

Non-isolating 2-bondage in graphs

By Marcin KRZYWKOWSKI

(Received Dec. 14, 2010)

(Revised Apr. 30, 2011)

Abstract. A 2-dominating set of a graph $G = (V, E)$ is a set D of vertices of G such that every vertex of $V(G) \setminus D$ has at least two neighbors in D . The 2-domination number of a graph G , denoted by $\gamma_2(G)$, is the minimum cardinality of a 2-dominating set of G . The non-isolating 2-bondage number of G , denoted by $b'_2(G)$, is the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\delta(G - E') \geq 1$ and $\gamma_2(G - E') > \gamma_2(G)$. If for every $E' \subseteq E$, either $\gamma_2(G - E') = \gamma_2(G)$ or $\delta(G - E') = 0$, then we define $b'_2(G) = 0$, and we say that G is a γ_2 -non-isolatingly strongly stable graph. First we discuss the basic properties of non-isolating 2-bondage in graphs. We find the non-isolating 2-bondage numbers for several classes of graphs. Next we show that for every non-negative integer there exists a tree having such non-isolating 2-bondage number. Finally, we characterize all γ_2 -non-isolatingly strongly stable trees.

1. Introduction.

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The degree of a vertex v , denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong if it is adjacent to at least two leaves. Let $\delta(G)$ mean the minimum degree among all vertices of G . The path (cycle, respectively) on n vertices we denote by P_n (C_n , respectively). A wheel W_n , where $n \geq 4$, is a graph with n vertices, formed by connecting a vertex to all vertices of a cycle C_{n-1} . Let T be a tree, and let v be a vertex of T . We say that v is adjacent to a tree H if there is a neighbor of v , say x , such that the tree resulting from T by removing the edge vx , and which contains the vertex x , is a tree H . Let $K_{p,q}$ denote a complete bipartite graph the partite sets of which have cardinalities p and q . By a star we mean a connected graph in which exactly one vertex has degree greater than one. By a double star we mean a graph that can be obtained from a star by joining a positive number of vertices to one of the leaves.

2010 *Mathematics Subject Classification.* Primary 05C05, 05C69.

Key Words and Phrases. 2-domination, bondage, non-isolating 2-bondage, graph, tree.

The research was supported by the Polish National Science Centre grant 2011/02/A/ST6/00201.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D , while it is a 2-dominating set, abbreviated 2DS, of G if every vertex of $V(G) \setminus D$ has at least two neighbors in D . The domination (2-domination, respectively) number of a graph G , denoted by $\gamma(G)$ ($\gamma_2(G)$, respectively), is the minimum cardinality of a dominating (2-dominating, respectively) set of G . Note that 2-domination is a type of multiple domination in which each vertex, which is not in the dominating set, is dominated at least k times for a fixed positive integer k . Multiple domination was introduced by Fink and Jacobson [3], and further studied for example in [1], [13]. For a comprehensive survey of domination in graphs, see [7], [8].

The bondage number $b(G)$ of a graph G is the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\gamma(G - E') > \gamma(G)$. If for every $E' \subseteq E$ we have $\gamma(G - E') = \gamma(G)$, then we define $b(G) = 0$, and we say that G is a γ -strongly stable graph. Bondage in graphs was introduced in [4], and further studied for example in [2], [5], [6], [9], [10], [11], [12] and [14].

We define the non-isolating 2-bondage number of a graph G , denoted by $b'_2(G)$, to be the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\delta(G - E') \geq 1$ and $\gamma_2(G - E') > \gamma_2(G)$. Thus $b'_2(G)$ is the minimum number of edges of G that have to be removed in order to obtain a graph with no isolated vertices, and with the 2-domination number greater than that of G . If for every $E' \subseteq E$, either $\gamma_2(G - E') = \gamma_2(G)$ or $\delta(G - E') = 0$, then we define $b'_2(G) = 0$, and we say that G is a γ_2 -non-isolatingly strongly stable graph.

First we discuss the basic properties of non-isolating 2-bondage in graphs. We find the non-isolating 2-bondage numbers for several classes of graphs. Next we show that for every non-negative integer there exists a tree having such non-isolating 2-bondage number. Finally, we characterize all γ_2 -non-isolatingly strongly stable trees.

2. Results.

We begin with the following observations.

Observation 1: *Every leaf of a graph G is in every $\gamma_2(G)$ -set.*

Observation 2: *If $H \subseteq G$ and $V(H) = V(G)$, then $\gamma_2(H) \geq \gamma_2(G)$.*

Observation 3: *For every positive integer n we have $\gamma_2(K_n) = \min\{2, n\}$.*

Observation 4: *If n is a positive integer, then $\gamma_2(P_n) = \lfloor n/2 \rfloor + 1$.*

Observation 5: *For every integer $n \geq 3$ we have $\gamma_2(C_n) = \lfloor (n+1)/2 \rfloor$.*

Observation 6: For every integer $n \geq 4$ we have

$$\gamma_2(W_n) = \begin{cases} 2 & \text{if } n = 4, 5; \\ \lfloor (n+1)/3 \rfloor + 1 & \text{if } n \geq 6. \end{cases}$$

Observation 7: Let p and q be positive integers such that $p \leq q$. Then

$$\gamma_2(K_{p,q}) = \begin{cases} \max\{q, 2\} & \text{if } p = 1; \\ \min\{p, 4\} & \text{if } p \geq 2. \end{cases}$$

Since the definition of the non-isolating 2-bondage does not allow isolated vertices in the searched subgraphs of a given graph, in this paper, we do not consider removing edges that produces an isolated vertex.

First we find the non-isolating 2-bondage numbers of complete graphs.

REMARK 8. For every positive integer n we have

$$b'_2(K_n) = \begin{cases} 0 & \text{if } n = 1, 2, 3; \\ \lfloor 2n/3 \rfloor & \text{otherwise.} \end{cases}$$

PROOF. Of course, $b'_2(K_1) = 0$, $b'_2(K_2) = 0$, and $b'_2(K_3) = 0$. Now assume that $n \geq 4$. Let $E(K_n) = \{v_1, v_2, \dots, v_n\}$. Let G be a graph with at least two vertices. Let us observe that $\gamma_2(G) = 2$ if and only if G has two vertices which are both adjacent to every vertex other than they. Let $E' \subseteq E(K_n)$. Let us observe that $\gamma_2(K_n - E') > 2$ if and only if at most one vertex of K_n is not incident to any edge of E' , and every edge of E' is adjacent to some other edge of E' . We want to choose a smallest set $E' \subseteq E(K_n)$ satisfying the condition above while $\delta(K_n - E') \geq 1$. Let us observe that the most efficient way of choosing edges of K_n is to choose for example edges $v_1v_2, v_2v_3, v_4v_5, v_5v_6$, and so on. In this way no vertex becomes isolated. Let k be a positive integer.

If $n = 3k$, then we remove $2k$ edges. Thus $b'_2(K_{3k}) = 2k = \lfloor 2n/3 \rfloor$. If $n = 3k + 1$, then we also remove $2k$ edges as one vertex can remain universal. We have $b'_2(K_{3k+1}) = 2k = \lfloor 2k + 2/3 \rfloor = \lfloor 2(3k + 1)/3 \rfloor = \lfloor 2n/3 \rfloor$. Now assume that $n = 3k + 2$. If we remove the edges $v_1v_2, v_2v_3, v_4v_5, v_5v_6, \dots, v_{2k-2}v_{2k-1}, v_{2k-1}v_{2k}$, then the vertices v_{3k+1} and v_{3k+2} remain universal. Therefore $b'_2(K_{3k+2}) > 2k$. Let us observe that removing the edges $v_1v_2, v_2v_3, v_4v_5, v_5v_6, \dots, v_{2k-2}v_{2k-1}, v_{2k-1}v_{2k}, v_{2k}v_{2k+1}$ increases the 2-domination number. This implies that $b'_2(K_{3k+2}) = 2k + 1 = \lfloor 2k + 4/3 \rfloor = \lfloor 2(3k + 2)/3 \rfloor = \lfloor 2n/3 \rfloor$. \square

Now we calculate the non-isolating 2-bondage numbers of paths.



REMARK 9. If n is a positive integer, then

$$b'_2(P_n) = \begin{cases} 0 & \text{for } n = 1, 2, 3; \\ 1 & \text{for } n \geq 4. \end{cases}$$

Now we investigate the non-isolating 2-bondage in cycles.

REMARK 10. For every integer $n \geq 3$ we have

$$b'_2(C_n) = \begin{cases} 0 & \text{if } n = 3; \\ 1 & \text{if } n \text{ is even;} \\ 2 & \text{otherwise.} \end{cases}$$

Now we calculate the non-isolating 2-bondage numbers of wheels.

REMARK 11. For every integer $n \geq 4$ we have

$$b'_2(W_n) = \begin{cases} 1 & \text{if } n = 5; \\ 2 & \text{if } n \neq 3k + 2; \\ 3 & \text{otherwise.} \end{cases}$$

PROOF. Let $E(W_n) = \{v_1v_2, v_1v_3, \dots, v_1v_n, v_2v_3, v_3v_4, \dots, v_{n-1}v_n, v_nv_2\}$. Since $W_4 = K_4$, by Remark 8 we get $b'_2(W_4) = b'_2(K_4) = \lfloor 8/3 \rfloor = 2$. By Observation 6 we have $\gamma_2(W_5) = 2$. Let us observe that $\gamma_2(W_5 - v_2v_3) = 3 > 2 = \gamma_2(W_5)$. Thus $b'_2(W_5) = 1$. Now let us assume that $n \geq 6$. If we remove an edge incident with v_1 , say v_1v_2 , then we get $\gamma_2(W_n - v_1v_2) = \gamma_2(W_n)$ as we can construct a $\gamma_2(W_n)$ -set that contains the vertices v_1 and v_2 ; such set is also a 2DS of the graph $W_n - v_1v_2$. If we remove an edge non-incident with v_1 , say v_2v_3 , then we get $\gamma_2(W_n - v_2v_3) = \gamma_2(W_n)$ as we can construct a $\gamma_2(W_n)$ -set that does not contain the vertices v_2 and v_3 ; such set is also a 2DS of the graph $W_n - v_2v_3$. This implies that $b'_2(W_n) = 0$ or $b'_2(W_n) \geq 2$. First assume that $n = 3k$ or $n = 3k + 1$. Let us remove the edges $v_{n-1}v_n$ and v_nv_2 . We find a relation between the numbers $\gamma_2(W_n - v_{n-1}v_n - v_nv_2)$ and $\gamma_2(W_{n-1})$. Let D be any $\gamma_2(W_n - v_{n-1}v_n - v_nv_2)$ -set. By Observation 1 we have $v_n \in D$. Let us observe that $D \setminus \{v_n\}$ is a 2DS of the graph W_{n-1} . Therefore $\gamma_2(W_{n-1}) \leq \gamma_2(W_n - v_{n-1}v_n - v_nv_2) - 1$. Using Observation 6 we get $\gamma_2(W_n - v_{n-1}v_n - v_nv_2) \geq \gamma_2(W_{n-1}) + 1 = \lfloor n/3 \rfloor + 2 = \lfloor (n+1)/3 \rfloor + 2 > \lfloor (n+1)/3 \rfloor + 1 = \gamma_2(W_n)$. Therefore $b'_2(W_n) = 2$ if $n = 3k$ or $n = 3k + 1$. Now assume that $n = 3k + 2$. It is not difficult to verify that now removing any two edges does not increase the 2-domination number. This implies

that $b'_2(W_n) = 0$ or $b'_2(W_n) \geq 3$. Let us remove the edges $v_{n-2}v_{n-1}$, $v_{n-1}v_n$, and v_nv_2 . We find a relation between the numbers $\gamma_2(W_n - v_{n-2}v_{n-1} - v_{n-1}v_n - v_nv_2)$ and $\gamma_2(W_{n-2})$. Let D be any $\gamma_2(W_n - v_{n-2}v_{n-1} - v_{n-1}v_n - v_nv_2)$ -set. By Observation 1 we have $v_{n-1}, v_n \in D$. Let us observe that $D \setminus \{v_{n-1}, v_n\}$ is a 2DS of the graph W_{n-2} . Therefore $\gamma_2(W_{n-2}) \leq \gamma_2(W_n - v_{n-2}v_{n-1} - v_{n-1}v_n - v_nv_2) - 2$. Now we get $\gamma_2(W_n - v_{n-2}v_{n-1} - v_{n-1}v_n - v_nv_2) \geq \gamma_2(W_{n-2}) + 2 = \lfloor (n-1)/3 \rfloor + 3 = \lfloor (n+2)/3 \rfloor + 2 > \lfloor (n+1)/3 \rfloor + 1 = \gamma_2(W_n)$. Therefore $b'_2(W_n) = 3$ if $n = 3k + 2$. \square

Now we investigate the non-isolating 2-bondage in complete bipartite graphs.

REMARK 12. Let p and q be positive integers such that $p \leq q$. Then

$$b'_2(K_{p,q}) = \begin{cases} 3 & \text{if } p = q = 3; \\ 5 & \text{if } p = q = 4; \\ p - 1 & \text{otherwise.} \end{cases}$$

PROOF. Let $E(K_{p,q}) = \{a_i b_j : 1 \leq i \leq p \text{ and } 1 \leq j \leq q\}$. If $p = 1$, then obviously $b'_2(K_{p,q}) = 0 = p - 1$ as removing an edge gives us an isolated vertex. Now assume that $p = 2$. By Observation 7 we have $\gamma_2(K_{2,q}) = 2$. Let us observe that $\gamma_2(K_{2,q} - a_1 b_1) = 3$ as the vertex b_1 has to belong to every 2DS of the graph $K_{2,q} - a_1 b_1$. Thus $b'_2(K_{2,q}) = 1 = p - 1$.

Now let us assume that $p = 3$. By Observation 7 we have $\gamma_2(K_{3,q}) = 3$. Let us observe that removing one edge does not increase the 2-domination number. This implies that $b'_2(K_{3,q}) = 0$ or $b'_2(K_{3,q}) \geq 2$. If $q = 3$, then it is easy to verify that removing any two edges does not increase the 2-domination number. This implies that $b'_2(K_{3,3}) = 0$ or $b'_2(K_{3,3}) \geq 3$. Let us observe that $\gamma_2(K_{3,3} - a_1 b_1 - a_1 b_2 - a_2 b_1) = 4 > 3 = \gamma_2(K_{3,3})$. Therefore $b'_2(K_{3,3}) = 3$. Now assume that $q \geq 4$. We have $\gamma_2(K_{3,q} - a_1 b_1 - a_2 b_1) = 4$ as the vertex b_1 has to belong to every 2DS of the graph $K_{3,q} - a_1 b_1 - a_2 b_1$. Thus $b'_2(K_{3,q}) = 2$ if $q \geq 4$.

Now assume that $p \geq 4$. By Observation 7 we have $\gamma_2(K_{p,q}) = 4$. If $q = 4$, then it is not difficult to verify that removing any four edges does not increase the 2-domination number. This implies that $b'_2(K_{4,4}) = 0$ or $b'_2(K_{4,4}) \geq 5$. We have $\gamma_2(K_{4,4} - a_1 b_1 - a_1 b_2 - a_1 b_3 - a_2 b_1 - a_3 b_1) = 5$ as the vertices a_1 and b_1 have to belong to every 2DS of the graph $K_{4,4} - a_1 b_1 - a_1 b_2 - a_1 b_3 - a_2 b_1 - a_3 b_1$. Thus $b'_2(K_{4,4}) = 5$. Now assume that $q \geq 5$. Let us observe that removing any $p - 2$ edges does not increase the 2-domination number. This implies that $b'_2(K_{p,q}) = 0$ or $b'_2(K_{p,q}) \geq p - 1$. We have $\gamma_2(K_{p,q} - a_1 b_1 - a_2 b_1 - \dots - a_{p-1} b_1) = 5$ as the vertex b_1 has to belong to every 2DS of the graph $K_{p,q} - a_1 b_1 - a_2 b_1 - \dots - a_{p-1} b_1$. Therefore $b'_2(K_{p,q}) = p - 1$ if $p \geq 4$ and $q \geq 5$. \square



A paired dominating set of a graph G is a dominating set of vertices whose induced subgraph has a perfect matching. The paired domination number of G , denoted by $\gamma_p(G)$, is the minimum cardinality of a paired dominating set of G . The paired bondage number, denoted by $b_p(G)$, is the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\delta(G - E') \geq 1$ and $\gamma_p(G - E') > \gamma_p(G)$. If for every $E' \subseteq E$, either $\gamma_p(G - E') = \gamma_p(G)$ or $\delta(G - E') = 0$, then we define $b_p(G) = 0$, and we say that G is a γ_p -strongly stable graph. Raczek [11] observed that if $H \subseteq G$, then $b_p(H) \leq b_p(G)$. Let us observe that no inequality of such type is true for the non-isolating 2-bondage. Consider the complete bipartite graphs $K_{3,3}$, $K_{3,5}$, and $K_{4,5}$. Of course, $K_{3,3} \subseteq K_{3,5} \subseteq K_{4,5}$. Using Remark 12 we get $b'_2(K_{3,3}) = 3 > 2 = b'_2(K_{3,5}) < 3 = b'_2(K_{4,5})$.

The authors of [4] proved that the bondage number of any tree is either one or two. Let us observe that for every non-negative integer there exists a tree having such non-isolating 2-bondage number. For positive integers k consider trees T_k of the form presented in Figure 1. It is not difficult to verify that $b'_2(T_k) = k - 1$.

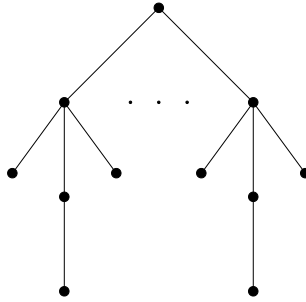


Figure 1. A tree T_k having $5k + 1$ vertices.

Hartnell and Rall [5] characterized all trees with bondage number equaling two. We characterize all trees with the non-isolating 2-bondage number equaling zero, that is, all γ_2 -non-isolatingly strongly stable trees.

We have the following property of γ_2 -non-isolatingly strongly stable trees.

LEMMA 13. *Let T be a tree with $b'_2(T) = 0$, and let x be a vertex of T which is neither a leaf nor a support vertex. Then $\gamma_2(T) = \gamma_2(T - x) + 1$.*

PROOF. The neighbors of x we denote by y_1, y_2, \dots, y_k . Let T_i mean the component of $T - x$ which contains the vertex y_i . Let $E_0 = \{xy_i : 3 \leq i \leq k\}$, $E_1 = E_0 \cup \{xy_2\}$, and $E_2 = E_0 \cup \{xy_1\}$. Since $b'_2(T) = 0$, we have $\gamma_2(T) = \gamma_2(T - E_0) = \gamma_2(T - E_1) = \gamma_2(T - E_2)$. By T'_i we denote the component of $T - E_i$ which contains the vertex x . For $i = 1, 2$, let D'_i be any $\gamma_2(T'_i)$ -set. By Observation 1 we have $x \in D'_i$. It is easy to observe that $D'_1 \cup D'_2$ is a 2DS of the tree T'_0 . Thus



$\gamma_2(T'_0) \leq \gamma_2(T'_1) + \gamma_2(T'_2) - 1$. Now let D_1 be any $\gamma_2(T_1)$ -set. Of course, $D_1 \cup \{x\}$ is a 2DS of the tree T'_1 . Thus $\gamma_2(T'_1) \leq \gamma_2(T_1) + 1$. Suppose that $\gamma_2(T'_1) < \gamma_2(T_1) + 1$. Now we get $\gamma_2(T) = \gamma_2(T - E_0) = \gamma_2(T'_0) + \gamma_2(T_3) + \gamma_2(T_4) + \dots + \gamma_2(T_k) \leq \gamma_2(T'_1) + \gamma_2(T'_2) - 1 + \gamma_2(T_3) + \gamma_2(T_4) + \dots + \gamma_2(T_k) < \gamma_2(T_1) + \gamma_2(T_2) + \gamma_2(T_3) + \gamma_2(T_4) + \dots + \gamma_2(T_k) = \gamma_2(T - E_2) = \gamma_2(T)$, a contradiction. Therefore $\gamma_2(T'_1) = \gamma_2(T_1) + 1$. Now we get $\gamma_2(T) = \gamma_2(T - E_1) = \gamma_2(T'_1) + \gamma_2(T_2) + \gamma_2(T_3) + \dots + \gamma_2(T_k) = \gamma_2(T_1) + \gamma_2(T_2) + \dots + \gamma_2(T_k) + 1 = \gamma_2(T - x) + 1$. \square

We have the following sufficient condition for that a subtree of a γ_2 -non-isolatingly strongly stable tree is also γ_2 -non-isolatingly strongly stable.

LEMMA 14. *Let T be a γ_2 -non-isolatingly strongly stable tree. Assume that $T' \neq K_1$ is a subtree of T such that $T - T'$ has no isolated vertices. Then $b_2(T') = 0$.*

PROOF. Let E_1 mean the minimum subset of the set of edges of T such that T' is a component of $T - E_1$. Now let E' be a subset of the set of edges of T' such that $\delta(T' - E') \geq 1$. The assumption $b_2(T') = 0$ implies that $\gamma_2(T - E_1 - E') = \gamma_2(T)$. We have $T - E_1 - E' = T' - E' \cup (T - T')$, and consequently, $\gamma_2(T - E_1 - E') = \gamma_2(T' - E') + \gamma_2(T - T')$. Now we get $\gamma_2(T' - E') = \gamma_2(T - E_1 - E') - \gamma_2(T - T') = \gamma_2(T) - \gamma_2(T) + \gamma_2(T') = \gamma_2(T')$. This implies that $b_2(T') = 0$. \square

Now we prove that attaching a path P_3 by joining it through the support vertex increases the 2-domination number of any graph by two.

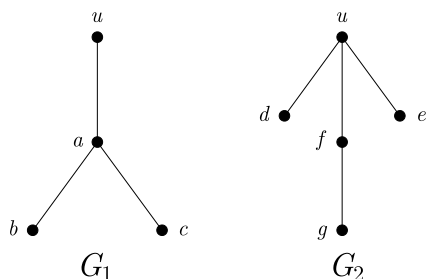
LEMMA 15. *Let G be a graph, and let H obtained from G by attaching a path P_3 by joining the support vertex to any vertex of G . Then $\gamma_2(H) = \gamma_2(G) + 2$.*

PROOF. Let $v_1v_2v_3$ mean the attached path. Let D' be any $\gamma_2(G)$ -set. It is easy to see that $D' \cup \{v_1, v_3\}$ is a 2DS of the graph H . Thus $\gamma_2(H) \leq \gamma_2(G) + 2$. Now let us observe that there exists a $\gamma_2(H)$ -set that does not contain the vertex v_2 . Let D be such a set. By Observation 1 we have $v_1, v_3 \in D$. Observe that $D \setminus \{v_1, v_3\}$ is a 2DS of the graph G . Therefore $\gamma_2(G) \leq \gamma_2(H) - 2$. This implies that $\gamma_2(H) = \gamma_2(G) + 2$. \square

Now we need to define trees G_1 and G_2 , see Figure 2. The tree G_1 is a star $K_{1,3}$ and the tree G_2 is a double star with five vertices.

For the purpose of characterizing all γ_2 -non-isolatingly strongly stable trees, that is, all trees T such that for every $E' \subseteq E$, either $\gamma_2(T - E') = \gamma_2(T)$ or $\delta(T - E') = 0$, we introduce a family \mathcal{T} of trees $T = T_k$ that can be obtained as follows. Let $T_1 \in \{P_1, P_2, P_3\}$. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.



Figure 2. The trees G_1 and G_2 .

- Operation \mathcal{O}_1 : Attach a vertex by joining it to a strong support vertex of T_k .
- Operation \mathcal{O}_2 : Attach a path P_3 by joining the support vertex to a leaf of $T_k \neq P_3$ the neighbor of which has degree at most two.
- Operation \mathcal{O}_3 : Attach a path P_3 by joining the support vertex to a vertex of T_k which is not a leaf.
- Operation \mathcal{O}_4 : Let x mean a vertex of T_k adjacent to a tree G_1 through the vertex u . Remove that tree G_1 and attach a tree G_2 by joining the vertex u to the vertex x .
- Operation \mathcal{O}_5 : Attach a path P_3 by joining the support vertex to a leaf of T_k the neighbor of which is adjacent to at least three leaves.

Now we characterize all γ_2 -non-isolatingly strongly stable trees.

THEOREM 16. *Let T be a tree. Then $b'_2(T) = 0$ if and only if $T \in \mathcal{T}$.*

PROOF. Let T be a tree of the family \mathcal{T} . We use the induction on the number k of operations performed to construct the tree T . If $T = P_1$, then obviously $b'_2(T) = 0$. If $T = P_2$, then also $b'_2(T) = 0$ as removing the edge gives us isolated vertices. Similarly, $b'_2(P_3) = 0$. Let $k \geq 2$ be an integer. Assume that the result is true for every tree $T' = T_k$ of the family \mathcal{T} constructed by $k - 1$ operations. Let $T = T_{k+1}$ be a tree of the family \mathcal{T} constructed by k operations.

First assume that T is obtained from T' by operation \mathcal{O}_1 . Let x mean the attached vertex, and let y mean its neighbor. Let D be any $\gamma_2(T)$ -set. By Observation 1 we have $x \in D$. Let us observe that $D \setminus \{x\}$ is a 2DS of the tree T' as the vertex y has at least two neighbors in $D \setminus \{x\}$. Therefore $\gamma_2(T') \leq \gamma_2(T) - 1$. Now let E' be a subset of the set of edges of T such that $\delta(T - E') \geq 1$. Since x is a leaf of T , we have $xy \notin E'$. The assumption $b'_2(T') = 0$ implies that $\gamma_2(T' - E') = \gamma_2(T')$. Let D' be any $\gamma_2(T' - E')$ -set. Of course, $D' \cup \{x\}$ is a 2DS of $T - E'$. Thus $\gamma_2(T - E') \leq \gamma_2(T' - E') + 1$. Now we get $\gamma_2(T - E') \leq$



$\gamma_2(T' - E') + 1 = \gamma_2(T') + 1 \leq \gamma_2(T)$. On the other hand, by Observation 2 we have $\gamma_2(T - E') \geq \gamma_2(T)$. This implies that $\gamma_2(T - E') = \gamma_2(T)$, and consequently, $b'_2(T) = 0$.

Now assume that T is obtained from T' by operation \mathcal{O}_2 . The leaf to which is attached P_3 we denote by x . Let y mean the neighbor of x . The attached path we denote by $v_1v_2v_3$. If $d_T(y) = 1$, then T' is a path P_2 . It is not difficult to verify that $b'_2(T) = 0$. Now assume that $d_T(y) = 2$. The neighbor of y other than x we denote by z . Since $T' \neq P_3$, we have $d_{T'}(z) \geq 2$. Let t mean a neighbor of z other than y . By Lemma 15 we have $\gamma_2(T) = \gamma_2(T') + 2$. Let E' be a subset of the set of edges of T such that $\delta(T - E') \geq 1$. Since v_1 and v_3 are leaves of T , we have $v_1v_2, v_2v_3 \notin E'$. If $xv_2 \in E'$, then $\gamma_2(T - E') = \gamma_2(P_3 \cup T' - (E' \setminus \{xv_2\})) = \gamma_2(T' - (E' \setminus \{xv_2\})) + \gamma_2(P_3) = \gamma_2(T') + 2 = \gamma_2(T)$. Now assume that $xv_2 \notin E'$. If $xy \notin E'$, then using Lemma 15 we get $\gamma_2(T - E') = \gamma_2(T' - E') + 2 = \gamma_2(T') + 2 = \gamma_2(T)$. Now assume that $xy \in E'$. By T'_y we denote the component of $T' - (E' \setminus \{xy\})$ which contains the vertex y . Let us observe that $T'_y \neq P_3$. Suppose that $T'_y = P_3$. Let $E'' = E' \setminus \{xy, zt\}$ and $E''' = E'' \cup \{yz\}$. Since $b'_2(T') = 0$, we have $\gamma_2(T' - E'') = \gamma_2(T')$ and $\gamma_2(T' - E''') = \gamma_2(T')$. This implies that $\gamma_2(T' - E') = \gamma_2(T' - E''')$. Let D''' be any $\gamma_2(T' - E''')$ -set. By Observation 1 we have $x, y, z \in D'''$. Let us observe that $D''' \setminus \{y\}$ is a 2DS of $T' - E''$. Consequently, $\gamma_2(T' - E'') \leq \gamma_2(T' - E''') - 1$, a contradiction. Therefore $T'_y \neq P_3$. Since $b'_2(T') = 0$, we have $\gamma_2(T' - (E' \setminus \{xy\}) - yz) = \gamma_2(T')$. Let D' be any $\gamma_2(T' - (E' \setminus \{xy\}) - yz)$ -set. By Observation 1 we have $x, y \in D'$. Let us observe that $D' \cup \{v_1, v_3\}$ is a 2DS of $T - E'$. Thus $\gamma_2(T - E') \leq \gamma_2(T' - (E' \setminus \{xy\}) - yz) + 2$. We get $\gamma_2(T - E') \leq \gamma_2(T' - (E' \setminus \{xy\}) - yz) + 2 = \gamma_2(T') + 2 = \gamma_2(T)$. Now we conclude that $\gamma_2(T - E') = \gamma_2(T)$. This implies that $b'_2(T) = 0$.

Now assume that T is obtained from T' by operation \mathcal{O}_3 . The vertex to which is attached P_3 we denote by x . Let $v_1v_2v_3$ mean the attached path. Let E' be a subset of the set of edges of T such that $\delta(T - E') \geq 1$. If $xv_2 \in E'$, then similarly as when considering the previous operation we get $\gamma_2(T - E') \leq \gamma_2(T)$. Now assume that $xv_2 \notin E'$. If the component of $T - E'$ which contains the vertex x is not a star $K_{1,3}$, then similarly as when considering the previous operation we get $\gamma_2(T - E') \leq \gamma_2(T)$. Now assume that the component of $T - E'$ which contains the vertex x is a star $K_{1,3}$. Let us observe that $b'_2(T' - x) = 0$. Suppose that $b'_2(T' - x) > 0$. This implies that there is a component of $T' - x$, say T_i , such that $b'_2(T_i) > 0$. Since x is not a leaf of T' , the graph $T' - T_i$ has no isolated vertices. By Lemma 14 we have $b'_2(T_i) = 0$, a contradiction. Therefore $b'_2(T' - x) = 0$. This implies that $\gamma_2(T' - x - (E' \cap E(T' - x))) = \gamma_2(T' - x)$. Let D' be any $\gamma_2(T' - x - (E' \cap E(T' - x)))$ -set. It is easy to observe that $D' \cup \{x, v_1, v_3\}$ is a 2DS of $T - E'$. Thus $\gamma_2(T - E') \leq \gamma_2(T' - x - (E' \cap E(T' - x))) + 3$. Using Lemmas 13 and 15 we get $\gamma_2(T - E') \leq \gamma_2(T' - x - (E' \cap E(T' - x))) + 3 = \gamma_2(T' - x) + 3 =$

$\gamma_2(T') + 2 = \gamma_2(T)$. Now we conclude that $\gamma_2(T - E') = \gamma_2(T)$, and consequently, $b'_2(T) = 0$.

Now assume that T is obtained from T' by operation \mathcal{O}_4 . Let us observe that there exists a $\gamma_2(T)$ -set that contains the vertex u . Let D be such a set. By Observation 1 we have $d, e, g \in D$. The set D is minimal, thus $f \notin D$. Let us observe that $D \cup \{b, c\} \setminus \{d, e, g\}$ is a 2DS of the tree T' . Therefore $\gamma_2(T') \leq \gamma_2(T) - 1$. Now let E' be a subset of the set of edges of T such that $\delta(T - E') \geq 1$. Since d, e , and g are leaves of T , we have $ud, ue, fg \notin E'$. First assume that $ux \in E'$. The assumption $b'_2(T') = 0$ implies that $\gamma_2(T' - (E' \cap E(T'))) = \gamma_2(T')$. We have $\gamma_2(G_1) = 3$ and $\gamma_2(G_2) = 4$. Now we get $\gamma_2(T - E') = \gamma_2(T' - (E' \cap E(T'))) - \gamma_2(G_1) + \gamma_2(G_2) = \gamma_2(T') - 3 + 4 = \gamma_2(T') + 1 \leq \gamma_2(T)$. Now assume that $ux \notin E'$. First assume that x is a leaf of $T' - (E' \cap E(T'))$. Since $b'_2(T') = 0$, we have $\gamma_2(T' - (E' \cap E(T'))) = \gamma_2(T')$. Let us observe that there exists a $\gamma_2(T' - (E' \cap E(T')))$ -set that contains the vertex u . Let D' be such a set. By Observation 1 we have $b, c, x \in D'$. The set D' is minimal, thus $a \notin D'$. Let us observe that $D' \setminus \{u, b, c\} \cup \{d, e, f, g\}$ is a 2DS of $T - E'$. Thus $\gamma_2(T - E') \leq \gamma_2(T' - (E' \cap E(T'))) + 1$. Now we get $\gamma_2(T - E') \leq \gamma_2(T' - (E' \cap E(T'))) + 1 = \gamma_2(T') + 1 \leq \gamma_2(T)$. Now assume that x is not a leaf of $T' - (E' \cap E(T'))$. Since $b'_2(T') = 0$, we have $\gamma_2(T' - (E' \cap E(T')) - ux) = \gamma_2(T')$. Let D' be any $\gamma_2(T' - (E' \cap E(T')) - ux)$ -set. By Observation 1 we have $b, c, u \in D'$. The set D' is minimal, thus $a \notin D'$. Let us observe that now also $D' \setminus \{u, b, c\} \cup \{d, e, f, g\}$ is a 2DS of $T - E'$. Thus $\gamma_2(T - E') \leq \gamma_2(T' - (E' \cap E(T)) - ux) + 1$. Now we get $\gamma_2(T - E') \leq \gamma_2(T' - (E' \cap E(T)) - ux) + 1 = \gamma_2(T') + 1 \leq \gamma_2(T)$. We conclude that $\gamma_2(T - E') = \gamma_2(T)$, and consequently, $b'_2(T) = 0$.

Now assume that T is obtained from T' by operation \mathcal{O}_5 . The leaf to which is attached P_3 we denote by x . Let y mean the neighbor of x . The attached path we denote by $v_1v_2v_3$. Let E' be a subset of the set of edges of T such that $\delta(T - E') \geq 1$. If $xv_2 \in E'$, then similarly as when considering the operation \mathcal{O}_2 we get $\gamma_2(T - E') \leq \gamma_2(T)$. Now assume that $xv_2 \notin E'$. If the component of $T - E'$ which contains the vertex x is not a star $K_{1,3}$, then similarly as when considering the operation \mathcal{O}_2 we get $\gamma_2(T - E') \leq \gamma_2(T)$. Now assume that the component of $T - E'$ which contains the vertex x is a star $K_{1,3}$. Since $b'_2(T') = 0$, we have $\gamma_2(T' - (E' \setminus \{xy\})) = \gamma_2(T')$. Let D' be any $\gamma_2(T' - (E' \setminus \{xy\}))$ -set. By Observation 1 we have $x \in D'$. Let us observe that $D' \cup \{v_1, v_3\}$ is a 2DS of $T - E'$ as the vertex y is adjacent to at least two leaves in $T - E'$. Thus $\gamma_2(T - E') \leq \gamma_2(T' - (E' \setminus \{xy\})) + 2$. Using Lemma 15 we get $\gamma_2(T - E') \leq \gamma_2(T' - (E' \setminus \{xy\})) + 2 = \gamma_2(T') + 2 = \gamma_2(T)$. Now we conclude that $\gamma_2(T - E') = \gamma_2(T)$. Consequently, $b'_2(T) = 0$.

Now assume that T is a γ_2 -non-isolatingly strongly stable tree. Let n mean the number of vertices of the tree T . We proceed by induction on this number. If $\text{diam}(T) = 0$, then $T = P_1 \in \mathcal{T}$. If $\text{diam}(T) = 1$, then $T = P_2 \in \mathcal{T}$. If

$\text{diam}(T) = 2$, then T is a star. If $T = P_3$, then $T \in \mathcal{T}$. If T is a star different from P_3 , then it can be obtained from P_3 by a proper number of operations \mathcal{O}_1 . Thus $T \in \mathcal{T}$. Now let us assume that $\text{diam}(T) = 3$. Thus T is a double star. Let a and b mean the support vertices of T . Without loss of generality we assume that $d_T(a) \leq d_T(b)$. If $T = P_4$, then by Remark 9 we have $b'_2(T) = 1 \neq 0$. Now assume that T is a double star different from P_4 . First assume that $d_T(a) = 1$. If $d_T(b) = 2$, then the tree T can be obtained from P_2 by operation \mathcal{O}_2 . Thus $T \in \mathcal{T}$. If $d_T(b) \geq 3$, then the tree T can be obtained from P_2 by first, operation \mathcal{O}_2 , and then a proper number of operations \mathcal{O}_1 performed on the strong support vertex. Thus $T \in \mathcal{T}$. Now assume that $d_T(a) \geq 2$. The tree T can be obtained from P_3 by first, operation \mathcal{O}_3 performed on the support vertex, and then possibly proper numbers of operations \mathcal{O}_1 performed on the support vertices. Thus $T \in \mathcal{T}$.

Now assume that $\text{diam}(T) \geq 4$. Thus the order of the tree T is an integer $n \geq 5$. The result we obtain by the induction on the number n . Assume that the lemma is true for every tree T' of order $n' < n$.

First assume that some support vertex of T , say x , is adjacent to at least three leaves. Let y mean a leaf adjacent to x . Let $T' = T - y$. Let D' be any $\gamma_2(T')$ -set. Of course, $D' \cup \{y\}$ is a 2DS of the tree T . Thus $\gamma_2(T) \leq \gamma_2(T') + 1$. Now let E' be a subset of the set of edges of T' such that $\delta(T' - E') \geq 1$. Since $b'_2(T) = 0$, we have $\gamma_2(T - E') = \gamma_2(T)$. Let D be any $\gamma_2(T - E')$ -set. By Observation 1 we have $y \in D$. Let us observe that $D \setminus \{y\}$ is a 2DS of $T' - E'$ as the vertex y is adjacent to at least two leaves in $T' - E'$. Therefore $\gamma_2(T' - E') \leq \gamma_2(T - E') - 1$. Now we get $\gamma_2(T' - E') \leq \gamma_2(T - E') - 1 = \gamma_2(T) - 1 \leq \gamma_2(T')$. On the other hand, by Observation 2 we have $\gamma_2(T' - E') \geq \gamma_2(T')$. This implies that $\gamma_2(T' - E') = \gamma_2(T')$, and consequently, $b'_2(T') = 0$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation \mathcal{O}_1 . Thus $T \in \mathcal{T}$. Henceforth, we can assume that every support vertex of T is adjacent to at most two leaves.

We now root T at a vertex r of maximum eccentricity $\text{diam}(T)$. Let t be a leaf at maximum distance from r , v be the parent of t , u be the parent of v , and w be the parent of u in the rooted tree. By T_x let us denote the subtree induced by a vertex x and its descendants in the rooted tree T .

First assume that $d_T(v) = 2$. Assume that among the descendants of u there is a support vertex, say x , different from v . It suffices to consider only the possibilities when x is adjacent to one or two leaves. First assume that x is adjacent to two leaves. Let $T' = T - T_x$. Lemma 14 implies that $b'_2(T') = 0$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation \mathcal{O}_3 . Thus $T \in \mathcal{T}$.

Now assume that x is adjacent to exactly one leaf. Let $T' = T - T_v$. Let us observe that there exists a $\gamma_2(T')$ -set that contains the vertex u . Let D' be such a set. It is easy to see that $D' \cup \{t\}$ is a 2DS of the tree T . Thus $\gamma_2(T) \leq \gamma_2(T') + 1$.

We have $T - uv = T' \cup P_2$. Now we get $\gamma_2(T - uv) = \gamma_2(T' \cup P_2) = \gamma_2(T') + \gamma_2(P_2) = \gamma_2(T') + 2 \geq \gamma_2(T) + 1 > \gamma_2(T)$. Therefore $b'_2(T) = 1$, a contradiction.

Now assume that every descendant of u excluding v is a leaf. First assume that u is adjacent to two leaves, say x and y . Let T' be a tree obtained from $T - T_u$ by attaching a tree G_1 by joining the vertex u to the vertex w . Let us observe that there exists a $\gamma_2(T')$ -set that contains the vertex u . Let D' be such a set. By Observation 1 we have $b, c \in D'$. The set D' is minimal, thus $a \notin D'$. Let us observe that $D' \setminus \{b, c\} \cup \{t, x, y\}$ is a 2DS of the tree T . Thus $\gamma_2(T) \leq \gamma_2(T') + 1$. Now let E' be a subset of the set of edges of T' such that $\delta(T' - E') \geq 1$. Since b and c are leaves of T' , we have $ab, ac \notin E'$. The assumption $b'_2(T) = 0$ implies that $\gamma_2(T - (E' \cap E(T))) = \gamma_2(T)$. Let us observe that there exists a $\gamma_2(T - (E' \cap E(T)))$ -set that contains the vertex u . Let D be such a set. By Observation 1 we have $t, x, y \in D$. The set D is minimal, thus $v \notin D$. Let us observe that $D \cup \{b, c\} \setminus \{t, x, y\}$ is a 2DS of $T' - E'$. Therefore $\gamma_2(T' - E') \leq \gamma_2(T - (E' \cap E(T))) - 1$. Now we get $\gamma_2(T' - E') \leq \gamma_2(T - (E' \cap E(T))) - 1 = \gamma_2(T) - 1 \leq \gamma_2(T')$. We conclude that $\gamma_2(T' - E') = \gamma_2(T')$, and consequently, $b'_2(T') = 0$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation \mathcal{O}_4 . Thus $T \in \mathcal{T}$.

Now assume that u is adjacent to exactly one leaf, say x . Let $E' = \{wu, uv\}$ and $T' = T - T_u$. Let D' be any $\gamma_2(T')$ -set. It is easy to observe that $D' \cup \{u, t, x\}$ is a 2DS of the tree T . Thus $\gamma_2(T) \leq \gamma_2(T') + 3$. We have $T - E' = T' \cup P_2 \cup P_2$. Now we get $\gamma_2(T - E') = \gamma_2(T' \cup P_2 \cup P_2) = \gamma_2(T') + 2\gamma_2(P_2) = \gamma_2(T') + 4 \geq \gamma_2(T) + 1 > \gamma_2(T)$. This implies that $b'_2(T) \in \{1, 2\}$, a contradiction.

Now assume that $d_T(u) = 2$. Let $T' = T - T_v$. Let D' be any $\gamma_2(T')$ -set. By Observation 1 we have $u \in D'$. It is easy to see that $D' \cup \{t\}$ is a 2DS of the tree T . Thus $\gamma_2(T) \leq \gamma_2(T') + 1$. We have $T - uv = T' \cup P_2$. Now we get $\gamma_2(T - uv) = \gamma_2(T' \cup P_2) = \gamma_2(T') + \gamma_2(P_2) = \gamma_2(T') + 2 \geq \gamma_2(T) + 1 > \gamma_2(T)$. Therefore $b'_2(T) = 1$, a contradiction.

Now assume that $d_T(v) = 3$. The leaf adjacent to v and different from t we denote by a . Assume that $d_T(u) \geq 3$. Let $T' = T - T_v$. Lemma 14 implies that $b'_2(T') = 0$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation \mathcal{O}_3 . Thus $T \in \mathcal{T}$.

Now assume that $d_T(u) = 2$. First assume that w is adjacent to two leaves. Let $T' = T - T_v$. Lemma 14 implies that $b'_2(T') = 0$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation \mathcal{O}_5 . Thus $T \in \mathcal{T}$.

Now assume that w is adjacent to exactly one leaf, say x . Let G' be a graph obtained from T by removing all edges incident to w excluding wx . Let D' be any $\gamma_2(G')$ -set. By Observation 1 we have $u, w, x \in D'$. Let us observe that $D' \setminus \{w\}$ is a 2DS of the tree T . Thus $\gamma_2(T) \leq \gamma_2(G') - 1$. Therefore $b'_2(T) > 0$, a

contradiction.

Now assume that there is a descendant of w , say k , such that the distance of w to the most distant vertex of T_k is three. It suffices to consider only the possibility when T_k is isomorphic to T_u . The descendant of k we denote by l , and the leaves adjacent to l we denote by m and p . Let G' be a graph obtained from T by removing all edges incident to w excluding wu . Let us observe that there exists a $\gamma_2(G')$ -set that contains the vertex u . Let D' be such a set. By Observation 1 we have $w, k \in D'$. Let us observe that $D' \setminus \{w\}$ is a 2DS of the tree T . Thus $\gamma_2(T) \leq \gamma_2(G') - 1$. This implies that $b'_2(T) > 0$, a contradiction.

Now assume that there is a descendant of w , say k , such that the distance of w to the most distant vertex of T_k is two. It suffices to consider only the possibilities when k is adjacent to one or two leaves. First assume that k is adjacent to two leaves. Let $T' = T - T_k$. Lemma 14 implies that $b'_2(T') = 0$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation \mathcal{O}_3 . Thus $T \in \mathcal{T}$.

Now assume that k is adjacent to exactly one leaf. Let $T' = T - T_k$. Similarly as when T_k is isomorphic to T_u we conclude that $b'_2(T) > 0$, a contradiction.

Now assume that $d_T(w) = 2$. Let $T' = T - T_v$. Lemma 14 implies that $b'_2(T') = 0$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation \mathcal{O}_2 . Thus $T \in \mathcal{T}$. \square

ACKNOWLEDGMENTS. Thanks are due to the anonymous referee for comments that helped to make the paper correct.

References

- [1] M. Blidia, M. Chellali and O. Favaron, Independence and 2-domination in trees, *Australas. J. Combin.*, **33** (2005), 317–327.
- [2] G. S. Domke and R. C. Laskar, The bondage and reinforcement numbers of γ_f for some graphs, *Discrete Math.*, **167/168** (1997), 249–259.
- [3] J. F. Fink and M. S. Jacobson, n -domination in graphs, In: *Graph Theory with Applications to Algorithms and Computer Science* (Kalamazoo, Mich., 1984), Wiley-Intersci. Publ., Wiley, New York, 1985, pp. 282–300.
- [4] J. F. Fink, M. S. Jacobson, L. F. Kinch and J. Roberts, The bondage number of a graph, *Discrete Math.*, **86** (1990), 47–57.
- [5] B. L. Hartnell and D. F. Rall, A characterization of trees in which no edge is essential to the domination number, *Ars Combin.*, **33** (1992), 65–76.
- [6] B. L. Hartnell and D. F. Rall, Bounds on the bondage number of a graph, *Discrete Math.*, **128** (1994), 173–177.
- [7] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Monogr. Textbooks Pure Apple. Math., **208**, Marcel Dekker, New York, 1998.
- [8] T. W. Haynes, S. T. Hedetniemi and P. J. Slater (eds.), *Domination in Graphs: Advanced Topics*, Monogr. Textbooks Pure Apple. Math., **209**, Marcel Dekker, New York, 1998.
- [9] L. Kang and J. Yuan, Bondage number of planar graphs, *Discrete Math.*, **222** (2000),



- 191–198.
- [10] H. Liu and L. Sun, The bondage and connectivity of a graph, *Discrete Math.*, **263** (2003), 289–293.
 - [11] J. Raczek, Paired bondage in trees, *Discrete Math.*, **308** (2008), 5570–5575.
 - [12] U. Teschner, New results about the bondage number of a graph, *Discrete Math.*, **171** (1997), 249–259.
 - [13] L. Volkmann, A Nordhaus-Gaddum-type result for the 2-domination number, *J. Combin. Math. Combin. Comput.*, **64** (2008), 227–235.
 - [14] Y. P. Wu and Q. Fan, The bondage number of four domination parameters for trees, *J. Math. (Wuhan)*, **24** (2004), 267–270.

Marcin KRZYWKOWSKI

Faculty of Electronics
Telecommunications and Informatics
Gdańsk University of Technology
Narutowicza 11/12
80–233 Gdańsk, Poland
E-mail: marcin.krzywkowski@gmail.com

