *L-vector field*  $f := L + F : \mathbb{H} \multimap \mathbb{H}$ ,  $F$  having a sublinear growth (i.e., there is a constant  $C > 0$  such that, for each  $u \in \mathbb{H}$  and  $y \in F(u)$ ,  $\|y\| \leq C(1 + \|u\|)$ ), the standard fixed point argument (see, e.g., [11], Theorem 5.2.2) implies that, for each  $x \in \mathbb{H}$ , there is a *mild solution* to the Cauchy problem

$$(1) \quad \begin{cases} u' \in f(u) & \text{a.e. on } \mathbb{R}; \\ u(0) = x, \end{cases}$$

i.e., a continuous function  $u : \mathbb{R} \rightarrow \mathbb{H}$  and a locally (Bochner) integrable function  $w : \mathbb{R} \rightarrow \mathbb{H}$  such that  $w(t) \in F(u(t))$  and  $u(t) = e^{tL}x + \int_0^t e^{(t-s)L}w(s) ds$  for all  $t \in \mathbb{R}$ .

Let  $S(x) \subset C(\mathbb{R}, \mathbb{H})$  <sup>(3)</sup> be the set of all solutions to (1),  $x \in \mathbb{H}$ .

Consider a map  $\varphi : \mathbb{H} \times \mathbb{R} \dashrightarrow \mathbb{H}$  given by the formula

$$(2) \quad \varphi(x, t) := \{u(t) \mid u \in S(x)\}, \quad x \in \mathbb{H}, t \in \mathbb{R}.$$

It is shown in [6],(Ex. 3.3) that  $\varphi$  is an admissible multivalued flow on  $\mathbb{H}$  (we say that  $\varphi$  is *generated by  $f$* ).

We consider here only flows generated by  $L$ -vector fields. In particular, if  $F$  is single-valued and locally Lipschitz, then  $f$  generates a usual (single-valued) flow.

Recall that a *suspension* of a pointed space  $(X, x_0)$  is the quotient space  $(SX, *) := (S^1 \times X)/(S^1 \times \{x_0\} \cup \{s_0\} \times X)$ , where  $S^1$  denotes a circle.

Let  $\nu : \mathbb{N} \rightarrow \mathbb{N}$  be a given map.

**Definition 3.4** A pair of sequences  $X = ((X_n, x_n)_{n=n_0}^\infty, (\gamma_n))$  is a *spectrum* provided the maps  $\gamma_n : S^{\nu(n)}X_n \rightarrow X_{n+1}$  are homotopy equivalences for some  $n_1 \geq n_0$  and each  $n \geq n_1$ .

We can define a natural notion of a *map of spectra*  $f : X \rightarrow X'$  as a sequence of maps  $f_n : X_n \rightarrow X'_{n+1}; n \geq n_0 = \max\{n(X), n(X')\}$  such that the diagrams

$$\begin{array}{ccc} S^{\nu(n)}X_n & \xrightarrow{S^{\nu(n)}f_n} & S^{\nu(n)}X'_n \\ \downarrow \gamma_n & \curvearrowright & \downarrow \gamma'_n \\ X_{n+1} & \xrightarrow{f_{n+1}} & X'_{n+1} \end{array}$$

are homotopy commutative for all  $n \geq n_0$ .

Two spectra are *homotopy equivalent* if there is  $n_1 \geq n_0$  such that  $f_n$  are homotopy equivalences for  $n \geq n_1$ . The equivalence class of this relation is called *the homotopy type of a spectrum  $X$*  and is denoted by  $[X]$ . One observes that the homotopy type of a spectrum  $X$  is determined by the homotopy type of the pointed space  $(X_n, x_n)$  with  $n$  sufficiently large.

We denote by  $\underline{0}$  the spectrum such that for each  $n \geq 0$  the space  $X_n$  consists only of a base point with the only maps  $\epsilon_n : X_n \rightarrow X_{n+1}$ . This is called a *trivial spectrum*.

<sup>3</sup> $C(\mathbb{R}, \mathbb{H})$  stands for the Fréchet space (i.e., locally convex metrizable and complete) of all continuous maps  $\mathbb{R} \rightarrow \mathbb{H}$  with the topology of the almost uniform convergence.

One can also define usual topological operations like a "wedge sum" and smash product of spectra and on their homotopy types (see [10], Sec.2 for details).

Let now  $f = L + K : \mathbb{H} \rightarrow \mathbb{H}$  be a single-valued  $L$ -vector field and let  $\varphi : \mathbb{H} \times \mathbb{R} \rightarrow \mathbb{H}$  be an  $L$ -flow generated by  $f$ .

Denote by  $\mathbb{H}^n := \bigoplus_{i=0}^n \mathbb{H}_i$  and by  $P_n : \mathbb{H} \rightarrow \mathbb{H}$  an orthogonal projection onto  $\mathbb{H}^n$ .

Let  $\mathbb{H}_n^\pm := \mathbb{H}_n \cap \mathbb{H}^\pm, n \geq 1$ , where  $\mathbb{H}^+$  and  $\mathbb{H}^-$  denote  $L$ -invariant subspaces of  $H$  corresponding to parts of the spectrum of  $L$  with positive and negative real parts, respectively. Define  $\nu : \mathbb{N} \cup \{0\} \rightarrow \mathbb{N} \cup \{0\}$  by  $\nu(n) = \dim \mathbb{H}_{n+1}^+$ .

Define  $f_n : \mathbb{H}^n \rightarrow \mathbb{H}^n$  by  $f_n(x) := Lx + P_n(K(x))$  and let  $\varphi_n : \mathbb{H}^n \times \mathbb{R} \rightarrow \mathbb{H}^n$  be a flow generated by  $f_n$ .

**Lemma 3.5** ([8], Lemma 4.1) *Let  $N \subset \mathbb{H}$  be an isolating neighborhood for  $\varphi$ . Then there exists  $n_0$  such that, for all  $n \geq n_0$ , the set  $N^n = N \cap \mathbb{H}^n$  is an isolating neighborhood for  $\varphi_n$ .*

Thus the isolated invariant set  $S_n = \text{Inv}(N^n, \varphi_n)$  admits an index pair  $(P_1, P_2)$ , (see [17]), i.e. a compact pair  $(P_1, P_2)$  such that

- (i) the set  $\overline{P_1} \setminus \overline{P_2}$  is an isolating neighborhood for  $S_n$  in  $N^n$ ;
- (ii) (positive invariance of  $P_2$  in  $P_1$ ) if  $x \in P_2$  with  $\varphi_n(x, t) \in P_1$  for every  $t \in [0, t_0]$ , then  $\varphi_n(x, t) \in P_2$  for every  $t \in [0, t_0]$ ;
- (iii) if  $x \in P_1$  and there is  $t \geq 0$  with  $\varphi_n(x, t) \notin P_1$ , then there exists  $0 \leq t_0 < t$  such that  $\varphi_n(x, t_0) \in P_2$ .

The classical homotopy Conley index of  $S_n$  is the homotopy type of the pointed space  $[P_1/P_2, *]$ . By the use of the continuation property of the classical Conley index it was proved in [8], that the family of such index pairs  $(P_1^n, P_2^n)$  for  $n \geq n_0$  forms a spectrum in the above sense. A homotopy type of this spectrum is called an  $\mathcal{LS}$ -index of the isolating neighborhood  $N$ .

Let us denote this index by  $h_{\mathcal{LS}}(N, \varphi)$ . The following two basic properties have been proved in [8].

**Proposition 3.6** (Nontriviality) *Let  $\varphi$  be an single-valued  $L$ -flow and  $N \subset \mathbb{H}$  an isolating neighborhood. If  $h_{\mathcal{LS}}(N, \varphi) \neq \underline{0}$ , then  $\text{Inv}(N, \varphi) \neq \emptyset$ .*

**Proposition 3.7** (Continuation) *Let  $\Lambda$  be a compact, connected and locally contractible metric space. Assume that  $\varphi_\lambda$  is a family of single-valued  $L$ -flows and let  $N \subset \mathbb{H}$  be an isolating*

neighborhood for the flow  $\varphi_\lambda$  for some  $\lambda \in \Lambda$ . Then there is a compact neighborhood  $C \subset \Lambda$  of  $\lambda$  such that

$$h_{\mathcal{LS}}(N, \varphi_\mu) = h_{\mathcal{LS}}(N, \varphi_\nu) \quad \text{for all } \mu, \nu \in C.$$

Let us now consider a multivalued  $L$ -vector field  $L + F : \mathbb{H} \multimap \mathbb{H}$ . Denote by  $a(F, \varepsilon)$  the set of all  $\varepsilon$ -approximations of  $F$  in the sense of Theorem 1.3.

**Proposition 3.8** *Let  $N = \overline{U} \subset \mathbb{H}$  be an isolating neighborhood for a multivalued flow generated by  $L + F$ . There exists an  $\varepsilon > 0$  such that for arbitrary  $f_0, f_1 \in a(F, \varepsilon)$   $N$  is an isolating neighborhood for a family of  $L$ -flows  $\eta_\lambda$  generated by the family of  $L$ -vector fields  $\Psi_\lambda = L + (1 - \lambda)f_0 + \lambda f_1$ .*

*Proof.* Let  $r > 0$  be such that  $N \subset B_r(0)$ , and find the Urysohn function  $u : \mathbb{H} \rightarrow [0, 1]$  such that  $u(x) = 0$  for  $x \in \overline{B_r(0)}$  and  $u(x) = 1$  for any  $x \in \mathbb{H} \setminus B_{2r}(0)$ .

Consider a homotopy  $h : \mathbb{H} \times [0, 1] \multimap \mathbb{H}$ ,

$$h(x, s) := Lx + (1 - u(x)) \left( \overline{\text{conv}F(B_s(x)) + \overline{B_s(0)}} \right) \cap \overline{\text{conv}F(\overline{B_{2r}(0)})}.$$

Since  $\overline{\text{conv}F(\overline{B_{2r}(0)})}$  is compact, the map  $h$  generates a family  $\eta$  of multivalued  $L$ -flows on  $\mathbb{H}$ . Notice that  $h(\cdot, 0) = L + F$  on  $\overline{B_r(0)}$ . From Proposition 3.2 it follows that the map  $s \mapsto \text{Inv}(N, \eta(\cdot, s))$  is usc with compact values.

Now, suppose the contrary to our claim. Then, for a sequence  $\varepsilon_n = \frac{1}{n}$ , there are approximations  $f_0^n, f_1^n \in a(F, \frac{1}{n})$  and numbers  $\lambda_n \in [0, 1]$  with  $\text{Inv}(N, \gamma_{f_{\lambda_n}}) \not\subset U$ , where  $\gamma_{f_{\lambda_n}}$  is the flow generated by  $L + (1 - \lambda_n)f_0 + \lambda_n f_1$ . This implies that there are points  $y_n \in \text{Inv}(N, \gamma_{f_{\lambda_n}}) \cap (N \setminus U)$ . Note that  $f_{\lambda_n}(\cdot) \subset \overline{\text{conv}F(B_{\frac{1}{n}}(\cdot)) + \overline{B_{\frac{1}{n}}(0)}}$  for every  $n \geq 1$ . Since the map  $s \mapsto \text{Inv}(N, \eta(\cdot, s))$  is usc with compact values, there exists a subsequence  $(f_k)$ , where  $f_k := f_{\lambda_{n_k}}$ , such that  $\text{Inv}(N, \gamma_{f_k}) \subset \text{Inv}(N, \varphi) + B_{\frac{1}{k}}(0)$  for every  $k \geq 1$ . Indeed, it is sufficient to notice that  $\text{Inv}(N, \gamma_{f_k}) \subset \text{Inv}(N, \eta(\cdot, \frac{1}{n_k}))$ .

Now, we can choose a sequence  $(z_k) \subset \text{Inv}(N, \varphi)$  such that  $|z_k - y_{n_k}| < \frac{1}{k}$ . Since the set  $\text{Inv}(N, \varphi)$  is compact, we can assume that  $z_k \rightarrow z_0 \in \text{Inv}(N, \varphi)$ . So,  $y_{n_k} \rightarrow z_0$ . But then  $z_0 \in \text{Inv}(N, \varphi) \cap (N \setminus U)$ ; a contradiction.  $\square$

The above proposition proves that the following crucial notion of this note does not depend on the approximation  $f$ .

**Definition 3.9** If  $N$  is an isolating neighborhood for an  $L$ -flow  $\varphi$  generated by  $L + F$ , then we define a *homotopy index*

$$h(N, \varphi) := h_{\mathcal{LS}}(N, \varphi_f),$$

where  $\varphi_f$  is the flow generated by  $L + f$ ;  $f \in a(F, \varepsilon)$  with  $\varepsilon > 0$  sufficiently small.



Now we establish some properties of the index. The first one is an obvious consequence of Proposition 3.6.

**Proposition 3.10** *If  $N$  is an isolating neighborhood for a multivalued  $L$ -flow  $\varphi$  and the homotopy index is nontrivial  $h(N, \varphi) \neq \underline{0}$ , then  $\text{Inv}(N, \varphi) \neq \emptyset$ .*

**Proposition 3.11** *If  $N_0, N_1$  are two isolating neighborhoods for an  $L$ -flow  $\varphi$  such that  $\text{Inv}(N_0, \varphi) \subset \text{int } N_1, \text{Inv}(N_1, \varphi) \subset \text{int } N_0$ , then  $h(N_0, \varphi) = h(N_1, \varphi)$ .*

**Proposition 3.12** *Let  $\varphi : \mathbb{H} \times [0, 1] \times \mathbb{R} \multimap \mathbb{H}$  be a family of multivalued  $L$ -flows and let  $N \subset \mathbb{H}$  be an isolating neighborhood for all  $\varphi(\cdot, \lambda), \lambda \in [0, 1]$ . Then  $h(N, \varphi(\cdot, 0)) = h(N, \varphi(\cdot, 1))$ .*

*Proof.* Consider the family of vector fields  $\tilde{F} : \mathbb{H} \times [0, 1] \multimap \mathbb{H}$  such that for every  $\lambda \in [0, 1]$  the multivalued flow  $\varphi(\cdot, \lambda, \cdot)$  is generated by the  $L$ -vector field  $L + \tilde{F}(\cdot, \lambda) : \mathbb{H} \multimap \mathbb{H}$ . Applying Theorem 1.3 to the map  $\tilde{F}$  we obtain, for every  $\varepsilon > 0$ , a locally Lipschitz compact single-valued map  $\tilde{f} : \mathbb{H} \times [0, 1] \rightarrow \mathbb{H}$  such that

$$(*) \quad \tilde{f}(x, \lambda) \in \overline{\text{conv}}\tilde{F}(\overline{B_\varepsilon}(x) \times \overline{B_\varepsilon}(\lambda)) + B_\varepsilon(0) \text{ for all } x \in \mathbb{H}, \lambda \in [0, 1].$$

Let us fix  $\lambda \in [0, 1]$ . We shall show that  $N$  is an isolating neighborhood for the flows generated by  $\tilde{f}(\cdot, \lambda')$ , where  $\lambda' \in (\lambda - \varepsilon, \lambda + \varepsilon)$ , if  $\varepsilon$  is small enough. Assume that  $\overline{N} \subset B_r(0)$ .

Let us define a homotopy  $h : \mathbb{H} \times [0, 1] \multimap \mathbb{H}$  by the formula

$$h(x, s) = Lx + u(x)[(\overline{\text{conv}}\tilde{F}(\overline{B_s}(x) \times \overline{B_s}(\lambda)) + \overline{B_s}(0)) \cap \overline{\text{conv}}\tilde{F}(\overline{B_{2r}}(0) \times [0, 1])],$$

where  $u : \mathbb{H} \rightarrow \mathbb{R}$  is an Urysohn function such that  $u(x) = 1$  for  $|x| \leq r$  and  $u(x) = 0$  for  $|x| \geq 2r$ .

Since  $h$  is a family of multivalued  $L$ -vector fields, it generates a family of multivalued  $L$ -flows  $\eta(\cdot, s)$ . Moreover,  $h(\cdot, 0) = L + \tilde{F}(\cdot, \lambda)$ . By Prop.3.2 the mapping  $s \mapsto \text{Inv}(N, \eta(\cdot, s))$  is usc and  $\text{Inv}(N, \eta(\cdot, 0)) \subset \text{int } N$ . Therefore there exists  $s > 0$  such that for all  $s' \leq s$  we have  $\text{Inv}(N, \eta(\cdot, s')) \subset \text{int } N$ . If we choose  $0 < \varepsilon_\lambda < \frac{s}{2}$ , then for all  $\lambda' \in [\lambda - \varepsilon_\lambda, \lambda + \varepsilon_\lambda]$  we obtain by Theorem 1.3 that for  $\varepsilon < \varepsilon_\lambda$

$$\tilde{f}(x, \lambda') \in \overline{\text{conv}}\tilde{F}(\overline{B_\varepsilon}(x) \times \overline{B_\varepsilon}(\lambda')) + \overline{B_\varepsilon}(0) \subset \overline{\text{conv}}\tilde{F}(\overline{B_s}(x) \times \overline{B_s}(\lambda)) + \overline{B_s}(0).$$

We can assume that also  $B_s(N) \subset B_r(N) \subset B_{2r}(0)$ . It follows that the map  $L + \tilde{f}(\cdot, \lambda')$  is a selection of  $h(\cdot, s)$ . Therefore for the  $L$ -flow  $\psi(\cdot, \lambda')$  generated by the vector field  $L + \tilde{f}(\cdot, \lambda')$  we have the inclusion  $\text{Inv}(N, \psi(\cdot, \lambda')) \subset \text{int } N$ , i.e.,  $N$  is an isolating neighborhood.

Intervals  $I_\lambda = (\lambda - \varepsilon_\lambda, \lambda + \varepsilon_\lambda) \cap [0, 1]$  form an open covering of  $[0, 1]$ . Choosing a finite subcovering  $I_{\lambda_1}, \dots, I_{\lambda_k}$  we find  $\bar{\varepsilon} < \min\{\varepsilon_{\lambda_i}\}$  such that, for  $\tilde{f}$  satisfying  $(*)$  with  $\varepsilon = \bar{\varepsilon}$ , the set

$N$  is an isolating neighborhood for flows generated by  $L + \tilde{f}(\cdot, \lambda)$  for all  $\lambda \in [0, 1]$ . Thus by Prop. 3.7 the homotopy index  $h(N, \psi(\cdot, \lambda))$  does not depend on  $\lambda$ .

On the other hand, the approximation  $\tilde{f}$  can be taken with an additional condition satisfied:

$$\tilde{f}(\cdot, i) \in a(\tilde{F}(\cdot, i), \varepsilon), \text{ for } i = 0, 1.$$

In order to assure this condition is satisfied, one repeats the proof of Theorem 1.3 with the following modification: For  $(x, \lambda)$  with  $\lambda \notin \{0, 1\}$  we take  $B_{\delta(x, \lambda)}(x, \lambda)$  such that  $B_{\delta(x, \lambda)}(x, \lambda) \cap (\mathbb{H} \times \{0, 1\}) = \emptyset$ ,  $B_{\delta(x, 0)}(x, 0) \cap (\mathbb{H} \times \{1\}) = \emptyset$ ,  $B_{\delta(x, 1)}(x, 1) \cap (\mathbb{H} \times \{0\}) = \emptyset$  and for a locally finite covering  $\{V_s\}$  of  $\mathbb{H} \times [0, 1]$  we choose  $(x_s, \lambda_s) \in V_s$  such that  $\lambda_s = i, i \in \{0, 1\}$ , if  $V_s \cap (\mathbb{H} \times \{i\}) \neq \emptyset$ .

This finishes the proof. □

**Proposition 3.13** *Let  $\varphi : \mathbb{H} \times \mathbb{R} \rightarrow \mathbb{H}$  be an  $L$ -flow and let  $N_1, N_2, N$  be isolating neighborhoods for  $\varphi$  such that  $N_1 \cap N_2 = \emptyset$ ,  $N_1 \cup N_2 \subset N$  and  $\text{Inv}(N, \varphi) \subset N_1 \cup N_2$ . Then  $h(N, \varphi) = h(N_1, \varphi) \vee h(N_2, \varphi)$ .*

*Proof.* The property follows from the obvious observation that for each  $n$   $\mathbb{H}^n \cap N_1 \cap N_2 = \emptyset$  and thus the appropriate index pairs  $(P, Q)$  defining the classical Conley index for the isolating neighborhood  $N \cap \mathbb{H}^n$  can be chosen in the form of disjoint sums  $(P_1 \cup P_2, Q_1 \cup Q_2)$ , where  $(P_1, Q_1), (P_2, Q_2)$  are index pairs for  $N_1, N_2$ , respectively. The rest is the definition of the wedge sum of spectra (see [10] for details). □

In [6] a cohomological version of the Conley index for multivalued  $L$ -flows in a Hilbert space was established starting from the finite-dimensional case given in [16]. Instead of the homotopy type of index pairs the authors consider the Alexander-Spanier cohomology groups of these pairs. Since all the maps in the spectra are homotopy equivalences for  $n$  large enough, the inverse limit of the groups is well-defined

$$CH^q(N, \varphi) = \varprojlim \{H^{q+\rho(n)}(Y_n, Z_n), \tilde{\gamma}_n\}.$$

Similarly as in the single-valued case (see [10]) we obtain

**Proposition 3.14** *Let  $N$  be an isolating neighbourhood for a multivalued  $L$ -flow  $\varphi$ . Then the cohomology index from [6] is equal to the cohomology of our spectrum:  $CH^q(N, \varphi) = H^q(h(N, \varphi))$  for all  $q \in \mathbb{Z}$ .*

As a by-product we obtain that the cohomology index of Mrozek ([16]) for a multivalued flow  $\varphi$  in  $\mathbb{R}^n$  generated by a differential inclusion is just a cohomology of the homotopy index considered by Kunze in [13].

An interesting question appears: can the homotopy index  $h(N, \varphi)$  be described using a behavior of an  $L$ -vector field  $L + F$  on the boundary of a prescribed set of constraints? In [7] the author gave a positive answer for differential inclusions in a finite dimensional space. We show that an infinite-dimensional version of this result is possible.

We will need the following extension result on graph approximations. Recall that  $P_n : \mathbb{H} \rightarrow \mathbb{H}^n$  denotes the ortogonal projection.

**Lemma 3.15** *Let  $B = \overline{B(0, r)} \subset \mathbb{H}$  be a closed ball in  $\mathbb{H}$  and  $B^n := P_n(B) \subset B$ . Let  $F : B \rightarrow \mathbb{H}$  be a compact upper semicontinuous map with convex values and  $F_n := P_n \circ F$ . Then, for every  $\varepsilon > 0$  there exists  $n_0 \geq 1$  such that for any  $n \geq n_0$  there exists a  $\delta_n > 0$  such that any continuous (locally Lipschitz)  $\delta_n$ -approximation  $f : B^n \rightarrow \mathbb{H}^n$  of  $F_n$  over  $B^n$  may be extended to a continuous (locally Lipschitz)  $\varepsilon$ -approximation  $g : B \rightarrow E$  of  $F$ , i.e.,  $g|_{B^n} = f$ .*

*Proof.* Let  $\varepsilon > 0$  be arbitrary. We will proceed in several steps.

*Step 1.* There exists a locally Lipschitz function  $\eta : B \rightarrow (0, \infty)$  such that, for every  $x \in \mathbb{H}$ , there is  $x' \in B(x, \varepsilon)$  such that  $B_{\varepsilon/2}(F(B_{\eta(x)}(x))) \subset B_\varepsilon(F(x'))$ .

Indeed, for each  $x \in B$  we choose  $0 < r_x < \varepsilon$  such that  $F(B_{2r_x}(x)) \subset B_{\varepsilon/2}(F(x))$ , since  $F$  is usc, and take a locally finite and locally Lipschitz partition of unity  $\{\lambda\}_{s \in S}$  subordinated to the covering  $\{B(x, r_x)\}_{x \in B}$ . For each  $s \in S$  denote  $r_s := r_{x_s}$ , where  $\text{supp } \lambda_s \subset B(x_s, r_{x_s})$  for some  $x_s \in B$ .

Define  $\eta : B \rightarrow (0, \infty)$ ,

$$\eta(x) := \sum_{s \in S} \lambda_s(x) r_s, \quad x \in B.$$

Obviously,  $\eta$  is locally Lipschitz. Let  $x \in B$ , and let  $S_x := \{s \in S; \lambda_s(x) > 0\}$ . Since the partition of unity is locally finite, we can find  $s \in S_x$  such that  $\eta(x) \leq r_s$ . Hence,  $\|x - x_s\| < r_s < \varepsilon$  and, for any  $y \in B_{\eta(x)}(x)$ ,  $\|y - x_s\| \leq \|y - x\| + \|x - x_s\| < 2r_s$ . Therefore  $B_{\eta(x)}(x) \subset B_{2r_s}(x_s)$  and

$$F(B_{\eta(x)}(x)) \subset F(B_{2r_s}(x_s)) \subset B_{\varepsilon/2}(F(x_s)).$$

Hence, putting  $x' := x_s$ , we obtain

$$B_{\varepsilon/2}(F(B_{\eta(x)}(x))) \subset B_\varepsilon(F(x')).$$

*Step 2.* For any  $(x, y) \in B \times B$  we define

$$U(x, y) := [\eta^{-1}((\eta(x)/2, \infty)) \cap B_{\eta(x)/2}(x)] \times B_{\varepsilon/2}(y)$$

and an open neighborhood of the graph of  $F$

$$\mathcal{U} := \bigcup_{(x,y) \in \text{Gr}(F)} U(x, y).$$

Notice that, if  $W \subset B$  is any subset, and a continuous map  $f : W \rightarrow \mathbb{H}$  satisfies  $\text{Gr}(f) \subset \mathcal{U}$ , then, for each  $x \in W$ , there exists  $(x', y') \in \text{Gr}(F)$  such that  $(x, f(x)) \in U(x, y)$ . Hence,  $f(x) \in B_{\varepsilon/2}(y')$  and  $\|x - x'\| < \eta(x')/2 < \eta(x)$ . This implies that  $f(x) \in B_{\varepsilon/2}(F(B_{\eta(x)}(x))) \subset B_{\varepsilon}(F(B_{\varepsilon}(x)))$ .

*Step 3.* There is  $n_0 \geq 1$  such that  $\|P_n(y) - y\| < \varepsilon/4$  for every  $n \geq n_0$  and  $y \in F(B)$ . Fix  $n \geq n_0$ , and define

$$\tilde{U}(x, y) := [\eta^{-1}((\eta(x)/2, \infty)) \cap B_{\eta(x)/2}(x)] \times B_{\varepsilon/4}(y)$$

and an open neighborhood of the graph of  $F_n$  in  $B \times \mathbb{H}$

$$\mathcal{U}_n := \bigcup_{(x,y) \in \text{Gr}(F_n)} \tilde{U}(x, y).$$

Notice that, if  $(u, v) \in \tilde{U}(x, y)$ , then  $v \in B_{\varepsilon/4}(y)$  and  $y = P_n(y')$  for some  $y' \in F(x)$ . Hence,  $\|v - y'\| \leq \|v - y\| + \|y - y'\| < \varepsilon/2$ . It implies that  $(u, v) \in U(x, y')$  and, consequently,  $\mathcal{U}_n \subset \mathcal{U}$ .

Using the partition of unity technique, as in Step 1, it is easy to find a continuous function  $\rho' : \mathbb{H}^n \rightarrow (0, \varepsilon)$  such that, any  $\rho'(\cdot)$ -approximation  $f : B^n \rightarrow \mathbb{H}^n$  of  $F_n$ , i.e.,  $f(x) \in B_{\rho'(x)}(F_n(B_{\rho'(x)}(x)))$  for any  $x \in B^n$ , satisfies  $\text{Gr}(f) \subset \mathcal{U}_n$ . Analogously, let  $\theta : \mathbb{H} \rightarrow (0, \varepsilon)$  be a continuous function such that any  $\theta(\cdot)$ -approximation  $f : B \rightarrow \mathbb{H}$  of  $F$ , satisfies  $\text{Gr}(f) \subset \mathcal{U}$  (comp. [12], Prop. 1.2).

Since  $B^n$  is compact, there exists  $0 < \delta = \delta_n < \min\{\rho'(x); x \in B^n\}$ .

*Step 4.* Now, let  $f : B^n \rightarrow \mathbb{H}^n$  be any locally Lipschitz  $\delta$ -approximation of  $F_n$  over  $B^n$ . Then  $\text{Gr}(f) \subset \mathcal{U}_n$ . Since  $B^n$  is a Lipschitz retract of  $B$ , there exists a locally Lipschitz extension  $k : B \rightarrow \mathbb{H}$  of  $f$ . Since  $\mathcal{U}_n \subset \mathcal{U}$  and  $\mathcal{U}$  is open in  $B \times \mathbb{H}$ , there is an open neighborhood  $\Omega$  of  $B^n$  in  $B$  such that  $(x, k(x)) \in \mathcal{U}$  for every  $x \in \Omega$ . Hence,

$$k(x) \in B_{\varepsilon/2}(F(B_{\eta(x)}(x))) \text{ for every } x \in \Omega.$$

Take an open set  $\Omega_0 \subset B$  with  $B^n \subset \Omega_0 \subset \overline{\Omega_0} \subset \Omega$  and a locally Lipschitz Urysohn function  $\beta : \mathbb{H} \rightarrow [0, 1]$  with  $\beta(\overline{\Omega_0}) = \{1\}$  and  $\beta(\mathbb{H} \setminus \Omega) = \{0\}$ . Take any locally Lipschitz  $\theta(\cdot)$ -approximation  $h : B \rightarrow \mathbb{H}$  of  $F$ , where  $\theta(\cdot)$  is from Step 3. Then  $\text{Gr}(h) \subset \mathcal{U}$ . Define  $g : B \rightarrow \mathbb{H}$ ,  $g(x) := \beta(x)k(x) + (1 - \beta(x))h(x)$  for every  $x \in B$ . Obviously,  $g|_{B^n} = f$ .

Take any  $x$  with  $\beta(x) > 0$ . Then  $x \in \Omega$ , and

$$\{k(x), h(x)\} \subset B_{\varepsilon/2}(F(B_{\eta(x)}(x))).$$

By Step 1,  $\{k(x), h(x)\} \subset B_{\varepsilon}(F(x'))$  for some  $x'$  with  $\|x - x'\| < \varepsilon$ . By the convexity of values of  $F$ ,

$$g(x) \in B_{\varepsilon}(F(x')) \subset B_{\varepsilon}(F(B_{\varepsilon}(x))).$$

If  $\beta(x) = 0$ , then, since  $\theta(x) < \varepsilon$  for every  $x \in B$ ,  $g(x) = h(x) \in B_\varepsilon(F(B_\varepsilon(x)))$ , too. Hence,  $g$  is the required approximation.  $\square$

In the sequel we will use the following

**Theorem 3.16** (comp. [7], Theorem 4.1) *Let  $K = \overline{\text{Int}K}$  be a subset of a finite dimensional space  $E$  and  $F : E \multimap E$  be an usc map with compact convex values and a sublinear growth and such that  $K^-(F)$  is a closed strong deformation retract of some open neighborhood  $V \subset K$  of  $K^-(F)$  in  $K$ . Assume that  $\text{int } T_K(x) \neq \emptyset$  for every  $x \in K \setminus K^-(F)$ , and  $T_K(\cdot)$  is lsc outside  $K^-(F)$ .*

Then

$$h(\text{Inv}(K, \varphi), \varphi) = [K/K^-(F), [K^-(F)]],$$

where  $\varphi$  is a multivalued flow generated by  $F$ , and  $h(\text{Inv}(K, \varphi), \varphi)$  is defined, if  $K$  is an isolating neighborhood, as the Conley index for any flow generated by a sufficiently close Lipschitz approximation of  $F$ .

Here  $T_K(x)$  denotes the Bouligand tangent cone:

$$T_K(x) := \{v \in \mathbb{R}^n \mid \liminf_{h \rightarrow 0^+} \frac{\text{dist}(x + hv, K)}{h} = 0\}.$$

Let  $L + F : \mathbb{H} \multimap \mathbb{H}$  be a multivalued  $L$ -vector field, and let  $\varphi$  be an  $L$ -flow generated by  $L + F$ . On the boundary of a set  $K \subset \mathbb{H}$  of constraints we consider the following *exit set*:

$$K^-(L + F) := \{x_0 \in \partial K \mid \forall x \in S(x_0) \forall t > 0 : x([0, t]) \not\subset K\}.$$

It means that all trajectories starting at points in  $K^-(L + F)$  immediately leave the set  $K$ . Assume that  $K$  is an isolating neighborhood for  $\varphi$ .

Suppose that the pair  $(K, K^-(L + F))$  generates a spectrum  $(K_n/K_n^-)$ , where  $K_n^-$  is the exit set for  $K_n = K \cap \mathbb{H}^n$  with respect to  $L + P_n F$ . Moreover, for some  $N \geq 1$  and each  $n \geq N$ , let the following regularity conditions be satisfied:

- (H1) Each  $K_n$  is *epi-Lipschitz* outside  $K_n^-$ , i.e.,  $\text{int } T_{K_n}(x) \neq \emptyset$  for every  $x \in K_n \setminus K_n^-$ .
- (H2)  $K_n$  is *sleek* outside  $K_n^-$ , i.e.,  $T_{K_n}(\cdot)$  is lsc on  $K_n \setminus K_n^-$ .
- (H3)  $K_n^-$  is a strong deformation retract of some open neighborhood  $V_n$  of  $K_n^-$  in  $K_n$ .

Denote by  $[K, L + F]$  the homotopy type of the spectrum  $(K_n/K_n^-)$ .

**Theorem 3.17** *Under the above assumptions,*

$$h(K, \varphi) = [K, L + F].$$

*Proof.* Let  $\varepsilon > 0$  be such that  $h(K, \varphi) := h_{\mathcal{LS}}(K, \varphi_f)$ , for any  $f \in a(F, \varepsilon)$ . Let  $B = \overline{B(0, r)}$  be a ball in  $\mathbb{H}$  such that  $K \subset \text{int } B$ , and let  $n_0 \geq N$  be such that  $K_n$  is an isolating neighborhood for each  $n \geq n_0$ , and  $n_0$  is as in Lemma 3.15. We want to find  $f \in a(F, \varepsilon)$  such that  $h_{\mathcal{LS}}(K, \varphi_f)$  is a homotopy type of a spectrum  $(Y_n/Z_n)_{n \geq n_0}$ , and  $([Y_n/Z_n, [Z_n]]) = ([K_n/K_n^-, [K_n]])$ .

From Theorem 3.16 it follows that there exists a  $\delta_{n_0}$ -approximation  $g_{n_0} : B^{n_0} \rightarrow \mathbb{H}^{n_0}$  of  $P_{n_0}F|_{B^{n_0}}$  such that its Conley index  $[Y_{n_0}/Z_{n_0}, [Z_{n_0}]]$  equals  $[K_{n_0}/K_{n_0}^-, [K_{n_0}]]$ . We extend  $g$  to an  $\varepsilon$ -approximation  $f : B \rightarrow \mathbb{H}$  of  $F$ . Since the spectrum  $(Y_n/Z_n)_{n \geq n_0}$  for  $L + f$  is uniquely determined up to a homotopy type by  $(Y_{n_0}/Z_{n_0}, [Z_{n_0}])$ , and  $(Y_{n_0}/Z_{n_0}, [Z_{n_0}])$  is homotopy equivalent to  $(K_{n_0}/K_{n_0}^-, [K_{n_0}])$ , we obtain  $h(K, \varphi) = [K, L + F]$ .  $\square$

## 4 Conley index for finite dimensional gradient differential inclusions

Let  $L : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a linear operator, and let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be a locally Lipschitz function satisfying

$$(3) \quad \sup_{y \in \partial f(u)} |y| \leq c(1 + |u|) \quad \text{for some } c \geq 0 \text{ and every } u \in \mathbb{R}^d.$$

Then the function  $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}$ ,

$$(4) \quad \Phi(u) = \frac{1}{2} \langle Lu, u \rangle + f(u), \quad \text{for } u \in \mathbb{R}^d,$$

is locally Lipschitz, and the Clarke generalized gradient

$$F(u) := \partial\Phi(u) = Lu + \partial f(u)$$

is well defined (see [1], [4] for definitions and properties of the gradient). Moreover,  $F : \mathbb{R}^d \multimap \mathbb{R}^d$  is usc with compact convex values and of sublinear growth. Hence, the differential inclusion  $\dot{x} \in F(x)$  generates a multivalued admissible flow (see Preliminaries). We say that  $\dot{x} \in \partial\Phi(x)$  is a *gradient differential inclusion*.

Assume that  $F : \mathbb{R}^d \multimap \mathbb{R}^d$  is of the form  $F(u) = Lu + \varphi(u)$  for some usc map  $\varphi$  with compact convex values and sublinear growth. We say that  $F$  has a *variational structure*, if there exists a locally Lipschitz function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  such that  $\partial\Phi(u) \subset F(u)$ , where  $\Phi$  is defined in (4). As we will see in the sequel, multivalued maps with a variational structure plays an important role in our investigations. If  $\mathbb{E}$  is a Hilbert space,  $P^d : \mathbb{H} \rightarrow \mathbb{H}^d$  is the orthogonal finite-dimensional projection, and  $F : \mathbb{H} \multimap \mathbb{H}$  is of the form  $F(u) = Lu + \partial f(u)$  for some linear bounded operator  $L : \mathbb{H} \rightarrow \mathbb{H}$  with  $L(\mathbb{H}^d) \subset \mathbb{H}^d$ , and a locally Lipschitz map  $f : \mathbb{H} \rightarrow \mathbb{R}$ ,

then the map  $F_d : \mathbb{H}^d \multimap \mathbb{H}^d$ ,  $F_d(x) := Lx + P_d(\partial f(i_d(x)))$ , where  $i_d : \mathbb{H}^d \hookrightarrow \mathbb{H}$  is the inclusion map, has a variational structure, since  $Lx + \partial(f \circ i_d)(x) \subset Lx + P_d(\partial f(i_d(x)))$  (see [1],[4]). Moreover,  $F_d$  need not be a generalized gradient of any locally Lipschitz function.

**Example 4.1** Consider a 1-Lipschitz function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,

$$f(x, y) = \begin{cases} |x| & \text{for } |x| \leq |y|, \\ |y| & \text{otherwise.} \end{cases}$$

Obviously,  $\partial(f \circ i)(x) = \{0\}$ , where  $i(x) := (x, 0)$ , while  $F_1(x) = P_1(\partial f(x, 0)) = \{0\}$  for  $x \neq 0$  and  $F_1(0) = P_1(\partial f(0, 0)) = [-1, 1]$ . Hence,  $F_1$  is not a generalized gradient of any locally Lipschitz function.

For multivalued flows generated by differential inclusions with compact convex valued right-hand sides a homotopy index has been constructed (see [13]). Below we repeat the construction and investigate the index in the context of gradient differential inclusions.

Let  $K$  be an isolated invariant set, and let  $N$  be its isolating neighborhood, i.e.,  $K = \text{Inv}(N, F) := \text{Inv}(N, \varphi)$ , where  $\varphi$  is a multivalued flow generated by the inclusion  $\dot{x} \in F(x)$ . By Proposition 3.2 it follows that  $K$  is a compact subset of  $\text{int } N$  (see also [13], Lemma 5.2.3). Now it is easy to prove ([13], Lemmas 5.2.5 and 5.3.1) that for each  $\varepsilon > 0$  there exists a smooth  $\varepsilon$ -approximation of  $F$  generating a global flow on  $\mathbb{R}^d$ , and there is  $0 < \delta < \varepsilon$  such that for each two such smooth  $\delta$ -approximations  $g_1, g_2$  the set  $N$  is an isolating neighborhood and they generate global flows with the same Conley homotopy index  $h(\text{Inv}(N, g_1), g_1) = h(\text{Inv}(N, g_2), g_2)$ .

**Definition 4.2** Let  $K$  be an isolated compact invariant set, and  $K = \text{Inv}(N, F)$ . By the *homotopy index* of  $K$  we mean the homotopy type

$$H(K, F) := h(\text{Inv}(N, g), g)$$

for sufficiently near smooth approximation  $g$  of  $F$ .

One can easily check that this definition does not depend on the choice of an isolating neighborhood  $N$  of  $K$  and the choice of  $g$ .

For  $F = \partial\Phi$ , where  $\Phi$  is of the form (4), the index can be described using smooth approximations of the given locally Lipschitz map  $f$ , as we can see below.

We say that  $\tilde{f} : U \rightarrow \mathbb{R}$  is a  $C_\varepsilon^\infty$ -approximation of a locally Lipschitz function  $f : U \rightarrow \mathbb{R}$ , where  $U \subset \mathbb{R}^d$  is open and  $\varepsilon : U \rightarrow (0, +\infty)$  is a continuous map, if

- (i)  $|f(x) - \tilde{f}(x)| < \varepsilon(x)$ , for every  $x \in U$ ,

(ii)  $\nabla \tilde{f}$  is an  $\varepsilon$ -approximation<sup>4</sup> of  $\partial f$ .

In the sequel we will apply only a simplified version of the following result with a constant function  $\varepsilon(x) = \varepsilon > 0$ .

**Proposition 4.3** ([3], Theorem 3.7) *Suppose  $f : U \rightarrow \mathbb{R}$ , where  $U \subset \mathbb{R}^d$  is open, is a locally Lipschitz function. Then for any continuous map  $\varepsilon : U \rightarrow (0, +\infty)$  there exists a  $C_\varepsilon^\infty$ -approximation of  $f$ .*

Assume that  $F : \mathbb{R}^d \multimap \mathbb{R}^d$ ,  $F(u) = Lu + \varphi(u)$  has a variational structure with a multivalued selection  $\partial\Phi = L + \partial f$ . If  $K = \text{Inv}(N, F)$ , then

$$H(K, F) = H(\text{Inv}(N, \partial\Phi), \partial\Phi) = h(\text{Inv}(N, \nabla \tilde{f}), \tilde{f})$$

for every sufficiently near  $C_\varepsilon^\infty$ -approximation  $\tilde{f}$  of  $f$ , since near approximations of  $\partial\Phi$  are near approximations of  $F$ .

The index given in Definition 4.2 has standard important properties collected in the following proposition.

**Proposition 4.4** *comp.* ([13], Theorems 5.3.1, 5.3.2, 5.3.4)

(Pr1) (EXISTENCE)<sup>5</sup> *If  $H(K, F) \neq \bar{0}$ , then  $K \neq \emptyset$ , i.e., there is a full trajectory in  $N$ , where  $N$  is an isolating neighborhood of  $K$ .*

(Pr2) (ADDITIVITY) *If  $K_1, K_2$  are disjoint isolated invariant sets, then  $K = K_1 \cup K_2$  is an isolated invariant set and*

$$H(K, F) = H(K_1, F) \vee H(K_2, F).$$

(Pr3) (CONTINUATION) *Let  $F : [0, 1] \times \mathbb{R}^d \multimap \mathbb{R}^d$  be a compact convex valued usc map with  $\sup_{y \in F(\lambda, u)} |y| \leq c(1 + |u|)$  for some  $c > 0$  and every  $(\lambda, u) \in [0, 1] \times \mathbb{R}^d$ . If  $K_\lambda = \text{Inv}(N, F(\cdot, \lambda)) \subset \text{int } N$  for every  $\lambda \in [0, 1]$ , then  $H(K_\lambda, F(\cdot, \lambda))$  is independent of  $\lambda \in [0, 1]$ .*

*Remark.* In the continuation property Theorem 5.3.4 in [13] the author assume that  $F(\lambda, \cdot)$  is usc, and  $F(\cdot, u)$  is continuous for every  $u \in \mathbb{R}^d$  and usc uniformly on bounded subsets of  $\mathbb{R}^d$ . As the author proves in Lemma 5.3.3, under these assumptions the map  $F$  is jointly

<sup>4</sup>It means that  $d_{F(B(x, \varepsilon(x)))}(f(x)) < \varepsilon(x)$  for every  $x \in U$ .

<sup>5</sup>This is also called the Ważewski property.



usc in every  $(\lambda, u)$ , so the assumption in (Pr3) is weaker than in Theorem 5.3.4 in [13]. Our formulation is suitable for gradient inclusions. Indeed, if  $\Phi : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}$  is of the form

$$(5) \quad \Phi(\lambda, u) = \frac{1}{2} \langle L_\lambda u, u \rangle + f(\lambda, u), \quad \text{for } u \in \mathbb{R}^d,$$

where  $\lambda \mapsto L_\lambda(\cdot)$  and  $f$  are locally Lipschitz, then the generalized gradient of  $\Phi$  with respect to the second variable satisfies

$$\partial_u \Phi(\lambda, u) = L_\lambda u + \partial_u f(\lambda, u),$$

and it is jointly usc in every  $(\lambda, u) \in [0, 1] \times \mathbb{R}^d$ . Note also that the continuation property is true if an isolating neighborhood ranges during a homotopy.

**Example 4.5** One can check that for a function  $\Phi : [0, 1] \times \mathbb{R}$ ,  $\Phi(\lambda, u) := f(\lambda, u) = |u|^{1+\lambda}$ , i.e., with  $L \equiv 0$ , the generalized gradient  $\partial_u \Phi$  is not continuous with respect to the first variable.

*Proof of Proposition 4.4.* For (Pr1) and (Pr2) see [13]. We prove (Pr3).

Let  $r > 0$  be such that  $N \subset B(0, r)$ . By the Cellina approximation theorem and a standard mollifiers technique (see [13], Lemma 5.3.4) it follows that for every  $\varepsilon > 0$  there exists a continuous bounded map  $f : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  such that  $f(\lambda, \cdot) \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$  and

$$f(\lambda, u) \in F([\lambda - \varepsilon, \lambda + \varepsilon] \cap [0, 1]) \times B(u, \varepsilon) + B(0, \varepsilon)$$

for every  $(\lambda, u) \in [0, 1] \times \overline{B(0, r)}$ . Moreover, using Lemma 2.1 in [7] we can choose an approximation  $f$  with

$$f(\lambda, u) \in F(\{\lambda\} \times B(u, \varepsilon)) + B(0, \varepsilon),$$

i.e., such that  $f(\lambda, \cdot)$  is an  $\varepsilon$ -approximation of  $F(\lambda, \cdot)$ . Since  $K_\lambda \subset \text{int } N$  for every  $\lambda \in [0, 1]$  and the set  $K = \bigcup_{\lambda \in [0, 1]} K_\lambda$  is closed (see Proposition 3.2),  $K$  is a compact subset of  $\text{int } N$ .

Hence, there is  $\delta > 0$  such that for every  $\delta$ -approximation  $f$  of  $F$  chosen as above one has  $\text{Inv}(N, f(\lambda, \cdot)) \subset \text{int } N$  for every  $\lambda \in [0, 1]$  (see [13], Lemma 5.3.5). Continuity of  $f$  implies that the corresponding flows  $\pi_\lambda$  continuously depend on  $\lambda$ . From the continuation property of the homotopy Conley index for flows one obtains

$$H(K_0, F(0, \cdot)) = h(\text{Inv}(N, f(0, \cdot)), f(0, \cdot)) = h(\text{Inv}(N, f(1, \cdot)), f(1, \cdot)) = H(K_1, F(1, \cdot)),$$

and the proof is finished. □

**Corollary 4.6** *If  $K_1, K_2$  are disjoint isolated invariant sets,  $K_1 \cup K_2 \subset K$ , and  $K = \text{Inv}(N, F)$ . If  $H(K, F) \neq H(K_1, F) \vee H(K_2, F)$ , then there exists a full trajectory in  $K$  which is not contained in  $K_1 \cup K_2$ . In particular, it is the case if  $H(K, F) = \bar{0}$  and  $H(K_i, F) \neq \bar{0}$ , for some  $i \in \{1, 2\}$ .*

**Corollary 4.7** *Assume that  $F : \mathbb{R}^d \multimap \mathbb{R}^d$ ,  $F(u) = Lu + \varphi(u)$  has a variational structure with a multivalued selection  $\partial\Phi = L + \partial f$ . Let  $p_1, p_2$  be critical points of  $\Phi$  in an isolating neighborhood  $N$  for  $F$ , which are isolated invariant sets for  $F$ . Assume that  $\langle \partial\Phi(u), \partial\Phi(u) \rangle^- > 0$  for every  $u \in N \setminus \{p_1, p_2\}$ , where  $\langle \partial\Phi(u), \partial\Phi(u) \rangle^- := \min\{\langle y, y' \rangle \mid y, y' \in \partial\Phi(u)\}$ .*

*If  $H(\text{Inv}(N, F), F) \neq H(\{p_1\}, F) \vee H(\{p_2\}, F)$ , then there exists a heteroclinic or homoclinic nontrivial orbit in  $N$ . In particular, if  $(\{p_1\}, \{p_2\})$  is an attractor-repeller pair, then there is a trajectory joining the equilibria.*

*Proof.* Notice that  $p_1, p_2$  are the only critical points of  $\Phi$  in  $N$ . From Lemma 4.5 in [7] it follows that  $\{p_1\}$  and  $\{p_2\}$  are isolated invariant sets for  $\partial\Phi$ . Therefore  $H(\{p_i\}, F) = H(\{p_i\}, \partial\Phi)$  for  $i \in \{1, 2\}$ . Since  $\partial\Phi$  is a selection of  $F$ , we have

$$\begin{aligned} H(\text{Inv}(N, \partial\Phi), \partial\Phi) &= H(\text{Inv}(N, F), F) \neq \\ &\neq H(\{p_1\}, F) \vee H(\{p_2\}, F) = H(\{p_1\}, \partial\Phi) \vee H(\{p_2\}, \partial\Phi). \end{aligned}$$

Now, Corollary 4.6 applies, and there is a full trajectory  $x(\cdot)$  for  $\partial\Phi$  (so, for  $F$ ) in  $N$  with  $x(0) = x_0 \in N$ . Lemma 4.4 in [7] shows that  $\omega(x_0) \cup \alpha(x_0) \subset \{p_1, p_2\}$ . The proof is finished.  $\square$

*Remark.* Note that the only critical points  $p_1, p_2$  of  $\Phi$  need not be isolated sets for  $F$ . For example, we can examine the map  $F : \mathbb{R} \multimap \mathbb{R}$ ,

$$F(x) = \begin{cases} 1 & \text{for } |x| > 1, \\ [-1, 1] & \text{for } |x| \leq 1, \end{cases}$$

with the selection  $\partial\Phi$ , where

$$\Phi(x) = \begin{cases} x + 2 & \text{for } x < -1, \\ -x & \text{for } |x| \leq 1, \\ x - 2 & \text{for } x > 1. \end{cases}$$

Obviously,  $\Phi$  has two critical points  $-1$  and  $1$  which are not isolated sets for  $F$ . Notice that  $H(\text{Inv}([-2, 2], F), F) = \bar{0}$  and  $H(\{-1\}, F) \vee H(\{1\}, F) = \Sigma^0 \vee \Sigma^1 \neq \bar{0}$ . Moreover,  $\langle \partial\Phi(x), \partial\Phi(x) \rangle^- = 1 > 0$  for  $x \notin \{-1, 1\}$ . One can easily find a trajectory joining  $1$  with  $-1$ .

## References

- [1] J.P. Aubin, *Optima and Equilibria*, Springer, Berlin/Heidelberg 1993.
- [2] A. Cellina, *Approximation of set-valued mappings and fixed point theorems*, Ann. Mat. Pura Appl. 82(1969), 17-24.
- [3] A. Ćwiszewski, W. Kryszewski, *Approximate smoothings of locally Lipschitz functionals*, Bolletino U.M.I. (8) 5-B (2002), 289-320.

- [4] F.H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, New York 1983.
- [5] C. Conley, *Isolated Invariant Sets and the Morse Index*, CBMS 38 (1978), AMS Publ., Providence/RI.
- [6] Z. Dzedzej, W. Kryszewski, *Conley type index applied to Hamiltonian inclusions*, J. Math. Anal. Appl. 347(2008), 96-112.
- [7] G. Gabor, *Homotopy index for multivalued flows on sleek sets*, Set-Valued Anaysis 13(2005),125-149.
- [8] K. Gęba, M. Izydorek, A. Pruszko, *The Conley index in Hilbert spaces and its applications*, Studia Math. 134(1999), 217-233.
- [9] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings*, Springer, Dordrecht/Boston/London 2006.
- [10] M. Izydorek, *A cohomological Conley index in Hilbert spaces and applications to strongly indefinite problems*, J. Diff. Equations 170(2001), 22-50.
- [11] M. Kamenski, V. Obukhovskii, P. Zecca, *Condensing Multivalued Maps and Semilinear Inclusions in Banach Spaces*, Walter de Gruyter, Berlin 2001.
- [12] W. Kryszewski, *Graph-approximation of set-valued maps on noncompact domains*, Topology Appl. 83(1998), 1-21.
- [13] M. Kunze, *Non - Smooth Dynamical Systems*, Lecture Notes in Math. 1744, Springer 2000.
- [14] M. Kunze, T. Küpper, Y. Li, *On Conley index theory for non-smooth dynamical systems*, Differential Integral Equations 13(2000), 479-502.
- [15] K. Mischaikow, M. Mrozek, *Conley Index*, Handbook of Dynamical Systems, Vol 2, (ed. B. Fiedler), 393-460.
- [16] M. Mrozek, *A cohomological index of Conley type for multivalued admissible flows*, J. Differential Equations 84 (1990), 15-51.
- [17] D. Salamon, *Connected simple systems and the Conley index of isolated invariant sets*, Trans. Amer. Math. Soc. **291** (1985), 1-41.

