



## On the metric dimension of corona product graphs

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### ABSTRACT

Given a set of vertices  $S = \{v_1, v_2, \dots, v_k\}$  of a connected graph  $G$ , the metric representation of a vertex  $v$  of  $G$  with respect to  $S$  is the vector  $r(v|S) = (d(v, v_1), d(v, v_2), \dots, d(v, v_k))$ , where  $d(v, v_i)$ ,  $i \in \{1, \dots, k\}$  denotes the distance between  $v$  and  $v_i$ .  $S$  is a resolving set for  $G$  if for every pair of distinct vertices  $u, v$  of  $G$ ,  $r(u|S) \neq r(v|S)$ . The metric dimension of  $G$ ,  $\dim(G)$ , is the minimum cardinality of any resolving set for  $G$ . Let  $G$  and  $H$  be two graphs of order  $n_1$  and  $n_2$ , respectively. The corona product  $G \odot H$  is defined as the graph obtained from  $G$  and  $H$  by taking one copy of  $G$  and  $n_1$  copies of  $H$  and joining by an edge each vertex from the  $i$ th-copy of  $H$  with the  $i$ th-vertex of  $G$ . For any integer  $k \geq 2$ , we define the graph  $G \odot^k H$  recursively from  $G \odot H$  as  $G \odot^k H = (G \odot^{k-1} H) \odot H$ . We give several results on the metric dimension of  $G \odot^k H$ . For instance, we show that given two connected graphs  $G$  and  $H$  of order  $n_1 \geq 2$  and  $n_2 \geq 2$ , respectively, if the diameter of  $H$  is at most two, then  $\dim(G \odot^k H) = n_1(n_2 + 1)^{k-1} \dim(H)$ . Moreover, if  $n_2 \geq 7$  and the diameter of  $H$  is greater than five or  $H$  is a cycle graph, then  $\dim(G \odot^k H) = n_1(n_2 + 1)^{k-1} \dim(K_1 \odot H)$ .

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### 1. Introduction

The concepts of resolvability and location in graphs were described independently by Harary and Melter [1], and Slater [2], to define the same structure in a graph. After these papers were published, several authors developed diverse theoretical works on this topic [3–10]. Slater described the usefulness of these ideas into long range aids to navigation [2]. Also, these concepts have some applications in chemistry for representing chemical compounds [11,12] or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [13]. Other applications of this concept to navigation of robots in networks and other areas appear in [6,8,14]. Some variations on resolvability or location have been appearing in the literature, like those about conditional resolvability [9], locating domination [15], resolving domination [16] and resolving partitions [5,17–19]. In this article we study the metric dimension of corona product graphs.

We begin by giving some basic concepts and notations. Let  $G = (V, E)$  be a simple graph of order  $n = |V|$ . Let  $u, v \in V$  be two different vertices in  $G$ , the distance  $d_G(u, v)$  between two vertices  $u$  and  $v$  of  $G$  is the length of a shortest path between  $u$  and  $v$ . If there is no ambiguity, we will use the notation  $d(u, v)$  instead of  $d_G(u, v)$ . The diameter of  $G$  is defined as  $D(G) = \max_{u, v \in V} \{d(u, v)\}$ . Given  $u, v \in V$ ,  $u \sim v$  means that  $u$  and  $v$  are adjacent vertices. Given a set of vertices  $S = \{v_1, v_2, \dots, v_k\}$  of a connected graph  $G$ , the *metric representation* of a vertex  $v \in V$  with respect to  $S$  is the vector  $r(v|S) = (d(v, v_1), d(v, v_2), \dots, d(v, v_k))$ . We say that  $S$  is a *resolving set* for  $G$  if for every pair of distinct vertices  $u, v \in V$ ,  $r(u|S) \neq r(v|S)$ . The *metric dimension* of  $G$  is the minimum cardinality of any resolving set for  $G$ , and it is denoted by  $\dim(G)$ .

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Let  $G$  and  $H$  be two graphs of order  $n_1$  and  $n_2$ , respectively. The corona product  $G \odot H$  is defined as the graph obtained from  $G$  and  $H$  by taking one copy of  $G$  and  $n_1$  copies of  $H$  and joining by an edge each vertex from the  $i$ th-copy of  $H$  with the  $i$ th-vertex of  $G$ . We will denote by  $V = \{v_1, v_2, \dots, v_{n_1}\}$  the set of vertices of  $G$  and by  $H_i = (V_i, E_i)$  the copy of  $H$  such that  $v_i \sim v$  for every  $v \in V_i$ . Notice that the corona graph  $K_1 \odot H$  is isomorphic to the join graph  $K_1 + H$ . For any integer  $k \geq 2$ , we define the graph  $G \odot^k H$  recursively from  $G \odot H$  as  $G \odot^k H = (G \odot^{k-1} H) \odot H$ . We also note that the order of  $G \odot^k H$  is  $n_1(n_2 + 1)^k$ .

## 2. Metric dimension of corona product graphs

We begin by presenting the following useful facts.

**Lemma 1.** Let  $G = (V, E)$  be a connected graph of order  $n \geq 2$  and let  $H$  be a graph of order at least two. Let  $H_i = (V_i, E_i)$  be the subgraph of  $G \odot H$  corresponding to the  $i$ th-copy of  $H$ .

- (i) If  $u, v \in V_i$ , then  $d_{G \odot H}(u, x) = d_{G \odot H}(v, x)$  for every vertex  $x$  of  $G \odot H$  not belonging to  $V_i$ .
- (ii) If  $S$  is a resolving set for  $G \odot H$ , then  $V_i \cap S \neq \emptyset$  for every  $i \in \{1, \dots, n\}$ .
- (iii) If  $S$  is a resolving set for  $G \odot H$  of minimum cardinality, then  $V \cap S = \emptyset$ .
- (iv) If  $H$  is a connected graph and  $S$  is a resolving set for  $G \odot H$ , then for every  $i \in \{1, \dots, n\}$ ,  $S \cap V_i$  is a resolving set for  $H_i$ .

**Proof.** (i) Let  $y = v_i \in V$ . The result directly follows from the fact that  $d_{G \odot H}(u, x) = d_{G \odot H}(u, y) + d_{G \odot H}(y, x) = d_{G \odot H}(v, y) + d_{G \odot H}(y, x) = d_{G \odot H}(v, x)$ .

(ii) We suppose  $V_i \cap S = \emptyset$  for some  $i \in \{1, \dots, n\}$ . Let  $x, y \in V_i$ . By (i) we have  $d_{G \odot H}(x, u) = d_{G \odot H}(y, u)$  for every vertex  $u \in S$ , which is a contradiction.

(iii) We will show that  $S' = S - V$  is a resolving set for  $G \odot H$ . Now let  $x, y$  be two different vertices of  $G \odot H$ . We have the following cases.

Case 1:  $x, y \in V_i$ . By (i) we conclude that there exist  $v \in V_i \cap S'$  such that  $d_{G \odot H}(x, v) \neq d_{G \odot H}(y, v)$ .

Case 2:  $x \in V_i$  and  $y \in V_j$ ,  $i \neq j$ . Let  $v \in V_i \cap S'$ . Then we have  $d_{G \odot H}(x, v) \leq 2 < 3 \leq d_{G \odot H}(y, v)$ .

Case 3:  $x, y \in V$ . Let  $x = v_i$  and let  $v \in V_i \cap S'$ . Then we have  $d_{G \odot H}(x, v) = 1 < 1 + d_{G \odot H}(y, x) = d_{G \odot H}(y, v)$ .

Case 4:  $x \in V_i$  and  $y \in V$ . If  $x \sim y$ , then  $y = v_i$ . Let  $v_j \in V$ ,  $j \neq i$ , and let  $v \in V_j \cap S'$ . Then we have  $d_{G \odot H}(x, v) = 1 + d_{G \odot H}(y, v) > d_{G \odot H}(y, v)$ . For  $x \not\sim y = v_l$  we take  $v \in V_l \cap S'$  and we obtain  $d_{G \odot H}(x, v) = d_{G \odot H}(x, y) + d_{G \odot H}(y, v) > d_{G \odot H}(y, v)$ .

Therefore,  $S'$  is a resolving set for  $G \odot H$ .

(iv) Let  $S_i = S \cap V_i$ . For  $x \in S_i$  or  $y \in S_i$  the result is straightforward. We suppose  $x, y \in V_i - S_i$ . Since  $S$  is a resolving set for  $G \odot H$ , we have  $r(x|S) \neq r(y|S)$ . By (i),  $d_{G \odot H}(x, u) = d_{G \odot H}(y, u)$  for every vertex  $u$  of  $G \odot H$  not belonging to  $V_i$ . So, there exists  $v \in S_i$  such that  $d_{G \odot H}(x, v) \neq d_{G \odot H}(y, v)$ . Thus, either  $(v \sim x \text{ and } v \not\sim y)$  or  $(v \not\sim x \text{ and } v \sim y)$ . In the first case we have  $d_{G \odot H}(x, v) = d_{H_i}(x, v) = 1$  and  $d_{G \odot H}(y, v) = 2 \leq d_{H_i}(y, v)$ . The case  $v \not\sim x$  and  $v \sim y$  is analogous. Therefore,  $S_i$  is a resolving set for  $H_i$ .  $\square$

**Theorem 2.** Let  $G$  and  $H$  be two connected graphs of order  $n_1 \geq 2$  and  $n_2 \geq 2$ , respectively. Then,

$$\dim(G \odot^k H) \geq n_1(n_2 + 1)^{k-1} \dim(H).$$

**Proof.** Let  $S$  be a resolving set of minimum cardinality in  $G \odot H$ . From Lemma 1(iii) we have that  $S \cap V = \emptyset$ . Moreover, by Lemma 1(ii) we have that for every  $i \in \{1, \dots, n_1\}$  there exist a nonempty set  $S_i \subset V_i$  such that  $S = \bigcup_{i=1}^{n_1} S_i$ . Now, by using Lemma 1(iv) we have that  $S_i$  is a resolving set for  $H_i$ . Hence,  $\dim(G \odot H) = |S| = \sum_{i=1}^{n_1} |S_i| \geq \sum_{i=1}^{n_1} \dim(H) = n_1 \dim(H)$ . As a result, the lower bound follows.  $\square$

**Theorem 3.** Let  $G$  be a connected graph of order  $n_1 \geq 2$  and let  $H$  be a graph of order  $n_2 \geq 2$ . If  $D(H) \leq 2$ , then

$$\dim(G \odot^k H) = n_1(n_2 + 1)^{k-1} \dim(H).$$

**Proof.** Let  $S_i \subset V_i$  be a resolving set for  $H_i$  and let  $S = \bigcup_{i=1}^{n_1} S_i$ . We will show that  $S$  is a resolving set for  $G \odot H$ . Let us consider two different vertices  $x, y$  of  $G \odot H$ . We have the following cases.

Case 1:  $x, y \in V_i$ . Since  $D(H_i) \leq 2$ , we have that  $r(x|S_i) \neq r(y|S_i)$  leads to  $r(x|S) \neq r(y|S)$ .

Case 2:  $x \in V_i$  and  $y \in V_j$ ,  $i \neq j$ . Let  $v \in S_i$ . Hence we have  $d(x, v) \leq 2 < 3 \leq d(y, v)$ .

Case 3:  $x, y \in V$ . Let  $x = v_i$ . Then for every vertex  $v \in S_i$  we have  $d(x, v) = 1 < d(y, x) + 1 = d(y, v)$ .

Case 4:  $x \in V_i$  and  $y \in V$ . If  $x \sim y$ , then let  $v \in S_j$ , for some  $j \neq i$ . So we have  $d(x, v) = 1 + d(y, v) > d(y, v)$ . Moreover, if  $x \not\sim y = v_j$ , for  $v \in S_j$  we have  $d(x, v) = d(x, y) + d(y, v) > d(y, v)$ .

Thus, for every different vertices  $x, y$  of  $G \odot H$ , we have  $r(x|S) \neq r(y|S)$ , as a consequence,  $\dim(G \odot H) \leq n_1 \dim(H)$ . Therefore, we have  $\dim(G \odot^k H) \leq n_1(n_2 + 1)^{k-1} \dim(H)$ . By Theorem 2 we conclude the proof.  $\square$

In order to show a consequence of the above theorem we present the following well-known result, where  $K_t$  denotes a complete graph of order  $t$ ,  $K_{s,t}$  denotes a complete bipartite graph of order  $s + t$  and  $N_t$  denotes an empty graph of order  $t$ .

**Lemma 4** ([6]). Let  $G$  be a connected graph of order  $n \geq 4$ . Then  $\dim(G) = n - 2$  if and only if  $G = K_{s,t}$ , ( $s, t \geq 1$ ),  $G = K_s + N_t$ , ( $s \geq 1, t \geq 2$ ), or  $G = K_s + (K_1 \cup K_t)$ , ( $s, t \geq 1$ ).

**Corollary 5.** Let  $G$  be a connected graph of order  $n_1 \geq 2$  and let  $H$  be a graph of order  $n_2 \geq 4$  and diameter  $D(H) \leq 2$ . Then

$$\dim(G \odot^k H) = n_1(n_2 + 1)^{k-1}(n_2 - 2)$$

if and only if  $H = K_{s,t}$ , ( $s, t \geq 1$ );  $H = K_s + N_t$ , ( $s \geq 1, t \geq 2$ ), or  $H = K_s + (K_1 \cup K_t)$ , ( $s, t \geq 1$ ).

We recall that the wheel graph of order  $n + 1$  is defined as  $W_{1,n} = K_1 \odot C_n$ , where  $K_1$  is the singleton graph and  $C_n$  is the cycle graph of order  $n$ . The metric dimension of the wheel  $W_{1,n}$  was obtained by Buczkowski et al. in [20].

**Remark 6** ([20]). Let  $W_{1,n}$  be a wheel graph. Then

$$\dim(W_{1,n}) = \begin{cases} 3 & \text{for } n = 3, 6, \\ 2 & \text{for } n = 4, 5, \\ \left\lfloor \frac{2n+2}{5} \right\rfloor & \text{otherwise.} \end{cases}$$

The fan graph  $F_{n_1, n_2}$  is defined as the graph join  $N_{n_1} + P_{n_2}$ , where  $N_{n_1}$  is the empty graph of order  $n_1$  and  $P_{n_2}$  is the path graph of order  $n_2$ . The case  $n_1 = 1$  corresponds to the usual fan graphs. Notice that, for the metric dimension of fan graphs, it is possible to find an equivalent result to Remark 6 which was obtained by Caceres et al. in [4].

**Remark 7** ([4]). Let  $F_{1,n}$  be a fan graph. Then

$$\dim(F_{1,n}) = \begin{cases} 1 & \text{for } n = 1, \\ 2 & \text{for } n = 2, 3, \\ 3 & \text{for } n = 6, \\ \left\lfloor \frac{2n+2}{5} \right\rfloor & \text{otherwise.} \end{cases}$$

As a particular case of Theorem 3 we obtain the following results.

**Corollary 8.** Let  $G$  be a connected graph of order  $n_1 \geq 2$ . If  $H$  is a wheel graph or a fan graph of order  $n_2 \geq 8$ , then

$$\dim(G \odot^k H) = n_1(n_2 + 1)^{k-1} \left\lfloor \frac{2n_2}{5} \right\rfloor.$$

**Theorem 9.** Let  $G$  be a connected graph of order  $n_1 \geq 2$  and let  $H$  be a graph of order  $n_2 \geq 2$ . Let  $\alpha$  be the number of connected components of  $H$  of order greater than one and let  $\beta$  be the number of isolated vertices of  $H$ . Then

$$\dim(G \odot^k H) \leq \begin{cases} n_1(n_2 + 1)^{k-1}(n_2 - \alpha - 1) & \text{for } \alpha \geq 1 \text{ and } \beta \geq 1, \\ n_1(n_2 + 1)^{k-1}(n_2 - \alpha) & \text{for } \alpha \geq 1 \text{ and } \beta = 0, \\ n_1(n_2 + 1)^{k-1}(n_2 - 1) & \text{for } \alpha = 0. \end{cases}$$

**Proof.** We suppose  $\alpha \geq 1$  and  $\beta \geq 1$ . Let  $A_i$  be the set of vertices of  $G \odot H$  formed by all but one of the vertices per each of the  $\alpha$  connected components of  $H_i$ . If  $\beta \geq 2$  we define  $B_i$  to be the set of vertices of  $G \odot H$  formed by all but one of the isolated vertices of  $H_i$ . If  $\beta = 1$ , we assume  $B_i = \emptyset$ . Let us show that  $S = \cup_{j=1}^{n_1} (A_j \cup B_j)$  is a resolving set for  $G \odot H$ . Let  $x, y$  be two different vertices of  $G \odot H$ . We suppose  $x, y \notin S$ . We have the following cases.

Case 1:  $x = v_i \in V$  and  $y \in V_i$ . For every vertex  $u \in V_j \cap S, j \neq i$ , we obtain  $d(y, u) = d(y, x) + d(x, u) > d(x, u)$ .

Case 2:  $x = v_i \in V$  and  $y \notin V_i$ . For every  $v \in S \cap V_i$  we have  $d(x, v) = 1 < d(y, v)$ .

Case 3:  $x \in V_i$  and  $y \in V_j, j \neq i$ . For every  $u \in V_i \cap S$  we have  $d(x, u) \leq 2 < 3 \leq d(y, u)$ .

Case 4:  $x, y \in V_i$ . We consider, without loss of generality, that  $x$  is not an isolated vertex in  $H_i$ . Then there exists  $v \in V_i \cap S$  such that  $v \sim x$ , so  $d(x, v) = 1 < 2 = d(y, v)$ .

Thus, for every two different vertices  $x, y$  of  $G \odot H$ , we obtain  $r(x|S) \neq r(y|S)$  and, as a consequence,  $\dim(G \odot H) \leq n_1(n_2 - \alpha - 1)$ .

As above, if  $\beta = 0$  then we take  $S = \cup_{j=1}^{n_1} A_j$  and we obtain  $\dim(G \odot H) \leq n_1(n_2 - \alpha)$  and if  $\alpha = 0$ , then we take  $S = \cup_{j=1}^{n_1} B_j$  and we obtain  $\dim(G \odot H) \leq n_1(n_2 - 1)$ . Note that if  $\alpha = 0$ , then it is not necessary to consider Case 4. Thus, the result follows.  $\square$

**Corollary 10.** Let  $G$  be a connected graph of order  $n_1 \geq 2$  and let  $H$  be an unconnected graph of order  $n_2 \geq 2$ . Then

$$\dim(G \odot^k H) = n_1(n_2 + 1)^{k-1}(n_2 - 1)$$

if and only if  $H \cong N_{n_2}$ .

**Proof.** In [21] the authors showed that  $\dim(G \odot N_{n_2}) = n_1(n_2 - 1)$ . Hence,  $\dim(G \odot^k N_{n_2}) = n_1(n_2 + 1)^{k-1}(n_2 - 1)$ . Moreover, by the above theorem, if  $H$  is unconnected and  $H \not\cong N_{n_2}$ , then  $\dim(G \odot^k H) \leq n_1(n_2 + 1)^{k-1}(n_2 - 2)$ .  $\square$

**Theorem 11.** Let  $G$  and  $H$  be two connected graphs of order  $n_1 \geq 2$  and  $n_2 \geq 3$ , respectively. Then

$$\dim(G \odot^k H) = n_1(n_2 + 1)^{k-1}(n_2 - 1)$$

if and only if  $H \cong K_{n_2}$ . Moreover, if  $H \not\cong K_{n_2}$ , then

$$\dim(G \odot^k H) \leq n_1(n_2 + 1)^{k-1}(n_2 - 2).$$

**Proof.** Since  $\dim(K_{n_2}) = n_2 - 1$ , by Theorem 3 we conclude  $\dim(G \odot^k K_{n_2}) = n_1(n_2 + 1)^{k-1}(n_2 - 1)$ . On the contrary, we suppose  $H \not\cong K_{n_2}$ . Given a set  $X$  of vertices of  $H$  and a vertex  $v$  of  $H$ ,  $N_X(v)$  denotes the set of neighbors that  $v$  has in  $X$ :  $N_X(v) = \{u \in X : u \sim v\}$ . Given two vertices  $a, b$  of  $H$ , let  $X_{a,b}$  be the set formed by all vertices of  $H$  different from  $a$  and  $b$ . Since  $H$  is a connected graph and  $H \neq K_{n_2}$ , there exist at least two vertices  $a, b$  of  $H$  such that  $N_{X_{a,b}}(a) \neq N_{X_{a,b}}(b)$ . Let  $a_i, b_i$  be the vertices corresponding to  $a, b$ , respectively, in the  $i$ th-copy  $H_i = (V_i, E_i)$  of  $H$ . Let  $S = \bigcup_{i=1}^{n_1} (V_i - \{a_i, b_i\})$ . We will show that  $S$  is a resolving set for  $G \odot H$ . Let  $x, y$  be two different vertices of  $G \odot H$  such that  $x, y \notin S$ . We have the following cases.

Case 1:  $x = a_i$  and  $y = b_i$ . Since  $N_{X_{a,b}}(a) \neq N_{X_{a,b}}(b)$  we have  $r(x|S) \neq r(y|S)$ .

Case 2:  $x = v_i \in V$  and  $y \in V_i$ . For every  $v \in V_j - \{a_j, b_j\}$ ,  $j \neq i$ , we have  $d(y, v) = d(y, x) + d(x, v) > d(x, v)$ . If  $x \in V_i$  and  $y \in V_j$ ,  $j \neq i$ , then for every  $v \in V_i - \{a_i, b_i\}$  we have  $d(x, v) \leq 2 < 3 \leq d(y, v)$ .

Case 3:  $x, y \in V$ . Say  $x = v_i$ . Then for every  $v \in V_i - \{a_i, b_i\}$  we have  $d(x, v) = 1 < d(y, v)$ .

Hence, for every two different vertices  $x, y$  of  $G \odot H$ , we obtain  $r(x|S) \neq r(y|S)$ . Thus,  $\dim(G \odot H) \leq n_1(n_2 - 2)$ . Therefore, the result follows.  $\square$

As we have shown in Corollary 5, the above bound is tight.

**Theorem 12.** Let  $G$  be a connected graph of order  $n_1 \geq 2$  and let  $H$  be a graph of order  $n_2 \geq 2$ . Then

$$\dim(G \odot^k H) \leq n_1(n_2 + 1)^{k-1} \dim(K_1 \odot H).$$

**Proof.** We denote by  $K_1 \odot H_i$  the subgraph of  $G \odot H$ , obtained by joining the vertex  $v_i \in V$  with all vertices of  $H_i$ . For every  $v_i \in V$ , let  $B_i$  be a resolving set of minimum cardinality of  $K_1 \odot H_i$  and let  $B = \bigcup_{i=1}^{n_1} B_i$ . By Lemma 1(ii) we have that  $v_i$  does not belong to any resolving set of minimum cardinality for  $K_1 \odot H_i$ . So,  $B$  does not contain any vertex from  $G$ . We will show that  $B$  is a resolving set for  $G \odot H$ . Let  $x, y$  be two different vertices in  $G \odot H$ . We consider the following cases.

Case 1:  $x, y \in V_i$ . There exists  $u \in B_i$  such that  $d_{K_1 \odot H_i}(x, u) \neq d_{K_1 \odot H_i}(y, u)$ , which leads to  $d_{G \odot H}(x, u) \neq d_{G \odot H}(y, u)$ .

Case 2:  $x \in V_i$  and  $y \in V_j$ ,  $i \neq j$ . Let  $v \in B_i$ . We have  $d_{G \odot H}(x, v) \leq 2 < 3 \leq d_{G \odot H}(y, v)$ .

Case 3:  $x, y \in V$ . Suppose now that  $x$  is adjacent to the vertices of  $H_i$ . Hence, for every vertex  $v \in B_i$  we have  $d_{G \odot H}(x, v) = 1 < d_{G \odot H}(y, x) + 1 = d_{G \odot H}(y, v)$ .

Case 4:  $x \in V_i$  and  $y \in V$ . If  $x \sim y$ , then for every vertex  $v \in B_j$ , with  $j \neq i$ , we have  $d_{G \odot H}(x, v) = 1 + d_{G \odot H}(y, v) > d_{G \odot H}(y, v)$ . Now, let us assume that  $x \not\sim y$ . Hence, there exists  $v \in B_j$  adjacent to  $y$ , with  $j \neq i$ . So, we have  $d_{G \odot H}(x, v) = d_{G \odot H}(x, y) + 1 = d_{G \odot H}(x, y) + d_{G \odot H}(y, v) > d_{G \odot H}(y, v)$ .

Thus, for every two different vertices  $x, y$  of  $G \odot H$ , we have  $r(x|B) \neq r(y|B)$  and, as a consequence,  $\dim(G \odot H) \leq n_1 \dim(K_1 \odot H)$ . Therefore, the result follows.  $\square$

**Theorem 13.** Let  $G$  be a connected graph of order  $n_1 \geq 2$  and let  $H$  be a graph of order  $n_2 \geq 7$ . If  $D(H) \geq 6$  or  $H$  is a cycle graph, then

$$\dim(G \odot^k H) = n_1(n_2 + 1)^{k-1} \dim(K_1 \odot H).$$

**Proof.** Let  $S$  be a resolving set of minimum cardinality in  $G \odot H$ . By Lemma 1(iii) we have  $S \cap V = \emptyset$ , as a consequence,  $S = \bigcup_{i=1}^{n_1} S_i$ , where  $S_i \subset V_i$ . Notice that, by Lemma 1(ii),  $S_i \neq \emptyset$  for every  $i \in \{1, \dots, n_1\}$ . Now we differentiate two cases in order to show that  $r(x|S_i) \neq (1, \dots, 1)$  for every  $x \in V_i - S_i$ .

Case 1:  $H$  is a cycle graph of order  $n_2 \geq 7$ . If  $r(a|S_i) = (1, 1)$  for some  $a \in V_i - S_i$ , then, since  $n_2 \geq 7$ , there exist two vertices  $x, y \in V_i - S_i$  such that  $d_{H_i}(x, v) > 1$  and  $d_{H_i}(y, v) > 1$ , for every  $v \in S_i$ . Hence,  $d_{G \odot H}(x, v) = d_{G \odot H}(y, v) = 2$  for every  $v \in S_i$ , which is a contradiction because, by Lemma 1(i),  $d_{G \odot H}(x, v) = d_{G \odot H}(y, v)$  for every vertex  $u$  of  $S$  not belonging to  $S_i$ .

Case 2:  $D(H) \geq 6$ . Let  $x, y \in V_i - S_i$ . Since  $S$  is a resolving set for  $G \odot H$ , we have  $r(x|S) \neq r(y|S)$ . As we have noted before, by Lemma 1(i) we have that  $d_{G \odot H}(x, u) = d_{G \odot H}(y, u)$  for every vertex  $u$  of  $G \odot H$  not belonging to  $V_i$ . So, there exists  $v \in S_i$  such that  $d_{G \odot H}(x, v) \neq d_{G \odot H}(y, v)$  and, as a consequence, either  $(v \sim x \text{ and } v \not\sim y)$  or  $(v \not\sim x \text{ and } v \sim y)$ . Now we suppose that there exists a vertex  $a \in V_i - S_i$  such that  $r(a|S_i) = (1, 1, \dots, 1)$ . If there exists a vertex  $b \in V_i - S_i$  such that  $d_{H_i}(b, u) > 1$ , for every  $u \in S_i$ , then for every  $w \in V_i - (S_i \cup \{a, b\})$ , there exists  $v \in S_i$  such that  $w \sim v$ . Then  $D(H_i) \leq 5$ . Moreover, if for every  $b \in V_i - S_i$  there exists  $v_b \in S_i$  such that  $v_b \sim b$ , then  $D(H) \leq 4$ . Therefore, if  $D(H) \geq 6$ , then  $r(a|S_i) \neq (1, 1, \dots, 1)$  for every  $a \in V_i - S_i$ .

Now, we denote by  $K_1 \odot H_i$  the subgraph of  $G \odot H$ , obtained by joining the vertex  $v_i \in V$  with all vertices of the  $i$ th-copy of  $H$ . In both the above cases we have  $r(v_i|S_i) = (1, 1, \dots, 1) \neq r(x|S_i)$  for every  $x \in V_i - S_i$ , so  $S_i$  is a resolving set for  $K_1 \odot H_i$ . Hence,  $\dim(K_1 \odot H_i) \leq |S_i|$ , for every  $i \in \{1, \dots, n_1\}$ . Thus,  $\dim(G \odot H) \geq n_1 \dim(K_1 \odot H_i)$  and, as a consequence,  $\dim(G \odot^k H) \geq n_1(n_2 + 1)^{k-1} \dim(K_1 \odot H)$ . We conclude the proof by [Theorem 12](#).  $\square$

**Corollary 14.** Let  $G$  be a connected graph of order  $n_1 \geq 2$ .

- (i) If  $n_2 \geq 7$ , then  $\dim(G \odot^k C_{n_2}) = n_1(n_2 + 1)^{k-1} \left\lfloor \frac{2n_2+2}{5} \right\rfloor$ .  
(ii) If  $n_2 \geq 7$ , then  $\dim(G \odot^k P_{n_2}) = n_1(n_2 + 1)^{k-1} \left\lfloor \frac{2n_2+2}{5} \right\rfloor$ .

All our previous results concern to  $G \odot H$  for  $H$  of order at least two. Now we consider the case  $H \cong K_1$ . We obtain a general bound for  $\dim(G \odot^k K_1)$  and, when  $G$  is a tree, we give the exact value for this parameter.

**Claim 15.** Let  $G$  be a simple graph. If  $v$  is a vertex of degree greater than one in  $G$ , then for every vertex  $u$  adjacent to  $v$  there exists a vertex  $x \neq u$ ,  $v$  of  $G$ , such that  $d(v, x) \neq d(u, x) + 1$ .

The following lemma obtained in [20] is useful to obtain the next result.

**Lemma 16** ([20]). If  $G_1$  is a graph obtained by adding a pendant edge to a nontrivial connected graph  $G$ , then  $\dim(G_1) \leq \dim(G) + 1$ .

**Theorem 17.** For every connected graph  $G$  of order  $n \geq 2$ ,

$$\dim(G \odot^k K_1) \leq 2^{k-1}n - 1.$$

**Proof.** If  $G \cong K_2$ , then  $\dim(K_2 \odot K_1) = \dim(P_4) = 1$ . So, let us suppose  $G \not\cong K_2$ . Let us suppose, without loss of generality, that  $v_n$  is a vertex of degree greater than one in  $G$  and let  $S = V - \{v_n\}$ . For every  $i \in \{1, \dots, n\}$ , let  $u_i$  be the pendant vertex of  $v_i$  in  $G \odot K_1$ . We will show that  $S$  is a resolving set for  $G \odot K_1$ . Let  $x, y$  be two different vertices of  $G \odot K_1$ . If  $x = u_i$  and  $y = u_j$ ,  $i \neq j$ , then we have either  $i \neq n$  or  $j \neq n$ . Let us suppose for instance  $i \neq n$ . So, we obtain that  $d(x, v_i) = 1 \neq d(y, v_i)$ . On the other hand, if  $x = v_n$  and  $y = u_i$ , then let us suppose  $d(x, v_i) = 1$ . Since  $v_n$  is a vertex of degree greater than one in  $G$ , by [Claim 15](#), there exists a vertex  $v_j \in S$  such that  $d(x, v_j) \neq d(v_i, v_j) + 1$ . So, we have  $d(x, v_j) \neq d(v_i, v_j) + 1 = d(v_i, v_j) + d(u_i, v_i) = d(y, v_i) + d(v_i, v_j) = d(y, v_j)$ . Therefore, for every different vertices  $x, y$  of  $G \odot K_1$  we have  $r(x|S) \neq r(y|S)$  and, as a consequence,  $\dim(G \odot K_1) \leq n - 1$ . Therefore,  $\dim(G \odot^k K_1) \leq 2^{k-1}n - 1$ .  $\square$

By [Lemma 16](#) we have  $\dim(K_n \odot K_1) \geq \dim(K_n) = n - 1$ . Thus, for  $k = 1$  the above bound is achieved for the graph  $G = K_n$ .

To present the next result, we need additional definitions. A vertex of degree at least 3 in a graph  $G$  will be called a *major vertex* of  $G$ . Any vertex  $u$  of degree one is said to be a *terminal vertex* of a major vertex  $v$  if  $d(u, v) < d(u, w)$  for every other major vertex  $w$  of  $G$ . The *terminal degree* of a major vertex  $v$  is the number of terminal vertices of  $v$ . A major vertex  $v$  is an *exterior major vertex* if it has positive terminal degree. Given a graph  $G$ ,  $n_1(G)$  denotes the number of vertices of degree one and  $\text{ex}(G)$  denotes the number of exterior major vertices of  $G$ .

**Lemma 18** ([6,1,2]). If  $T$  is a tree that is not a path, then  $\dim(T) = n_1(T) - \text{ex}(T)$ .

**Theorem 19.** For any tree  $T$  of order  $n \geq 3$ ,

$$\dim(T \odot^k K_1) = \begin{cases} n_1(T) & \text{for } k = 1, \\ 2^{k-2}n & \text{for } k \geq 2. \end{cases}$$

**Proof.** If  $T$  is a path of order  $n \geq 3$ , then we have  $\dim(T \odot K_1) = 2 = n_1(T)$ . Now, if  $T$  is not a path, then by using [Lemma 18](#), since  $T \odot K_1$  is a tree,  $n_1(T \odot K_1) = n$  and  $\text{ex}(T \odot K_1) = n - n_1(T)$ , we obtain the result for  $k = 1$ . Since for every tree  $T$  of order  $n$  we have  $n_1(T \odot K_1) = n$ , we obtain the result for  $k \geq 2$ .  $\square$

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