

The Conley index, cup-length and bifurcation

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Abstract. A module structure of the cohomology Conley index is used to define a relative cup-length. This invariant is applied then to prove a multiplicity theorem for periodic solutions to Hamiltonian systems.

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1. Introduction

In this paper, we consider a module structure of the Conley index of smooth flows in \mathbb{R}^n . If $\bar{\Omega}$ is an isolating neighbourhood and (P_1, P_2) is a regular index pair in Ω , then the cohomology $H^*(P_1, P_2)$ is a module over $H^*(\bar{\Omega})$. We define a notion of relative cup-length of $H^*(P_1, P_2)$ over $H^*(\bar{\Omega})$. This notion can be used to derive several results on nontrivial structure of invariant sets. As an example we prove a theorem on a minimal number of periodic solutions to Hamiltonian systems. A natural action of the group S^1 on the space of periodic functions is being used. Some other applications of this tool to bifurcation theory are presented in the PhD thesis of the last author [17].

It is worth mentioning that the concept is not completely new. One can find a cup-length applied to Conley theory in [2] and [4]. A variant of a relative version appeared in [16]. We believe that our approach should also be useful for other problems considered in the bifurcation theory.

The paper is organized as follows. Section 2 contains an abstract algebraic definition of the relative cup-length and simple properties. In Section 3 we recall basic concepts from Conley index theory (main source is [13]) and specify the abstract notion to it. In Section 4 we prove an abstract result on

a number of critical points for gradient-like flows. Section 5 contains a reduction procedure for bifurcation problems. In the latter sections this procedure is applied to Hamiltonian systems.

2. Relative cup-length

Throughout this section we assume that $A \subset X \subset Y$ are compact metric spaces and we denote by H^* the Alexander–Spanier cohomology with the coefficients in the fixed abelian group G . The cup product (see [15, Sec. 5.6])

$$\smile : H^k(X) \times H^l(X, A) \rightarrow H^{k+l}(X, A)$$

endows $H^*(X, A)$ with a structure of an $H^*(X)$ -module. If $k : X \rightarrow Y$ denotes the inclusion map, then the formula

$$\beta \cdot \alpha := k^*(\beta) \smile \alpha$$

defines on $H^*(X, A)$ a structure of $H^*(Y)$ -module. The following remark is a simple consequence of the naturality property of the cup product (see [6, Prop. 3.10]).

Remark 2.1. If $B \subset A$ is compact, then

$$H^*(X, A) \rightarrow H^*(X, B) \rightarrow H^*(A, B)$$

is an exact sequence of $H^*(Y)$ -modules, where the maps are induced by inclusions.

Definition 2.1. Let $\beta \in H^p(Y)$, $p > 0$, $\beta \neq 0$, and let $A \subset X \subset Y$ be CW-complexes. The *relative cup-length* of β with respect to (X, A) is the number $\chi(\beta; X, A) \in \mathbb{N}$ defined as follows:

- $\chi(\beta; X, A) = 0$ if $H^*(X, A) = 0$;
- $\chi(\beta; X, A) = 1$ if $H^*(X, A) \neq 0$ and $\beta \cdot \alpha = 0$ for every $\alpha \in H^*(X, A)$;
- $\chi(\beta; X, A) = k \geq 2$ if there exists $\alpha_0 \in H^*(X, A)$ such that $\beta^{k-1} \cdot \alpha_0 \neq 0$ and $\beta^k \cdot \alpha = 0$ for every $\alpha \in H^*(X, A)$.

Definition 2.2. The *relative cup-length* of the $H^*(Y)$ -module $H^*(X, A)$ is the number given by

$$\Upsilon(X, A; Y) := \max\{\chi(\beta; X, A); 0 \neq \beta \in H^k(Y), k > 0\}.$$

If $H^k(Y) = \{0\}$ for $k > 0$ but $H^*(X, A)$ is nonzero, we set $\Upsilon(X, A; Y) = 1$, and if $H^l(X, A)$ are trivial for all $l \geq 0$, then $\Upsilon(X, A; Y) := 0$.

Lemma 2.2. If $B \subset A \subset X \subset Y$, then

$$\Upsilon(X, B; Y) \leq \Upsilon(X, A; Y) + \Upsilon(A, B; Y).$$

Proof. Let $k_1 = \Upsilon(X, A; Y)$, $k_2 = \Upsilon(A, B; Y)$, $0 \neq \alpha \in H^p(X, B)$, $p \geq 0$, $0 \neq \beta \in H^q(Y)$, $q > 0$. Consider the following inclusions:

$$i : (X, B) \rightarrow (X, A), \quad j : (A, B) \rightarrow (X, B).$$



Since $k_2 = \Upsilon(A, B; Y)$, $j^*(\beta^{k_2} \cdot \alpha) = 0$. By Remark 2.1, there exists $\gamma \in H^*(X, A)$ such that $\beta^{k_2} \cdot \alpha = i^*(\gamma)$. Therefore,

$$\beta^{k_1+k_2} \cdot \alpha = i^*(\beta^{k_1} \cdot \gamma).$$

But $\beta^{k_1} \cdot \gamma = 0$ by definition of k_1 , and thus $\beta^{k_1+k_2} \cdot \alpha = 0$. This means that

$$\Upsilon(X, B; Y) \leq k_1 + k_2,$$

which ends the proof. □

Lemma 2.3. *If $A \subset X \subset Y_1 \subset Y_2$, then*

$$\Upsilon(X, A; Y_2) \leq \Upsilon(X, A; Y_1).$$

Proof. Consider the following inclusions:

$$s : X \hookrightarrow Y, \quad k : A \hookrightarrow X, \quad t : A \hookrightarrow Y.$$

If $\beta \in H^{>0}(Y_2)$, $\alpha \in H^*(X, A)$, then $\beta\alpha = t^*(\beta) \smile \alpha = k^*(s^*(\beta)) \smile \alpha$. Hence $\chi(X, A; \beta) = \chi(X, A; s^*(\beta))$ for all $\beta \in H^{>0}(Y_2)$. Since $t = k \circ s$, the condition $t^*(\beta) \smile \alpha \neq 0$ implies $s^*(\beta) \smile \alpha \neq 0$, and our inequality follows. □

Recall that the *cross product* is defined by the formula

$$a \times b := p_1^*(a) \smile p_2^*(b),$$

where p_1, p_2 denote projections $(X, A) \times (Y, B)$ onto (X, A) and (Y, B) . For algebraic properties of the maps

$$\begin{aligned} \times &: H^k(X; R) \times H^l(Y; R) \rightarrow H^{k+l}(X \times Y; R), \\ \times &: H^k(X, A; R) \times H^l(Y, B; R) \rightarrow H^{k+l}(X \times Y, X \times B \cup A \times Y; R) \end{aligned}$$

see, e.g., [6] or [1, pp. 240–242].

Let $\sigma :=$ generator $H^1(I, \partial I)$, $I := [-1, 1]$.

The formula

$$\mathfrak{S}(a) := a \times \sigma$$

defines a mapping

$$\mathfrak{S} : H^k(X, A) \rightarrow H^{k+1}((X, A) \times (I, \partial I)) = H^{k+1}(X \times I, X \times \partial I \cup A \times I).$$

The following lemma holds (cf. [6, Thm. 3.21] for more general version).

Lemma 2.4. *If $X \subset Y$, then \mathfrak{S} is an isomorphism of $H^*(Y)$ -modules. More exactly,*

$$\mathfrak{S}(b \cdot a) = p^*(b) \cdot \mathfrak{S}(a),$$

where p denotes the projection $Y \times I$ onto Y .

Proof. Let $b \in H^*(Y)$, $a \in H^*(X, A)$. Consider the following projections:

$$\begin{aligned} p_1 &: (X \times I, A \times I) \rightarrow (X, A), \\ p_2 &: (X \times I, X \times \partial I) \rightarrow (I, \partial I), \\ \bar{p}_1 &: X \times I \rightarrow X. \end{aligned}$$



The following diagram is commutative, where $i_1(x, t) = (i(x), t)$:

$$\begin{array}{ccc} X \times I & \xrightarrow{i_1} & Y \times I \\ \downarrow \bar{p}_1 & & \downarrow p \\ X & \xrightarrow{i} & Y \end{array}$$

Using this diagram and the naturality and associativity properties of the cup product (see [1, p. 239]), we obtain

$$\begin{aligned} \mathfrak{S}(b \cdot a) &= (b \cdot a) \times \sigma = p_1^*(i^*(b) \smile a) \smile p_2^*(\sigma) = \bar{p}_1^*(i^*(b)) \smile p_1^*(a) \smile p_2^*(\sigma) \\ &= \bar{p}_1^*(i^*(b)) \smile \mathfrak{S}(a) = i_1^*(p^*(b)) \smile \mathfrak{S}(a) = p^*(b) \cdot \mathfrak{S}(a). \quad \square \end{aligned}$$

Theorem 2.5. *The following formula holds:*

$$\Upsilon((X, A) \times (I, \partial I); Y) = \Upsilon(X, A; Y).$$

Proof. Let us notice that formally $X \times I \subset Y \times I$ and thus $H^*(X \times I, X \times \partial I \cup A \times I)$ is an $H^*(Y \times I)$ -module, but $p^* : H^*(Y) \rightarrow H^*(Y \times I)$ is an isomorphism which gives the naturally isomorphic $H^*(Y)$ -module structure: $b \odot a := p^*(b) \cdot a$ for $b \in H^*(Y)$ and $a \in H^*(X \times I, X \times \partial I \cup A \times I)$. Taking this into account, the desired equality follows directly from Lemma 2.4. \square

3. Conley index and the relative cup-length

In this section, we recall the basic notions of the Conley index theory; the reader can refer to [9] and [13] for details. Let X be a locally compact metric space. A continuous map $\eta : X \times \mathbb{R} \rightarrow X$ is a *flow* if it satisfies the conditions

$$\begin{aligned} \eta(x, 0) &= x, \\ \eta(x, t + s) &= \eta(\eta(x, t), s). \end{aligned}$$

A set $S \subset X$ is an *invariant set* for the flow η if

$$\eta(S, \mathbb{R}) := \bigcup_{t \in \mathbb{R}} \eta(S, t) = S.$$

For an arbitrary set $N \subset X$ one can define its invariant part

$$\text{Inv}(N, \eta) := \{x \in N \mid \eta(x, \mathbb{R}) \subset N\}.$$

A compact set $N \subset X$ is an *isolating neighbourhood* if $\text{Inv}(N, \eta) \subset \text{int } N$. A set S is called an *isolated invariant set* if there is an isolating neighbourhood N such that $S = \text{Inv}(N, \varphi)$. A flow $\eta : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is *generated* by a smooth vector field $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ if $\eta(x, t)$ is the solution of the Cauchy problem $\dot{u} = -F(u), u(0) = x$ evaluated at time t . Such a flow is a *gradient flow* if $F = \nabla f$ for some smooth function $f : \mathbb{R}^n \rightarrow \mathbb{R}$.

Let S be an isolated invariant set for the flow η . A compact pair $N_0 \subset N_1$ of subsets of X is called an *index pair* for S if the following hold:

- (a) $\overline{\text{int}(N_1 \setminus N_0)}$ is an isolating neighbourhood for S ;
- (b) N_0 is *positively invariant* relative to N_1 ; i.e., if $x \in N_0$ and $\eta(x, [0, t]) \subset N_1$, then $\eta(x, [0, t]) \subset N_0$;



- (c) N_0 is an *exit set* for N_1 ; i.e., if $x \in N_1$ and $t_1 > 0$ such that $\eta(x, t_1) \notin N_1$, then there exists $t_0 \in [0, t_1]$ for which $\eta([0, t_0], x) \subset N_1$ and $\eta(x, t_0) \in N_0$.

The following result implies the correctness of the definition of the *homotopy Conley index* (cf. [9, Thms. 2.2.1 and 2.2.2] or [13, Thms. 23.7 and 23.12]).

Theorem 3.1. *Let S be an isolated invariant set for the flow η . Then there exists an index pair for S . Moreover, if (N_1, N_0) and (N'_1, N'_0) are index pairs for S , then the pointed topological spaces*

$$(N_1/N_0, [N_0]) \quad \text{and} \quad (N'_1/N'_0, [N'_0])$$

are homotopically equivalent.

Definition 3.1. Let S be an isolated invariant set for the flow η . The *homotopy Conley index* of S is the homotopy type of the pointed space

$$h(S) = h(S, \eta) := [N_1/N_0, [N_0]],$$

where (N_1, N_0) is an index pair for S .

It is useful to consider the *cohomology Conley index* defined by

$$CH^*(S) := H^*(N, L) = H^*(N/L),$$

where H^* denotes the Alexander–Spanier cohomology and (N, L) is an index pair for S . The last equality means that we identify $H^*(N, L)$ and $H^*(N/L)$ via the isomorphism induced by the quotient map.

It is convenient to extend the index to an index of isolating neighbourhood: if N is an isolating neighbourhood for η , then the *homotopy* (resp., *cohomology*) *Conley index* of N is defined as

$$h(N) = h(N, \eta) := h(\text{Inv}(N, \eta)),$$

(resp., $CH^*(N) = CH^*(N, \eta) := CH^*(\text{Inv}(N, \eta))$).

Before giving the definition of the relative cup-length of Conley index, we need some useful lemmas. If (N_0, N_1) is an index pair and $t \geq 0$, then, following [13], we set

$$N_1^t := \{x \in N_1; \eta(x, [-t, 0]) \subset N_1\},$$

$$N_0^{-t} := \{x \in N_1; \text{there are } x' \in N_0 \text{ and } t' \in [0, t] \\ \text{with } \eta(x', [-t', 0]) \subset N_1 \text{ and } \eta(x', t) = x\}.$$

For $t \geq 0$, define a map

$$g : N_1/N_0^{-t} \rightarrow N_1^t/(N_0 \cap N_1^t)$$

by

$$g([x]) := \begin{cases} [\eta(x, t)] & \text{if } \eta(x, [0, t]) \subset N_1 \setminus N_0; \\ * & \text{otherwise.} \end{cases}$$



It is known (see [13, Lem. 23.14]) that g is a homeomorphism. Therefore, it induces an isomorphism

$$g^* : H^*(N_1^t, N_0 \cap N_1^t) \rightarrow H^*(N_1, N_0^{-t}).$$

Lemma 3.2. *Assume that N is an isolating neighbourhood for η and (N_1, N_0) is an index pair for $S \subset N$. If $N_1 \subset N$, then the inclusion $i : (N_1, N_0 \cap N_1^t) \rightarrow (N_1, N_0^{-t})$ induces an isomorphism*

$$i^* = (g^*)^{-1} : H^*(N_1, N_0^{-t}) \rightarrow H^*(N_1, N_0 \cap N_1^t).$$

Proof. Consider the following diagram, where the vertical arrows denote the quotient maps.

$$\begin{array}{ccc} (N_1, N_0^{-t}) & \xleftarrow{i} & (N_1, N_0 \cap N_1^t) \\ \downarrow & & \downarrow \\ N_1/N_0^{-t} & \xrightarrow{g} & N_1/(N_0 \cap N_1^t) \end{array}$$

From the definition of g , it is obvious that the diagram is homotopy commutative and the conclusion follows. □

Definition 3.2. Let N be an isolating neighbourhood for the flow η . We define the *relative cup-length* of η with respect to N as

$$\Upsilon(\eta, N) := \Upsilon(N_1, N_0; N),$$

where (N_1, N_0) is an index pair for S .

The following lemma states that $\Upsilon(\eta, N)$ is well defined.

Lemma 3.3. *Let N be an isolating neighbourhood for η and let $S \subset N$ be an isolated invariant set. If (N_1, N_0) and (\bar{N}_1, \bar{N}_0) are index pairs for S such that $N_1, \bar{N}_1 \subset N$, then*

$$\Upsilon(\bar{N}_1, \bar{N}_0; N) = \Upsilon(N_1, N_0; N).$$

Proof. As in the proof of [13, Lem. 23.17], we consider the following sequence of maps, where j, \hat{i}, \hat{i}_1 are defined by inclusion maps of pairs of spaces and g, \hat{g} are as above. All of them are homotopy equivalences of pointed spaces, as it is proved in detail in [13]:

$$\begin{array}{ccccc} N_1/N_0 & \xrightarrow{j} & N_1/N_0^{-t} & \xrightarrow{g} & N_1^t/(N_0 \cap N_1^{-t}) \\ & & & & \downarrow \hat{i}_1 \\ \bar{N}_1/\bar{N}_0 & \xleftarrow{\hat{i}} & \bar{N}_1^t/(\bar{N}_0 \cap \bar{N}_1^t) & \xleftarrow{\hat{g}} & \bar{N}_1/\bar{N}_0^{-t} \end{array}$$

By Lemma 3.2 and definition of j , it follows that the following sequence of isomorphisms

$$\begin{array}{ccccc}
 H^*(N_1, N_0) & \xleftarrow{\approx} & H^*(N_1, N_0^{-t}) & \xleftarrow{\approx} & H^*(N_1^t, N_0 \cap N_1^{-t}) \\
 & & & & \uparrow \approx \\
 H^*(\overline{N}_1, \overline{N}_0) & \xrightarrow{\approx} & H^*(\overline{N}_1^t, \overline{N}_0 \cap \overline{N}_1^t) & \xrightarrow{\approx} & H^*(\overline{N}_1, \overline{N}_0^{-t})
 \end{array}$$

are all induced by inclusions. Therefore, they all are isomorphisms of $H^*(N)$ -modules and the conclusion follows. \square

One of the main properties of the Conley index is the continuation. The same holds true for the relative cup-length.

Lemma 3.4. *Consider a continuous family of flows $\eta_\lambda : X \times \mathbb{R} \rightarrow X; \lambda \in [0, 1]$. Let $N \subset X$ be an isolating neighbourhood for all flows η_λ . Then*

$$\Upsilon(\eta_0, N) = \Upsilon(\eta_1, N).$$

Proof. Similarly as in the proof of Lemma 3.3 we shall use parts of the proof of [13, Thm. 23.31]. Given $\mu \in [0, 1]$, there exists a neighbourhood W of μ in $[0, 1]$ with the property that for all $\lambda \in W$, we can find pairs $(N_1, N_0) \subset (P_1^\lambda, P_0^\lambda) \subset (\overline{N}_1, \overline{N}_0)$ such that $(N_1, N_0), (\overline{N}_1, \overline{N}_0)$ are index pairs for η_μ in N , and $(P_1^\lambda, P_0^\lambda)$ is an index pair for η_λ in N (see [13, Lem. 23.28]). Then it is shown in the proof of [13, Thm. 23.31] that the inclusion $i : (N_1, N_0) \rightarrow (P_1^\lambda, P_0^\lambda)$ induces a homotopy equivalence of pointed spaces N_1/N_0 and P_1^λ/P_0^λ . The same argument applies to show that $i^* : H^*(P_1^\lambda, P_0^\lambda) \approx H^*(N_1, N_0)$ is an isomorphism of $H^*(N)$ -modules. Therefore, $\Upsilon(\eta_\lambda, N) = \Upsilon(\eta_\mu, N)$. Since $[0, 1]$ is compact and connected, this completes the proof. \square

One easily sees that the continuation holds for more general parameter space Λ as in [13].

4. Gradient-like flows

Throughout this section, as before, η denotes a flow on a locally compact metric space X .

Let N be an isolating neighbourhood for η and let $\varphi : \text{int } N \rightarrow \mathbb{R}$ be continuous. The flow η is called *gradient-like* with respect to φ if $\eta(x, [0, t]) \subset \text{int } N$ and $\eta(x, t) \neq x$ imply $\varphi(\eta(x, t)) > \varphi(x)$. We define the *critical level set* of φ with respect to η as

$$\text{Crit}(\varphi, \eta) := \varphi(\{x \in U; \eta(x, t) = x \text{ for all } t \in \mathbb{R}\}).$$

In other words, $c \in \text{Crit}(\varphi, \eta)$ if and only if there is $x \in N$ which is a rest point of the flow and $\varphi(x) = c$.

The aim of this section is to give a proof of the following theorem.



Theorem 4.1. *Assume that X is locally contractible and N is an isolating neighbourhood for η . If η is gradient-like with respect to $\varphi : \text{int } N \rightarrow \mathbb{R}$ and $\text{Crit}(\varphi, \eta)$ is finite, then*

$$\# \text{Crit}(\varphi, \eta) \geq \Upsilon(\eta, N).$$

Before giving the proof of the theorem we shall recall some definitions and results concerning Morse decompositions.

Recall that the *omega limit set* of $x \in X$ is given by

$$\omega(x) := \bigcap_{t>0} \text{cl}(\eta(x, [t, \infty)))$$

and the *alpha limit set* is

$$\alpha(x) := \bigcap_{t<0} \text{cl}(\eta(x, (-\infty, t])).$$

Assume that S is an isolated invariant set for η . A *Morse decomposition* of S is a finite collection, $\{M_i : 1 \leq i \leq n\}$, of disjoint compact invariant subsets of S which can be ordered (M_1, M_2, \dots, M_n) in such a way that if $x \in S \setminus \bigcup\{M_i : 1 \leq i \leq n\}$, then there are indices $i < j$ such that $\omega(x) \subset M_i$ and $\alpha(x) \subset M_j$. Such an ordering will be called *admissible*. The elements M_i of the Morse decomposition of S will be called *Morse sets* of S . For an admissible ordering (M_1, \dots, M_n) of a Morse decomposition S , define subsets M_{ij} , $i < j$, by

$$M_{ij} := \{x \in S : \omega(x) \cup \alpha(x) \subset M_i \cup M_{i+1} \cup \dots \cup M_j\}.$$

The proof of the following existence theorem can be found in [13, Thm. 23.7] or in [12, Cor. 4.4].

Theorem 4.2. *Let S be an isolated invariant set for η and (M_1, M_2, \dots, M_n) an admissible ordering of a Morse decomposition of S . Then there exists an increasing sequence of compact sets (a (Morse) filtration of S),*

$$N_0 \subset N_1 \subset \dots \subset N_n$$

such that for any $i < j$, the pair (N_j, N_{i-1}) is an index pair for M_{ij} . In particular, (N_n, N_0) is an index pair for S , and (N_j, N_{j-1}) is an index pair for M_j .

Furthermore, given any isolating neighbourhood N of S , and any neighbourhood U of S , the sets N_j can be chosen so that $\text{cl}(N_n \setminus N_0) \subset U$ and each N_j is positively invariant relative to N .

Proof of Theorem 4.1. Let

- $S := \text{Inv } N$;
- $\text{Crit}(\varphi, \eta) = \{c_1 < c_2 < \dots < c_k\}$;
- $M_i := \text{Crit}(\varphi, \eta) \cap \varphi^{-1}(c_i)$.

Choose

$$N_0 \subset N_1 \subset \dots \subset N_n$$



satisfying the conditions of Theorem 4.2. Lemma 2.2 implies

$$\Upsilon(N_i, N_0; N) \leq \Upsilon(N_{i-1}, N_0; N) + \Upsilon(N_i, N_{i-1}; N) \tag{1}$$

for $i = 1, 2, \dots, k$. Since M_i is finite and X is locally contractible, we can find a neighbourhood $U \subset N$ of M_i consisting of pairwise disjoint contractible sets. Then we find an index pair (N'_i, N'_{i-1}) in U . Therefore, $H^*(N'_i, N'_{i-1})$ has a trivial structure as an $H^*(N)$ -module. Thus by Lemma 3.3, we obtain

$$\Upsilon(N_i, N_{i-1}; N) \leq 1.$$

Therefore,

$$\Upsilon(\eta, N) = \Upsilon(N_k, N_0; N) \leq k. \tag{□}$$

5. Bifurcation

Throughout this section we let E_1, E_0 be Banach spaces, H a Hilbert space and we assume that $E_1 \subset E_0 \subset H$, where the embeddings are continuous.

We assume also that a compact Lie group G acts orthogonally on H , and the action on E_1, E_0 is by isometries (i.e., the norms on E_1, E_0 are G -invariant).

Definition 5.1. Given an open $\Omega \subset E$ and a continuous $f : \Omega \rightarrow E_0$, we say that f is a *generalized gradient map* if there is an open $\Omega_0 \subset E_0$, with $\Omega \subset \Omega_0$, and a C^1 -function $\varphi : \Omega_0 \rightarrow \mathbb{R}$ such that

$$D\varphi(x)(y) = \langle f(x), y \rangle \quad \text{for all } x \in \Omega, y \in E_0.$$

Here $\langle \cdot, \cdot \rangle$ denotes the scalar product in H . Similarly, in the case of an open $\Omega \subset E \times \mathbb{R}$ and a continuous $f : \Omega \rightarrow E_0$, we say that f is a *generalized gradient map* if $f_\lambda : \Omega_\lambda \rightarrow E_0$ is a generalized gradient map. Here $\Omega_\lambda = \{x \in E; (x, \lambda) \in \Omega\}$.

If X is a Banach space, we denote the open ϵ -ball in X by $B_X(\epsilon) := \{x \in X; \|x\| < \epsilon\}$ and $B_X(x_0, \epsilon) := \{x \in X; \|x - x_0\| < \epsilon\}$.

If $V \subset E$ is a finite-dimensional linear subspace, then there is the *orthogonal decomposition determined by V*

$$E_1 = W_1 \oplus V, \quad E_0 = W_0 \oplus V, \tag{2}$$

where $W_0 := \{x \in E_0; \langle x, y \rangle = 0 \text{ for all } y \in V\}$, $W_1 := E_1 \cap W_0$.

Definition 5.2. Let $[\lambda_1, \lambda_2] \subset \mathbb{R}$. We say that a C^1 -gradient equivariant map

$$f : \Omega_f \rightarrow E_0,$$

where $\Omega_f \subset E \oplus \mathbb{R}$ is open G -invariant, $\{0\} \times [\lambda_1, \lambda_2] \subset \Omega_f$, defines a *bifurcation problem* on $[\lambda_1, \lambda_2]$ if

$$f(0, \lambda) = 0 \quad \text{for } (0, \lambda) \in \Omega_f$$

and

$$D_x f(0, \lambda_i) : E \approx E_0, \quad i = 1, 2.$$

We shall also simply say that f is a *bifurcation problem*.



Definition 5.3. Let $f_i : \Omega_i \rightarrow E_0$, $i = 1, 2$, be two bifurcation problems on $[\lambda_1, \lambda_2]$. We say that f_1 and f_2 are *equivalent* if there exists an equivariant diffeomorphism

$$\Psi : \Omega_1 \rightarrow \Omega_2$$

such that

$$f_2 = f_1 \circ \Psi.$$

Theorem 5.1. Let $f : \Omega_f \rightarrow E_0$ be a bifurcation problem on $[\lambda_1, \lambda_2]$. If there exist decompositions

$$E_1 = V \oplus W_1, \quad E_0 = V \oplus W_0, \quad f(x, y, \lambda) = (f_1(x, y, \lambda), f_2(x, y, \lambda)),$$

such that

$$Df_2(0, \lambda)|_{W_1} : W_1 \approx W_0 \quad \text{for } \lambda \in [\lambda_1, \lambda_2],$$

then there exist

- (1) an open invariant $\Omega \subset \Omega_f$, $\{0\} \times [\lambda_1, \lambda_2] \subset \Omega$;
- (2) $g : \Omega_g \rightarrow E_0$ — a bifurcation problem on $[\lambda_1, \lambda_2]$;

such that

- (a) $f|_{\Omega}$ is a bifurcation problem on $[\lambda_1, \lambda_2]$ equivalent to g ;
- (b) $g(V \cap \Omega_g) \subset V$ and $g^{-1}(0) \subset V$;
- (c) if $D_1 f_2(0, 0, \lambda) = 0$, then $D_1 g(0, 0, \lambda) = D_1 f_1(0, 0, \lambda)$.

The proof is based on the following two theorems.

Theorem 5.2 (Equivariant implicit function theorem). Let V_1, V_2, W be Banach G -spaces, $\Omega \subset V_1 \times V_2$ a G -invariant open set, $(x_0, 0) \in \Omega$ and $F : \Omega \rightarrow W$ be continuously differentiable G -map. Assume that $F(x_0, 0) = 0$ and

$$D_2 F(x_0, 0) : V_2 \rightarrow W$$

is a G -equivariant Banach space isomorphism. Then there exist $\epsilon_1, \epsilon_2 > 0$, $B_{V_1}(x_0, \epsilon_1) \times B_{V_2}(\epsilon_2) \subset \Omega$, and a continuously differentiable G -equivariant map $\psi : B_{V_1}(x_0, \epsilon_1) \rightarrow B_{V_2}(\epsilon_2)$ such that

$$F(x, \psi(x)) = 0 \tag{3}$$

and

$$D\psi(x) = -(D_2 F(x, \psi(x)))^{-1} D_1 F(x, \psi(x)) \tag{4}$$

for all $x \in B_{V_1}(x_0, \epsilon_1)$. Furthermore, for every $x \in B_{V_1}(x_0, \epsilon_1)$, $\psi(x)$ is the only solution of (3) in $B_{V_2}(\epsilon_2)$.

Proof. The theorem is an equivariant reformulation of [7, Thm. 10.1]. Since the mapping

$$G(x, y) := y - L_0^{-1} F(x, y), \quad L_0 := D_2 F(x_0, 0),$$

defined in [7, p. 134], is in our case equivariant, the proof carries over directly. \square

Theorem 5.3 (Parametrized equivariant implicit function theorem). *Let V_1, V_2, W be Banach G -spaces, $\Omega \subset V_1 \times V_2 \times \mathbb{R}$ a G -invariant open set, $(0, 0, \lambda) \in \Omega$ for $\lambda \in [\lambda_1, \lambda_2]$. Assume that $F : \Omega \rightarrow W$ is a continuously differentiable G -map, $F(0, 0, \lambda) = 0$ if $(0, 0, \lambda) \in \Omega$ and*

$$D_2F(0, 0, \lambda) : V_2 \rightarrow W$$

is a G -equivariant Banach space isomorphism if $(0, 0, \lambda) \in \Omega$. Then there exist $\epsilon_1, \epsilon_2 > 0$, $B_{V_1}(\epsilon_1) \times B_{V_2}(\epsilon_2) \times (\lambda_1 - \epsilon_1, \lambda_2 + \epsilon_1) \subset \Omega$, and a continuously differentiable G -equivariant map $\psi : B_{V_1}(\epsilon_1) \times (\lambda_1 - \epsilon_1, \lambda_2 + \epsilon_1) \rightarrow B_{V_2}(\epsilon_2)$ such that

$$F(x, \psi(x, \lambda), \lambda) = 0 \tag{5}$$

and

$$D\psi(x, \lambda) = -(D_2F(x, \psi(x, \lambda)))^{-1}D_1F(x, \psi(x, \lambda)) \tag{6}$$

for all $x \in B_{V_1}(\epsilon_1) \times [\lambda_1, \lambda_2]$. Furthermore, for every $(x, \lambda) \in B_{V_1}(\epsilon_1) \times [\lambda_1, \lambda_2]$, $\psi(x, \lambda)$ is the only solution of (5) in $B_{V_2}(\epsilon_2)$.

Proof. The theorem follows from Theorem 5.2. One should consider $V_1 \oplus \mathbb{R}$ instead of V_1 and then use the compactness of $[\lambda_1, \lambda_2]$. \square

Proof of Theorem 5.1. We apply Theorem 5.3 to the map f_2 and obtain a G -equivariant mapping

$$\psi : B_V(\epsilon_1) \times (\lambda_1 - \epsilon_1, \lambda_2 + \epsilon_1) \rightarrow B_{W_0}(\epsilon_2).$$

Observe that for each $\lambda \in [\lambda_1, \lambda_2]$, the following holds true:

$$\text{if } x \in B_V(\epsilon_1), y \in B_{W_0}(\epsilon_2) \text{ then } f_2(x, y, \lambda) = 0 \iff y = \psi(x, \lambda).$$

Taking ϵ_2 smaller if necessary, we define a G -equivariant diffeomorphism

$$\Psi : B_V(\epsilon_1) \times (\lambda_1 - \epsilon_1, \lambda_2 + \epsilon_1) \rightarrow \Omega_f$$

by the following formula:

$$\Psi(x, y, \lambda) := (x, y + \psi(x, \lambda), \lambda).$$

The desired map g is given by

$$g := f \circ \Psi.$$

Since V is finite dimensional, for $\epsilon > 0$ small enough, we have

$$g^{-1}(0) \cap (B_V(\epsilon) \times B_{W_1}(\epsilon) \times [\lambda_1, \lambda_2]) \subset B_V(\epsilon) \times [\lambda_1, \lambda_2].$$

Considering $Dg(0, 0)$ in a block form, we obtain the last assertion. \square



6. Bifurcation in \mathbb{R}^n

In this section to simplify the notation, we consider a finite-dimensional bifurcation problem on $I = [-1, 1]$ defined by a map f . More precisely, we assume that $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ is a C^1 -map, $f(0, \lambda) = 0$ for $\lambda \in \mathbb{R}$ and $Df(0, \pm 1) : \mathbb{R}^n \approx \mathbb{R}^n$.

Let $A_\lambda := D_x f(0, \lambda)$. Then $f(x, \lambda) = A_\lambda(x) + f_0(x, \lambda)$. For $\tau \in [0, 1]$, we set

$$f_\tau(x, \lambda) := A_\lambda(x) + \tau f_0(x, \lambda).$$

Assume further that there exist $\rho, C > 0$ such that

$$\langle f_\tau(x, 1), x \rangle \geq C|x|^2 \quad \text{for } |x| \leq 2\rho \tag{7}$$

and

$$\langle f_\tau(x, -1), x \rangle \leq -C|x|^2 \quad \text{for } |x| \leq 2\rho. \tag{8}$$

For $\alpha > 0$ and $0 < \epsilon < \rho$, define

$$F_\tau : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$$

by $F_\tau(x, \lambda) := (f_\tau(x, \lambda), \alpha(|x| - \epsilon))$. Let

$$\Omega = \{x \in \mathbb{R}^n; |x| \leq 2\rho\} \times [-1, 1]$$

and $M := \sup\{|f_\tau(x, \lambda)|; (x, \lambda) \in \Omega, \tau \in [0, 1]\}$.

Lemma 6.1. *If*

$$\alpha \geq \frac{2M}{\rho(\rho - \epsilon)},$$

then there exists $\delta > 0$ such that $\delta < \epsilon$ and for all $\tau \in [0, 1]$, the set $N := \{(x, \lambda) \in \Omega; |x| \geq \delta\}$ is an isolating neighbourhood for the flow generated by F_τ .

Proof. First we prove that Ω is an isolating neighbourhood. We fix τ and let $\eta(x, \lambda, t) = (\eta_1(x, \lambda, t), \eta_2(x, \lambda, t)) \in \mathbb{R}^n \times \mathbb{R}$ denote the flow generated by F_τ . It is enough to show that for all $(x, \lambda) \in \partial\bar{\Omega}$,

- (a) there exists $T > 0$ such that either $\eta(x, \lambda, T) \notin \bar{\Omega}$ or $\eta(x, \lambda, -T) \notin \bar{\Omega}$.

Let $K := \{(x, \lambda) \in \bar{\Omega}; |x| = 2\rho, \lambda \in [-1, 1]\}$. If $(x, \lambda) \in \partial\Omega \setminus K$, then (a) follows immediately from the definition of F_τ .

To complete the proof of our first claim we start from the following simple observations:

- (b) if $\eta(x, \lambda, t) \in \bar{\Omega}$ for all $t \in [0, T]$, then

$$|\eta_1(x, \lambda, t) - \eta_1(x, \lambda, 0)| \leq TM \quad \text{for } t \in [0, T];$$

- (c) if $\eta(x, \lambda, t) \in A := \{(x, \lambda) \in \bar{\Omega}; |x| \geq \rho\}$ for all $t \in [0, T]$, then

$$\eta_2(x, \lambda, t) - \eta_2(x, \lambda, 0) \geq (\rho - \epsilon)\alpha t \quad \text{for } t \in [0, T].$$



Let $(x_0, \lambda_0) \in K$ and

$$T_1 := \inf\{t \in (0, \infty); \eta(x_0, \lambda_0, t) \notin A\}.$$

Since every point of A leaves A in a finite time, $T_1 < \infty$. (One can call T_1 the *exit time* of (x_0, λ_0) from A .) Let $(x_1, \lambda_1) := \eta(x_0, \lambda_0, T_1)$. If $(x_1, \lambda_1) \in \partial\Omega$, then (a) holds. Suppose that $(x_1, \lambda_1) \in \Omega$. Then $|x_1| = \rho$, $\lambda_1 \in (-1, 1)$ and (b) implies $\rho \leq MT_1$. Applying (c), one obtains

$$\lambda_1 \geq \lambda_0 + \alpha(\rho - \epsilon) \frac{\rho}{M} > \lambda_0 + 2 > 1.$$

We have obtained a contradiction. Therefore, Ω is an isolating neighbourhood for all η_τ and thus the invariant part

$$\text{Inv}(\Omega, \eta) = \bigcup_{\tau \in [0,1]} \text{Inv}(\Omega, \eta_\tau) \subset \text{int}(\Omega)$$

is compact. Moreover, one easily checks that it is disjoint with $\{0\} \times [-1, 1]$. Thus there exists $\epsilon > \delta \geq 0$ such that $\text{Inv}(\Omega) \in \text{int}(N)$. This proves that N is an isolating neighbourhood for all η_τ . \square

Assume now that $V = (\mathbb{R}^n, \varphi)$ is an orthogonal representation of a compact Lie group G ; i.e., $\varphi : G \rightarrow O(n)$ is a group homomorphism. Let $S(V) := \{x \in V; |x| = 1\}$. The use of V instead of \mathbb{R}^n is a bit of notation abuse—we try to emphasize that $S(V)$ is a G -space.

Lemma 6.2. *Let $f : \Omega_f \rightarrow \mathbb{R}^n$ be a gradient equivariant bifurcation problem on $[-1, 1]$ and $A_\lambda := D_x f(0, \lambda)$, $\lambda \in [-1, 1]$. Assume that there is $C > 0$ such that*

$$\langle A_1(x), x \rangle \geq C|x|^2 \quad \text{for } x \in \mathbb{R}^n \tag{9}$$

and

$$\langle A_{-1}(x), x \rangle \leq -C|x|^2 \quad \text{for } x \in \mathbb{R}^n. \tag{10}$$

Then for sufficiently small ϵ , the number of zero G -orbits of f in $S(\mathbb{R}^n, \epsilon) \times (-1, 1)$ is not less than the cup-length of $S(V)/G$.

Proof. We keep the notation from the beginning of this section. From (9) and (10), it follows that there exists $\rho > 0$ such that assumptions (7) and (9) are satisfied. Now for $\epsilon < \rho$, we find α and δ as in Lemma 6.1 and obtain an isolating neighbourhood $N = \{(x, \lambda); \delta \leq |x| \leq 2\rho, -1 \leq \lambda \leq 1\}$ which is clearly an invariant set with respect to the action of G (trivial on the parameter space). By Lemma 3.4, it is enough to calculate the equivariant Conley index (and the relative cup-length) for the flow generated by the map $g(x, \lambda) := (D_x f(0, \lambda)(x), \alpha(|x| - \epsilon)) = (A_\lambda x, \alpha(|x| - \epsilon))$.

Now we can make another simplification. Consider a map $B : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ given by $B(x, \lambda) = \lambda x$ and a family of flows generated by vector fields $F_\tau : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$, $F_\tau(x, \lambda) = (\tau A_\lambda x + (1 - \tau)B(x, \lambda), \alpha(|x| - \epsilon))$ with $\tau \in [0, 1]$. It is easy to verify that N is an isolating neighbourhood for this



family of flows. Thus we can do all the calculations for $\tau = 0$. We can easily find an index pair: $N_1 := N$ and

$$\begin{aligned} N_0 := & \{(x, 1) : 1 \geq |x| \geq \epsilon\} \cup \{(x, \lambda); |x| = 2\rho, \lambda \in [0, 1]\} \\ & \cup \{(x, \lambda); |x| = \delta, \lambda \in [-1, 0]\} \\ & \cup \{(x, -1); \delta \leq |x| \leq \epsilon\}. \end{aligned}$$

Since all the sets are G -invariant, their quotient sets constitute an index pair for the flow generated on the orbit space. N is equivariantly homotopy equivalent to $S(V) \times [-1, 1] \times [-1, 1]$ and $N_0 \approx S(V) \times L$, where $L : \{(t, s) \in \partial([-1, 1] \times [-1, 1]); ts \geq 0\}$. Therefore, $\bar{N}_1 := N_1/G \approx S(V)/G \times [-1, 1] \times [-1, 1]$, $\bar{N}_0 := N_0/G \approx S(V)/G \times L$. Their quotient $\bar{N}_1/\bar{N}_0 \approx S(V)/G \wedge S^1$. Thus, by Theorem 2.5, $\Upsilon(\bar{N}_1, \bar{N}_0; \bar{N}_1)$ is equal to the cup-length of $S(V)/G$.

Now we can apply Theorem 4.1, since the gradient flow generated by f gives rise to a gradient-like flow on the orbit space and the critical points of this flow are images of the zero G -orbits of f . □

7. Bifurcations of periodic solutions to Hamiltonian systems

By

$$J : \mathbb{R}^{2N} = \mathbb{R}^N \oplus \mathbb{R}^N \rightarrow \mathbb{R}^N \oplus \mathbb{R}^N = \mathbb{R}^{2N}$$

we denote a linear automorphism given by the matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Throughout this section we assume that $H : \mathbb{R}^{2N} \rightarrow \mathbb{R}$ is a C^2 -function (Hamiltonian) such that

- (H1) $H(0) = 0, \nabla H(0) = 0$;
- (H2) the Hessian $\nabla^2 H(0)$ is nondegenerate.

The main object of our investigation is periodic solutions to the following equation:

$$\dot{u}(t) = J\nabla H(u(t)). \tag{11}$$

We shall use the following Banach spaces:

- (1) $\mathcal{E}_0 := C(S^1, \mathbb{R}^{2N})$. The elements of \mathcal{E}_0 are identified with continuous functions

$$u : \mathbb{R} \rightarrow \mathbb{R}^{2N}, \quad u(t + 2\pi) = u(t), \quad \|u\| := \sup\{|u(t)|; t \in \mathbb{R}\}.$$

- (2) $\mathcal{E} := C^1(S^1, \mathbb{R}^{2N})$. As a linear space \mathcal{E} is a subspace of \mathcal{E}_0 . The norm in \mathcal{E} is defined by a formula

$$\|u\|_1 := \|u\| + \|\dot{u}\|.$$

The above automorphism J defines also automorphisms of our Banach spaces

$$J : \mathcal{E} \rightarrow \mathcal{E}, \quad J : \mathcal{E}_0 \rightarrow \mathcal{E}_0.$$



More precisely,

$$J \left(\sum_{i=1}^{2N} u_i \mathbf{e}_i \right) := \sum_{i=1}^{2N} u_i J(\mathbf{e}_i),$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{2N}\}$ is the standard basis of \mathbb{R}^{2N} .

In the space \mathcal{E}_0 we have a continuous inner product

$$\langle u, v \rangle := \sum_{j=1}^{2N} \int_0^{2\pi} u_j(t)v_j(t)dt, \tag{12}$$

where

$$u = \sum_{j=1}^{2N} u_j \mathbf{e}_j, \quad v = \sum_{j=1}^{2N} v_j \mathbf{e}_j.$$

(In other words, we consider \mathcal{E}_0 as a subspace of $\mathcal{L}^2(S^1, \mathbb{R}^{2N})$.)

The formula

$$\mathcal{L}(u) := J(\dot{u})$$

defines a bounded linear operator

$$\mathcal{L} : \mathcal{E} \rightarrow \mathcal{E}_0.$$

Denote by

$$\mathcal{H} : \mathcal{E} \rightarrow \mathcal{E}_0$$

a map (nonlinear in general) given by a formula

$$(\mathcal{H}(u))(t) := \nabla H(u(t)).$$

Our further considerations are based on the following well-known remark.

Define a map

$$f : \mathcal{E} \times (0, \infty) \rightarrow \mathcal{E}_0, \quad f(u, \lambda) := \mathcal{L}(u) + \lambda \mathcal{H}(u). \tag{13}$$

Remark 7.1. A function $u \in \mathcal{E}$ is a periodic solution to equation (11) of period $\frac{2\pi}{\lambda}$ if and only if $f(u, \lambda) = 0$. The map f is (generalized) gradient in the sense introduced in Definition 5.1 with respect to the potential $\chi(u) := \int_0^{2\pi} u(t)dt$.

A change of variables $t \mapsto \lambda t$ gives the first part of the remark. The second part is well known.

Let $A := \nabla^2 H(0)$. The map JA is a *Hamiltonian* (i.e., $(JA)^T J + J(JA) = 0$). Observe that in [8] the notion *Hamiltonian matrix* is used.

Now we describe briefly the spectral decomposition of JA . We try to follow the notation of [8, Sec. 3.3], where further details can be found.

The eigenvalues of JA fall into three groups (because of (H2)):

- (1) the pure imaginary $\pm i\omega_1, \dots, \pm i\omega_r$;
- (2) the real eigenvalues $\alpha_1, \dots, \alpha_s$;
- (3) the truly complex $\pm\beta_1 \pm i\gamma_1, \dots, \pm\beta_t \pm i\gamma_t$.



This defines a direct sum decomposition

$$\mathbb{R}^{2N} = \mathbb{V} \oplus \mathbb{X} \oplus \mathbb{Y}, \tag{14}$$

where their complexifications are composed of generalized eigenspaces as follows:

$$\begin{aligned} \mathbb{V}^c &= \bigoplus_{j=1}^r (\eta^\dagger(i\omega_j) \oplus \eta^\dagger(-i\omega_j)), \\ \mathbb{X}^c &= \bigoplus_{j=1}^s (\eta^\dagger(\alpha_j) \oplus \eta^\dagger(-\alpha_j)), \\ \mathbb{Y}^c &= \bigoplus_{j=1}^t (\eta^\dagger(\beta_j + i\gamma_j) \oplus \eta^\dagger(\beta_j - i\gamma_j) \oplus \eta^\dagger(-\beta_j + i\gamma_j) \oplus \eta^\dagger(-\beta_j - i\gamma_j)). \end{aligned}$$

We are especially interested in part (1):

$$\begin{aligned} \sigma_0(JA) &= \sigma(JA) \cap \{i\mathbb{R}\} \\ &= \{\pm i\omega_1, \pm i\omega_2, \dots, \pm i\omega_r\}, \quad 0 < \omega_1 < \omega_2 < \dots < \omega_r. \end{aligned} \tag{15}$$

Denote by $\mathbb{V}_j, \mathbb{U}_j$ the subspaces of \mathbb{R}^{2N} such that

$$\mathbb{V}_j^c := \eta^\dagger(i\omega_j) \oplus \eta^\dagger(-i\omega_j), \quad \mathbb{U}_j^c := \eta(i\omega_j) \oplus \eta(-i\omega_j).$$

Obviously,

$$\mathbb{V} = \bigoplus_{j=1}^r \mathbb{V}_j \tag{16}$$

and each summand is A -invariant (and so are \mathbb{X}, \mathbb{Y} and \mathbb{U}_j).

Denote by $A^c : \mathbb{C}^{2N} \rightarrow \mathbb{C}^{2N}$ the complexification of A and let $\mathbb{U}_j \subset \mathbb{R}^{2N}, j = 1, \dots, r$, denote the subspace such that

$$\mathbb{U}_j^c = \text{Ker}(A + i\omega_j) \oplus \text{Ker}(A - i\omega_j).$$

Let $d_j := \frac{1}{2} \dim \mathbb{U}_j$. Clearly d_j is an integer. Let

$$d = d(A) := d_1 + d_2 + \dots + d_r. \tag{17}$$

In order to make our setup precise, we introduce the following terminology. If $u : \mathbb{R} \rightarrow \mathbb{R}^{2N}$ is a periodic C^1 -solution to (11) and $\tau \in \mathbb{R}$, then we let u_τ denote the periodic solution to (11) defined by $u_\tau(t) := u(t + \tau), t \in \mathbb{R}$. We say that two periodic solutions u, v to (11) are *geometrically distinct* if $u_\tau \neq v$ for all $\tau \in \mathbb{R}$.

Now we can formulate the main result of this section.

Theorem 7.2. *If H satisfies (H1) and (H2), then there exists an $\epsilon_0 > 0$ such that $0 < \epsilon < \epsilon_0$ implies the existence of at least d geometrically distinct periodic solutions to (11) in $\{u \in C(\mathbb{R}, \mathbb{R}^{2N}); \|u\| = \epsilon\}$.*



8. Proof of Theorem 7.2

Define the operators $\mathcal{A}, \mathcal{D}_\lambda : \mathcal{E} \rightarrow \mathcal{E}_0$ by

$$\mathcal{A}(u)(t) := A(u(t)) = (\nabla^2 H(0))(u(t)), \quad \mathcal{D}_\lambda u := J\dot{u} + \lambda A(u). \tag{18}$$

Note that $\mathcal{D}_\lambda = Df(0, \lambda)$. For any subspace $\mathbb{Z} \subset \mathbb{R}^{2N}$, we denote

$$\mathcal{E}(\mathbb{Z}) := C^1(S^1, \mathbb{Z}), \quad \mathcal{E}_0(\mathbb{Z}) := C^0(S^1, \mathbb{Z}).$$

We consider the above function spaces together with the S^1 -action defined by

$$(\gamma u)(t) := u(t - \theta) \quad \text{for } \gamma := e^{i\theta}.$$

Notation. Throughout this section we tacitly assume that the considered maps are S^1 -equivariant and gradient (in the generalized sense, see Definition 5.1). The gradient structure should be clear from the context.

Define an equivalence relation in the set $\mathfrak{S} := \{\omega_1, \omega_2, \dots, \omega_q\}$ by

$$\omega_j \sim \omega_k \iff n\omega_j = m\omega_k, \quad n, m \in \mathbb{N}.$$

This relation divides \mathfrak{S} into pairwise disjoint classes

$$\mathfrak{S} = \bigcup_{k=1}^p \mathfrak{S}_k.$$

For $k \in \{1, 2, \dots, q\}$, set $\mathcal{J}_k := \{j \in \{1, \dots, r\}; \omega_j \in \mathfrak{S}_k\}$, $\mathfrak{D}_k := \mathcal{D}_{\lambda_k}$,

$$\mathbb{W}_k := \bigoplus_{j \in \mathcal{J}_k} \mathbb{V}_j, \quad b_k := \sum_{j \in \mathcal{J}_k} d_j,$$

$$\mathcal{W}_k := \mathcal{E}(\mathbb{W}_k) \cap \text{Ker } \mathfrak{D}_k.$$

For each k , let ν_k denote the greatest real number such that for every $\omega \in \mathfrak{S}_k$ there is $n \in \mathbb{N}$ such that $\omega = n\nu_k$ and let $\lambda_k := \nu_k^{-1}$.

Suppose $\omega_j \in \mathfrak{S}_k$ and let $n_j := \frac{\omega_j}{\nu_k} \in \mathbb{N}$.

If $z = x + iy \in \mathbb{C}^{2N}$, $x, y \in \mathbb{R}^{2N}$, is an eigenvector corresponding to the eigenvalue $i\omega_j$ of $(JA)^c$, then $(JA)^c(x) = -\omega_j y$, $(JA)^c(y) = \omega_j x$ and thus $x - iy$ is an eigenvector corresponding to the eigenvalue $-i\omega_j$. Therefore, vectors x, y span a subspace of \mathbb{R}^{2N} which is invariant for A . Let $\mathbf{z}_p = \mathbf{x}_p + i\mathbf{y}_p$, $p := 1, \dots, d_j$, be a basis of $\text{Ker}((JA)^c + i\omega_j I)$. Then the vectors

$$\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2, \dots, \mathbf{x}_{d_j}, \mathbf{y}_{d_j} \tag{19}$$

form a basis of \mathbb{U}_j . Let $\mathbf{c}_j(t) := \cos(n_j t)$, $\mathbf{s}_j(t) := \sin(n_j t)$. Denote by \mathcal{U}_j the $2d_j$ -dimensional subspace of $C^1(S^1, \mathbb{U}_j) \subset \mathcal{E}$ spanned by

$$\mathbf{c}_j \mathbf{x}_p + \mathbf{s}_j \mathbf{y}_p, \quad \mathbf{s}_j \mathbf{x}_p - \mathbf{c}_j \mathbf{y}_p, \quad p = 1, \dots, d_j.$$

Then

$$\mathcal{U}_j = \mathcal{E}(\mathbb{V}_j) \cap \text{Ker } \mathfrak{D}_k \quad \text{and} \quad \mathcal{W}_k = \bigoplus_{j \in \mathcal{J}_k} \mathcal{U}_j.$$



Remark 8.1. The assignments

$$\mathbf{c}_j \mathbf{x}_p + \mathbf{s}_j \mathbf{y}_p \mapsto e_p, \quad \mathbf{s}_j \mathbf{x}_p - \mathbf{c}_j \mathbf{y}_p \mapsto i e_p, \quad p = 1, \dots, d_j,$$

where $\{e_p\}$ denotes the standard basis of \mathbb{C}^{d_j} , define an isomorphism of real linear spaces

$$\mathcal{A}_j : \mathcal{U}_j \rightarrow \mathbb{C}^{d_j}.$$

Lemma 8.2. *The cup-length of $S(\mathcal{W}_k)$ equals b_k .*

Proof. Consider the complex linear space

$$\mathbf{V} := \bigoplus_{j \in \mathcal{J}_k} \mathbb{C}^{d_j}$$

whose points we write as $z = (z_1, \dots, z_q), z_j \in \mathbb{C}^{d_j}$. Let \mathbf{Y} and \mathbf{Z} denote, respectively, the representations of S^1 on \mathbf{V} determined by

$$\begin{aligned} \gamma(z_1, \dots, z_q) &:= (\gamma^{d_1} z_1, \dots, \gamma^{d_q} z_q), \\ \gamma(z_1, \dots, z_q) &:= (\gamma^{b_k} z_1, \dots, \gamma^{b_k} z_q). \end{aligned}$$

To avoid misunderstandings we denote by \mathbf{X} the standard representation of S^1 on \mathbf{V} . Let $\alpha : S(\mathbf{X}) \rightarrow S(\mathbf{Y}), \beta : S(\mathbf{Y}) \rightarrow S(\mathbf{Z})$ denote the S^1 -equivariant maps between unit spheres in corresponding representations defined by

$$\begin{aligned} \alpha(z_1, \dots, z_q) &:= (z_1^{d_1}, \dots, z_q^{d_q}), \\ \beta(z_1, \dots, z_q) &:= (z_1^{b_k - d_1}, \dots, z_q^{b_k - d_q}). \end{aligned}$$

Obviously $S(\mathbf{X})/S^1 = CP^{d-1}$. A slight modification of the arguments given in [6, Sec. 3.2] permits to prove that $S(\mathbf{Z})$ is diffeomorphic to CP^{a-1} and $\beta \circ \alpha$ induces a monomorphism of cohomology rings

$$(\beta \circ \alpha)^* : H^*(S(\mathbf{Z})/S^1) \rightarrow H^*(S(\mathbf{X})/S^1).$$

Therefore,

$$\alpha^* : H^*(S(\mathbf{Y})/S^1) \rightarrow H^*(S(\mathbf{X})/S^1)$$

is also a monomorphism. Thus the cup-length of $S(\mathbf{Y})$ equals b_k . Since

$$\bigoplus_{j \in \mathcal{J}_k} \mathcal{A}_j : \mathcal{W}_k \rightarrow \mathbf{Y}$$

is an isomorphism of real representations of S^1 , the proof is completed. \square

Let

$$\begin{aligned} \mathcal{W}_k^{\perp,0} &:= \{w \in \mathcal{E}_0(\mathcal{W}_k); \langle w, v \rangle = 0 \text{ for } v \in \mathcal{W}_k\}, \\ \mathcal{W}_k^\perp &:= \mathcal{E}(\mathbb{W}_k) \cap \mathcal{W}_k^{\perp,0}. \end{aligned}$$

From (12) and (18), we have

$$\langle u, \mathfrak{D}_k(v) \rangle = \langle \mathfrak{D}_k(u), v \rangle = 0 \quad \text{for } u \in \mathcal{W}_k, v \in \mathcal{E}(\mathbb{W}_k).$$



Therefore, $\mathfrak{D}_k(\mathcal{W}_k^\perp) \subset \mathcal{W}_k^{\perp,0}$. Since \mathfrak{D}_k , as an operator from $\mathcal{E}(\mathbb{W}_k)$ into $\mathcal{E}_0(\mathbb{W}_k)$, is Fredholm of index 0, it maps isomorphically \mathcal{W}_k^\perp onto $\mathcal{W}_k^{\perp,0}$. Applying Theorem 5.1, we obtain an $\epsilon > 0$ and a mapping

$$g : B(\mathcal{E}, \epsilon) \times [\lambda_k - \delta, \lambda_k + \delta] \rightarrow \mathcal{E}_0,$$

where $B(\mathcal{E}, \epsilon) := \{u \in \mathcal{E}; \|u\| < \epsilon\}$, such that

- f and g determine equivalent bifurcation problems on $[\lambda_k - \delta, \lambda_k + \delta]$;
- $g(B(\mathcal{W}_k, \epsilon) \times [\lambda_k - \delta, \lambda_k + \delta]) \subset \mathcal{W}_k$;
- $Dg(0, \lambda) = Df(0, \lambda)$ for $\lambda \in [\lambda_k - \delta, \lambda_k + \delta]$.

Setting $\varphi(w, \lambda) := g(w, \lambda)$, $w \in \mathcal{W}_k$, we obtain

$$\varphi : \mathcal{W}_k \times [\lambda_k - \delta, \lambda_k + \delta] \rightarrow \mathcal{W}_k,$$

which determines a finite-dimensional bifurcation problem on $[\lambda_k - \delta, \lambda_k + \delta]$ (one may call it a *reduction* of f to \mathcal{W}_k). Applying Lemmas 8.2 and 6.2, we obtain the following conclusion.

Conclusion 8.3. *For each $k \in \{1, \dots, q\}$, there exist $\delta, \epsilon > 0$ such that*

- (a) *the mapping φ defines a bifurcation problem on $[\lambda_k - \delta, \lambda_k + \delta]$;*
- (b) *$f^{-1}(0) \cap (S(\mathcal{E}, \epsilon) \times [\lambda_k - \delta, \lambda_k + \delta])$ contains at least b_k different S^1 -orbits.*

Now, to complete the proof of Theorem 7.2, it is enough to observe that, for sufficiently small δ and ϵ , different S^1 orbits in

$$f^{-1}(0) \cap \left(\bigcup_{k=1}^q S(\mathcal{E}, \epsilon) \times [\lambda_k - \delta, \lambda_k + \delta] \right)$$

correspond to geometrically distinct solutions to (11).

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