Postulates for measures of genuine multipartite correlations

Charles H. Bennett,[1] Andrzej Grudka,[2,3] Michał Horodecki,[2] Paweł Horodecki,[4] and Ryszard Horodecki[2]

[1]*IBM T. J. Watson Research Center, Yorktown Heights, New York 10598, USA*
[2]*Institute of Theoretical Physics and Astrophysics, University of Gdańsk, PL-80-952 Gdańsk, Poland*
[3]*Faculty of Physics, Adam Mickiewicz University, PL-61-614 Poznań, Poland*
[4]*Faculty of Applied Physics and Mathematics, Technical University of Gdańsk, PL-80-952 Gdańsk, Poland*

A lot of research has been done on multipartite correlations, but the problem of satisfactorily defining genuine multipartite correlations—those not trivially reducible to lower partite correlations—remains unsolved. In this paper we propose three reasonable postulates which each measure or indicator of genuine multipartite correlations (or genuine multipartite entanglement) should satisfy. We also introduce the concept of degree of correlations, which gives partial characterization of multipartite correlations. Then, we show that covariance does not satisfy two postulates and hence it cannot be used as an indicator of genuine multipartite correlations. Finally, we propose a candidate for a measure of genuine multipartite correlations based on the work that can be drawn from a local heat bath by means of a multipartite state.

I. INTRODUCTION

One of the most important problems in quantum-information theory is the problem of quantifying correlations. Henderson and Vedral [1] raised the problem of separating total correlations in a bipartite state into quantum and classical parts (see also in this context Refs. [2–4]). They also proposed a measure of purely classical bipartite correlations. It was shown in Ref. [5] that there exist bipartite states which have almost maximal entanglement of formation and almost no mutual information and, hence, almost no classical correlations. In a series of papers a thermodynamical approach to quantifying correlations was developed [6–8]. It is well known that bits of information can be used to extract work from a heat bath [9]. If we have a bipartite quantum state one can ask how much work can be extracted from the heat bath under different classes of operations. In particular, Ref. [6] defined quantum-information deficit as the difference between globally and locally (with the use of local operations and classical communication) extractable work from the heat bath. Recently it was shown that under some restricted scenario of work extraction there exist quantum states for which quantum-information deficit is equal to quantum mutual information [10]. As a result, all correlations behave as if they were exclusively quantum.

The problem of coexistence of quantum and classical correlations in multipartite systems was considered in [11]. It was shown that there exist n-qubit states which consist of an equal mixture of two W states with an odd number of qubits, for which n-party covariance defined as $\mathrm{Cov}(X_1, \ldots, X_n) = \langle (X_1 - \langle X_1 \rangle) \ldots (X_n - \langle X_n \rangle) \rangle$ is zero for all choices of local observables X_i and the state is genuinely entangled. Based on these observations the authors argued that genuine multipartite correlations can exist without classical correlations. However, the conclusion is based on the assumption that covariance is an indicator of genuine multipartite correlations.

In this paper we formulate three postulates which any measure or indicator of genuine multipartite correlations should satisfy. We show that covariance does not satisfy two postulates when applied to more than two-partite systems. Our counterexamples are the states considered in Ref. [11]. Hence,

the vanishing of covariance for multipartite states does not imply the absence of genuine multipartite classical correlations but rather shows that covariance cannot be an indicator of genuine multipartite classical correlations. As a by-product we obtain a protocol of distillation of W states from a wide class of states (for distillation of W states from generic states see [12]).

The paper is organized as follows. In Sec. II, we formulate reasonable postulates for measures or indicators of genuine multipartite correlations and introduce the concept of degree of correlations. In Sec. III we show that covariance does not satisfy the postulates. In Sec. IV we discuss the relation between multipartite correlations and work extraction. In Sec. V we draw our conclusions.

II. GENUINE *n*-PARTITE CORRELATIONS

We do not know what it means that a state has genuine multipartite correlations. Hence we give reasonable postulates which each measure or indicator of genuine multipartite correlations should satisfy. In Sec. II A we formulate postulates for genuine n- and k-partite correlations of an n-partite state. In Sec. II B we introduce the degree of correlations as an indicator of genuine k-partite correlations for an arbitrary multipartite state. The postulates apply to correlations in general; hence, in particular, they apply also to genuine multipartite entanglement or genuine multipartite classical correlations.

A. Genuine *n*- and *k*-partite correlations of an *n*-partite state

Each measure or indicator of genuine n-partite correlations for an n-partite state should satisfy the following postulates.

Postulate 1. If an n-partite state does not have genuine n-partite correlations and one adds a party in a product state, then the resulting $n + 1$ partite state does not have genuine n-partite correlations.

Postulate 2. If an n-partite state does not have genuine n-partite correlations, then local operations and unanimous postselection (which mathematically correspond to the operation $\Lambda_1 \otimes \Lambda_2 \otimes \cdots \otimes \Lambda_n$, where n is the number of parties and each Λ_i is a trace nonincreasing operation acting

on the ith party's subsystem) cannot generate genuine n-partite correlations.

Postulate 3. If an n-partite state does not have genuine n-partite correlations, then if one party splits his subsystem into two parts, keeping one part for himself and using the other to create a new $n+1$-st subsystem, then the resulting $n+1$-partite state does not have genuine $n+1$-partite correlations.

One can also require (compare Postulate 2) for each measure $C(\rho)$ of genuine n-partite correlations that it does not increase on average under local operations; i.e.,

$$
C(\rho) \geqslant \sum_{i_1, i_2, \ldots, i_n} p_{i_1, i_2, \ldots, i_n}
$$

$$
C\big(E_{i_1}^1 E_{i_2}^2 \ldots E_{i_n}^n \rho E_{i_1}^{1\dagger} E_{i_2}^{2\dagger} \ldots E_{i_n}^{n\dagger} / p_{i_1, i_2, \ldots, i_n}\big), \tag{1}
$$

where E_i^j are Krauss operators acting on jth subsystem satisfying $\sum_i E_i^{j\dagger} E_i^j \leqslant I$ and $p_{i_1, i_2, \ldots, i_n} = \mathrm{Tr}(E_{i_1}^1 \otimes E_{i_2}^2 \otimes \cdots \otimes E_{i_n}^n \rho E_{i_1}^{1\dagger} \otimes E_{i_2}^{2\dagger} \otimes \cdots \otimes E_{i_n}^{n\dagger})$. It seems that this requirement should be added to the postulates of Henderson and Vedral [1] for measures of classical bipartite correlations.

Let us now propose a definition of genuine multipartite correlations.

Definition 1. A state of n particles has genuine n-partite correlations if it is nonproduct in every bipartite cut.

Below we prove that the genuine multipartite correlations defined here satisfy Postulates 1–3.

Observation 1. If genuine n-partite correlations are defined as in Definition 1 then they satisfy Postulates 1–3 (Fig. 1).

Proof. It is clear that genuine n-partite correlations satisfy Postulate 1.

To show that they satisfy Postulate 2 we observe that an n-partite state which does not have genuine n-partite correlations is of the form

$$
\rho = \rho^{(n_1)} \otimes \rho^{(n_2)}, \tag{2}
$$

where $\rho^{(n_1)}$ and $\rho^{(n_2)}$ are states of n_1 and n_2 particles, respectively ($n_1 + n_2 = n$). It is clear that this product form is preserved by the operation $\Lambda_1 \otimes \Lambda_2 \otimes \cdots \otimes \Lambda_n$.

To show that genuine n-partite correlations satisfy Postulate 3 we observe that, if a state does not have genuine n-partite correlations before splitting, then after splitting it has the form

$$
\rho = \rho'^{(n_1+1)} \otimes \rho'^{(n_2)}, \tag{3}
$$

where $\rho'^{(n_1+1)}$ and $\rho'^{(n_2)}$ are states of n_1+1 and n_2 particles, respectively, or

$$
\rho = \rho'^{(n_1)} \otimes \rho'^{(n_2+1)}, \tag{4}
$$

where $\rho'^{(n_1)}$ and $\rho'^{(n_2+1)}$ are states of n_1 and n_2+1 particles, respectively. Hence, we see that it does not have genuine $n+1$-partite correlations. This ends the proof of Observation 1.

It is useful to define k-partite genuine correlations for n-partite states not only for $k = n$ as we did above but also for any $k \leqslant n$.[1] A suitable definition is the following

_____

[1] We have used this definition implicitly in a previous version. We are grateful to Michael Seevinck for pointing out that it is not stated formally.
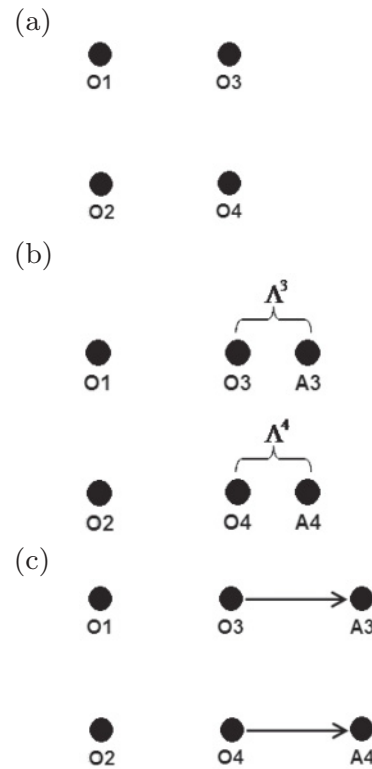

FIG. 1. Illustration of Observation 1. (a) Four parties share a joint state ρ. (b1) The third and the fourth party add ancillas in a state which is product with the original system and the other ancilla. After this step the parties share the state $\rho \otimes \rho_{A3} \otimes \rho_{A4}$. (b2) The third and the fourth party perform local operations on their original qubits and ancillas. After this step the parties share the state $\mathrm{id}_{O1} \otimes \mathrm{id}_{O2} \otimes \Lambda^1_{O3A3} \otimes \Lambda^2_{O4A4}(\rho \otimes \rho_{A3} \otimes \rho_{A4})$, where id_i denotes the identity map acting on qubit i and Λ^a_{ij} denotes any completely positive map acting on qubits i and j. (c) The third and the fourth party send ancillas to the fifth and sixth party, respectively. If the initial state has degree of correlations less than n then the final state cannot have degree of correlations greater than or equal to $n+2$.

Definition 2. A state of n particles has genuine k-partite correlations if there exists a k-particle subset whose reduced state has genuine k-partite correlations (according to Definition 1).

Remark. The postulates apply also to entanglement. However, Definitions 1 and 2 do not. More precisely, they do apply to pure state entanglement; i.e., one can say that n-partite pure state has genuine n-partite entanglement if and only if it is nonproduct with respect to any bipartite cut. To obtain a definition of genuine multipartite entanglement for mixed states, we proceed in a standard way [13]; i.e., we say that an n-partite state ρ has genuine n-partite entanglement if it is not a mixture of pure states that do not have genuine n-partite entanglement. This is a counterpart of Definition 1, which then determines the counterpart of Definition 2. Thus, to rule out correlations that do not represent entanglement, we can add a fourth postulate, saying that by mixing states which do not have n-partite entanglement we cannot obtain genuine n-partite entanglement. The resulting notion of genuine n-partite entanglement is slightly different from the notion of n-partite entanglement of Ref. [14]. For example, an n-particle

state which contains k-partite entanglement according to Ref. [14] cannot contain m-partite entanglement for $m < k$, as opposed to our genuine multipartite entanglement. For example, if we have a 5-partite state which is a product of a Greenberger-Horne-Zeilinger (GHZ) state and a singlet state, then it contains both 2- as well as 3-partite genuine entanglement, while it contains only 3-partite entanglement according to Ref. [14].

B. Degree of correlations

We introduce the concept of degree of correlations. We do not define it first but rather require that it should satisfy the following postulates.

Postulate 1′. If one adds a party in a product state with the remainder of the system then the degree of correlations cannot change.

Postulate 2′. Local operations and postselection cannot increase the degree of correlations. In particular local unitary operations cannot change the degree of correlations.

Postulate 3′. If one party splits his subsystem into two, i.e., sends part of his subsystem to a new party who is not correlated with the remainder of the system, then the degree of correlations can increase at most by 1.

Let us now propose a definition of the degree of correlations.

Definition 3. A state has degree of correlations equal to n if there exists a subset of n particles which has genuine n-partite correlations and there does not exist a subset of m particles which has genuine m-partite correlations for any $m > n$.

Example 1. An n-partite state of the form

$$\varrho = \frac{1}{2} \sum_{i=0}^{1} |ii \ldots i\rangle \langle ii \ldots i| \qquad (5)$$

has genuine 2-, 3-,...,n-partite correlations and degree of correlations equal to n.

Example 2. An n-partite state of the form

$$\varrho = \frac{2}{2^n} \sum_{i_1+i_2+\cdots+i_n=0 \bmod 2}^{1} |i_1 i_2 \ldots i_n\rangle \langle i_1 i_2 \ldots i_n| \qquad (6)$$

has only genuine n-partite genuine correlations and degree of correlations equal to n. It does not have genuine 2-,3-,...,$n-1$-partite correlations.

We can now find the form of a state which has degree of correlations equal to n.

Observation 2. A state which has degree of correlations equal to n is of the form

$$\rho = \rho^{(n)} \otimes \rho^{(m_1)} \otimes \cdots \otimes \rho^{(m_M)}, \qquad (7)$$

where $\rho^{(n)}$, $\rho^{(m_1)},\ldots,$ $\rho^{(m_M)}$ are states of $n, m_1, \ldots, m_M$ particles which are nonproduct in any bipartite cut and $n \geqslant m_1, \ldots, m_M$.

Proof. It is clear that the arbitrary state can be written in this form for some n. The state has degree of correlations at least equal to n because the reduced state of first n particles is nonproduct in any bipartite cut. To show that the degree of correlations cannot be greater than n it is enough to notice that if we trace some particles then if the state is product in some

cut before we trace particles then it will be product in this cut after we trace particles.

Below we prove that the degree of correlations defined earlier satisfies Postulates 1′–3′.

Observation 3. If genuine n-partite correlations and degree of correlations are defined as in Definition 1 and Definition 2 then the degree of correlations satisfies Postulates 1′–3′.

Proof. It is clear that the degree of correlations satisfies Postulate 1′.

To show that it satisfies Postulate 2′ we use the fact proved in Observation 2 that a state which has degree of correlations equal to n is of the form

$$\rho = \rho^{(n)} \otimes \rho^{(m_1)} \otimes \cdots \otimes \rho^{(m_M)}, \qquad (8)$$

where $\rho^{(n)}$, $\rho^{(m_1)},\ldots,$ $\rho^{(m_M)}$ are states of $n, m_1, \ldots, m_M$ particles, respectively, which are nonproduct in any bipartite cut and $n \geqslant m_1, \ldots, m_M$. It is clear that this product form is preserved by the operation $\Lambda_1 \otimes \Lambda_2 \otimes \cdots \otimes \Lambda_k$, where $k = n + m_1 + \cdots + m_M$ is the number of parties.

To show that degree of correlations satisfies Postulate 3′ we observe that a state which has degree of correlations equal to n after sending part of a one party's system is of the following form

$$\rho = \rho'^{(n+1)} \otimes \rho'^{(m_1)} \otimes \cdots \otimes \rho'^{(m_M)}, \qquad (9)$$

where $\rho'^{(n+1)}$, $\rho'^{(m_1)},\ldots,$ $\rho'^{(m_M)}$ are states of $n + 1, m_1, \ldots, m_M$ particles or

$$\rho = \rho'^{(n)} \otimes \rho'^{(m_1+1)} \otimes \cdots \otimes \rho'^{(m_M)}, \qquad (10)$$

where $\rho'^{(n)}$, $\rho'^{(m_1+1)},\ldots,$ $\rho'^{(m_M)}$ are states of $n, m_1 + 1, \ldots, m_M$ particles and so on. Hence, we see that it has a degree of correlations of at most $n + 1$.

Using Postulates 1–3 we can prove the following observation.

Observation 4. If the initial state has degree of correlations less than n and then if the parties add k ancillas in a product state, perform local operations on their particles and their ancillas, and send ancillas to k new parties in such a way that each new party receives an ancilla from only one party, then the final state cannot have degree of correlations greater or equal to $n + k$.

Proof. First, k parties add ancillas in a product state. The degree of correlations cannot change (Postulate 1′). Next, k parties apply local operations to their original qubits and added ancillas. The degree of correlations cannot increase (Postulate 2′). Finally, k parties send ancillas to k new parties. The degree of correlations can increase at most by k (it can increase by 1 for each sent ancilla) (Postulate 3′).

III. COVARIANCE DOES NOT SATISFY POSTULATES

A. Postulate 2

Before we show that covariance does not satisfy Postulate 2 we present a purification protocol which allows the distillation of W states from certain mixed states. The protocol consists of two steps. In the first step each party performs a measurement on their particle—the so-called local filtering [15–17]. The measurements performed by all parties are independent; i.e.,

they are not conditioned on the results of the measurements performed by other parties. In the second step the parties postselect a state. The postselected state can have fidelity with the W state as close to 1 as one wants. However, the probability of distilling such a state decreases with fidelity. This protocol is a multipartite version of the so-called quasidistillation process [18,19].

Let us consider an n-partite state which is a mixture of a W state and a normalized state $\rho^{(1)}$ with support contained in the $2^n - n - 1$-dimensional Hilbert space spanned by all vectors which have two or more 1's, i.e.,

$$\rho = p|W\rangle\langle W| + (1-p)\rho^{(1)}, \qquad (11)$$

where

$$|W\rangle = \frac{1}{\sqrt{n}}(|10\cdots0\rangle + |01\cdots0\rangle + \cdots + |00\cdots1\rangle). \quad (12)$$

Let each party perform a measurement described by the following Kraus operators:

$$\begin{aligned} E_S &= |0\rangle\langle0| + \sqrt{\epsilon}|1\rangle\langle1|, \\ E_F &= \sqrt{1-\epsilon}|1\rangle\langle1|. \end{aligned} \qquad (13)$$

The action of $E_S^{\otimes n}$ on states with m 1's and $n-m$ 0's is given by the following formula:

$$E_S^{\otimes n}|1\rangle^{\otimes m}|0\rangle^{\otimes(n-m)} = \sqrt{\epsilon}^m|1\rangle^{\otimes m}|0\rangle^{\otimes(n-m)}. \qquad (14)$$

A similar result applies for all permutations of m 1's and $n-m$ 0's.

Hence, if each party obtains S as the result of the measurement, then the post-measurement state is proportional to

$$\rho' = E_S^{\otimes n}\rho E_S^{\otimes n} = \epsilon p|W\rangle\langle W| + \epsilon^2(1-p)\rho'^{(1)}, \quad (15)$$

where $\rho'^{(1)}$ is an unnormalized state with the sum of eigenvalues less than or equal to 1 and with support contained in the $(2^n - n - 1)$-dimensional Hilbert space orthogonal to the W state. The probability that each party obtains S as the result of the measurement is

$$q = \text{Tr}(\rho') > \epsilon p. \qquad (16)$$

The fidelity of the postmeasurement state with the W state is

$$F = \frac{\langle W|\rho'|W\rangle}{\text{Tr}(\rho')} > \frac{\epsilon p}{\epsilon p + \epsilon^2(1-p)}. \qquad (17)$$

If ϵ is small then the fidelity is close to 1. Hence, with small probability it is possible to distill a state close to W. More precisely, the probability that we distill a W state with fidelity F satisfies

$$q > \frac{p^2}{(1-p)}\frac{1-F}{F}. \qquad (18)$$

Consider now as an example the n-partite state from [11] which is an equal mixture of W and $\overline{W}$ states consisting of an odd number of qubits; i.e.,

$$\rho = \frac{1}{2}|W\rangle\langle W| + \frac{1}{2}|\overline{W}\rangle\langle\overline{W}|, \qquad (19)$$

where

$$|\overline{W}\rangle = \frac{1}{\sqrt{n}}(|01\cdots1\rangle + |10\cdots1\rangle + \cdots + |11\cdots0\rangle). \quad (20)$$

One can show that for this state all n-partite covariances vanish [11] (for completeness we show it in the Appendix).

Let the parties apply the just described protocol to the state of Eq. (19). If each party obtains S as the result of measurement then the postmeasurement state is proportional to

$$\rho' = \epsilon\frac{1}{2}|W\rangle\langle W| + \epsilon^{n-1}\frac{1}{2}|\overline{W}\rangle\langle\overline{W}|. \qquad (21)$$

The probability that each party obtains S as the result of measurement is

$$q = \frac{1}{2}\epsilon(1 + \epsilon^{n-2}). \qquad (22)$$

The fidelity of the postmeasurement state with the W state is

$$F = \frac{\langle W|\rho'|W\rangle}{\text{Tr}(\rho')} = \frac{1}{1 + \epsilon^{n-2}}. \qquad (23)$$

Let us take $\epsilon = 1 - \frac{1}{\sqrt{n}}$ and calculate the limit of q and F for large n. We have

$$q \approx \frac{1}{2}\left(1 - \frac{1}{\sqrt{n}}\right)(1 + e^{-\sqrt{n}}) \qquad (24)$$

and

$$F \approx \frac{1}{1 + e^{-\sqrt{n}}}. \qquad (25)$$

More generally, the probability that we distill the state W with fidelity F is

$$q = \frac{1}{2F}\left(\frac{1-F}{F}\right)^{\frac{1}{n-2}}. \qquad (26)$$

In Fig. 2, we show how the probability of distillation depends on the fidelity of the distilled state for mixtures of two W states. For large n, the probability of obtaining a state close to W is close to $\frac{1}{2}$. Hence, asymptotically the measurement *effectively projects* the initial state on the W state.

We have thus shown that we can transform with local operations and postselection the state of Eq. (19) into a state of the form

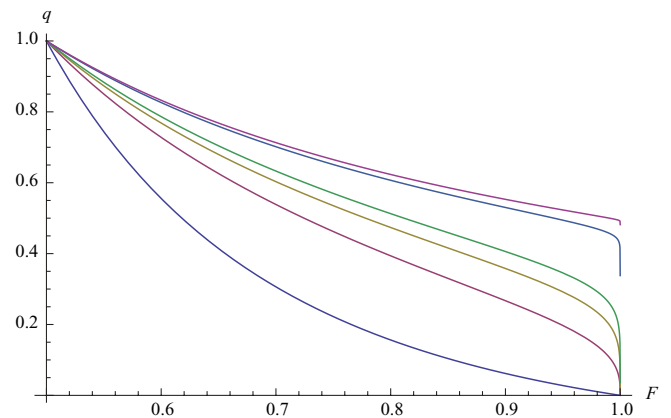$$\rho = F|W\rangle\langle W| + (1-F)|\overline{W}\rangle\langle\overline{W}|, \qquad (27)$$

FIG. 2. (Color online) Probability of distillation (q) versus fidelity (F). From bottom to top: 3-qubit state, 5-qubit state, 7-qubit state, 9-qubit state, 49-qubit state, and 499-qubit state.
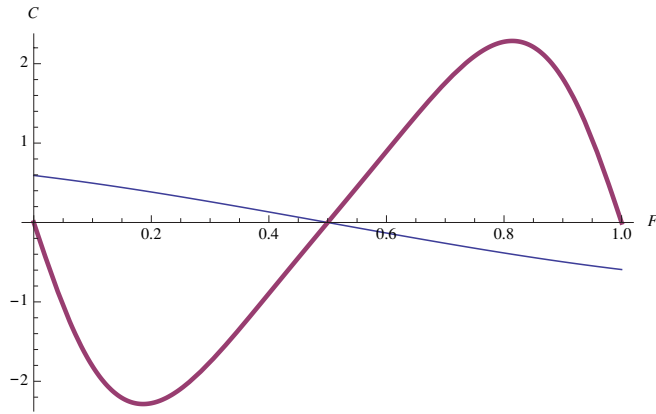
FIG. 3. (Color online) Covariance (C) versus fidelity (F) for a 3-qubit state (thin line) and a 9-qubit state (thick line).

where $0 < F < 1$ can be arbitrary close to 1. The n-partite covariance $\mathrm{Cov}(\sigma_z^1, \ldots, \sigma_z^n)$ of this state is given by the following expression (we show it in the Appendix):

$$
\begin{aligned}
&\mathrm{Cov}(\sigma_z^1, \ldots, \sigma_z^n) \\
&= \mathrm{Tr}[(\sigma_z - \langle \sigma_z \rangle)^{\otimes n} \rho] \\
&= -F(1 - \langle \sigma_z \rangle)^{n-1}(1 + \langle \sigma_z \rangle) \\
&\quad + (1 - F)(-1)^{n-1}(1 + \langle \sigma_z \rangle)^{n-1}(1 - \langle \sigma_z \rangle),
\end{aligned}
\tag{28}
$$

where

$$
\begin{aligned}
\langle \sigma_z \rangle &= \mathrm{Tr}(\sigma_z \rho) \\
&= F\,\mathrm{Tr}(\sigma_z |W\rangle\langle W|) + (1 - F)\,\mathrm{Tr}(\sigma_z |\overline{W}\rangle\langle\overline{W}|) \\
&= (2F - 1)\frac{n - 2}{n}.
\end{aligned}
\tag{29}
$$

In Fig. 3, we present how it depends on the fidelity F for a 3-qubit state and a 9-qubit state. For both states the covariance vanishes for $F = \frac{1}{2}$.

Let us summarize this result. The parties start with a state for which all covariances vanish. Then they apply local filtering and postselect a state. They can choose measurements in such a way that the postselected state has nonvanishing covariance. Hence, we have shown that covariance does not satisfy our second postulate. This shows that covariance should not be regarded as an indicator of genuine multipartite classical correlations.

B. Postulate 3

Suppose that n parties share a state of Eq. (19). Let each party add to their original qubit an auxiliary qubit in state $|0\rangle$, perform CNOT gate (the original qubit is control qubit and the auxiliary qubit is target qubit), and send the ancilla to new party. Each new party receives an ancilla from only one party. The $2n$-partite state is

$$
\rho' = \tfrac{1}{2}|W'\rangle\langle W'| + \tfrac{1}{2}|\overline{W}'\rangle\langle\overline{W}'|,
\tag{30}
$$

where

$$
\begin{aligned}
|W'\rangle = \frac{1}{\sqrt{n}}(&|10\ldots 0; 10\ldots 0\rangle + |01\ldots 0; 01\ldots 0\rangle \\
&+ \cdots + |00\ldots 1; 00\ldots 1\rangle)
\end{aligned}
\tag{31}
$$

and

$$
\begin{aligned}
|\overline{W}'\rangle = \frac{1}{\sqrt{n}}(&|01\ldots 1; 01\ldots 1\rangle + |10\ldots 1; 10\ldots 1\rangle \\
&+ \cdots + |11\ldots 0; 11\ldots 0\rangle).
\end{aligned}
\tag{32}
$$

One can show that the following covariance,

$$
\mathrm{Cov}(\sigma_z^1, \ldots, \sigma_z^{2n}) = \mathrm{Tr}[(\sigma_z - \langle\sigma_z\rangle)^{\otimes 2n}\rho'],
\tag{33}
$$

is equal to 1. We see that covariance does not satisfy Postulate 3 when applied to more than two-partite states.

IV. MULTIPARTITE CORRELATIONS AND WORK EXTRACTION

In this section we investigate the relation between multipartite correlations and the amount of work that can be extracted from the environment [6,8,9,20]. Let us suppose that two parties share a quantum state ρ_{AB}. It is well known that this state can be used to extract work from the environment in many different ways. We consider two scenarios. In the first one the parties are allowed to perform closed local operations; i.e., they can perform local unitary operations and local dephasing (CLO) [8]. In the second one they are allowed to perform closed local operations and send classical communication, i.e. transfer subsystems through completely dephasing channels (CLOCC). If for the state ρ_{AB} the parties can extract more work with CLOCC than with CLO, then the state ρ_{AB} has classical correlations. Let us now consider multipartite states. By analogy we expect that if the parties can extract more work with CLOCC and with sending classical information across any bipartite cut than with CLOCC and without sending classical information across at least one cut, then the state has genuine multipartite classical correlations. We shall denote the difference between extractable work in those two scenarios minimized overall bipartite cuts by δW.

The aforementioned procedure is closely related to the procedure which is used in the definition of quantum-information deficit [6] and quantum discord [2]. Quantum-information deficit is defined as the minimal difference between work which can be extracted from the environment by a party who has the whole state ρ_{AB} and can perform CLO and by a pair of parties who share a state ρ_{AB} and can perform CLOCC. One-way quantum-information deficit (i.e., when one allows classical communication only in one direction) is equal to quantum discord optimized over von Neumann measurements [21]. It should be noted however that both quantum-information deficit and quantum discord measure only quantum correlations and not all correlations.

Example 3. Here we compare the amount of work extractable with the use of an equal mixture of two tripartite W states when three parties cooperate, i.e., there is classical communication across any cut, and when two parties cooperate, i.e., there is no classical communication across one cut. Let us first suppose that three parties cooperate. If the parties dephase their qubits in a $\{|0\rangle, |1\rangle\}$ basis and the first two parties send their qubits to the third party (this is equivalent to sending

qubits down a completely dephasing channel), then the third party will hold the state

$$\sigma_{123} = \frac{1}{6}(|001\rangle\langle001| + |010\rangle\langle010| + |100\rangle\langle100|$$
$$+ |110\rangle\langle110| + |101\rangle\langle101| + |011\rangle\langle011|). \quad (34)$$

They can now extract $3 - \log_2 6 \approx 0.4150$ bits of work. Another protocol is the following: one party measures in a chosen basis and tells the result to other parties, who then draw work from the resulting state they share. If the basis is $|0\rangle, |1\rangle$, the result is the same as above. The complementary basis $\{|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\}$ gives 0.4499, while the optimal basis is $\{\sqrt{\frac{1}{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle, \sqrt{\frac{2}{3}}|0\rangle - \sqrt{\frac{1}{3}}|1\rangle\}$, providing 0.4502 bits of work. We do not know whether by general CLOCC protocol one can extract more work.

Let us now suppose that only two parties cooperate, i.e., the first one and the second one. The reduced state of the third party is maximally mixed and they cannot extract any work at all. The reduced state of the first two parties is Bell diagonal

$$\rho_{12} = \text{Tr}_3\rho = \frac{2}{3}|\Psi^+\rangle\langle\Psi^+| + \frac{1}{6}|\Phi^+\rangle\langle\Phi^+| + \frac{1}{6}|\Phi^-\rangle\langle\Phi^-|, \quad (35)$$

where

$$|\Psi^+\rangle = \frac{1}{2}(|01\rangle + |10\rangle)$$
$$|\Phi^\pm\rangle = \frac{1}{2}(|00\rangle \pm |11\rangle). \quad (36)$$

If the first party dephases their qubit in a $\{|+\rangle, |-\rangle\}$ basis and sends their qubit to the second party, then the second party after applying local unitary operation will hold the state

$$\sigma_{12} = \frac{5}{12}|01\rangle\langle01| + \frac{5}{12}|10\rangle\langle10| + \frac{1}{12}|00\rangle\langle00| + \frac{1}{12}|11\rangle\langle11|. \quad (37)$$

They can now extract $1 - H(\frac{5}{6}) \simeq 0.3499$ bits of work, where $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is binary entropy. This is maximal work which can be extracted with the help of one-way classical communication even if one takes into account positive operator-valued measures (POVMs) and the asymptotic limit of many copies [8]. However, again, it is not known if one can extract more work with the help of two-way classical communication.

To summarize, we showed that if the communication through an A:BC cut is allowed, then we can extract at least 0.4502 bits of work, while if it is not allowed, we are able to provide a protocol which extracts 0.3499 bits of work (this concerns all possible cuts, as the state is permutationally symmetric). If the latter protocol were optimal, we would have $\delta W \gtrsim 0.1$. This supports the existence of genuine tripartite correlations in the state.

Example 4. On the other hand if the parties can extract the same amount of work with CLOCC and with sending classical information across any bipartite cut as with CLOCC and without sending classical information across at least one cut, then we cannot conclude that the state does not have genuine multipartite classical correlations. Let us consider the following tripartite state:

$$\rho_{123} = \frac{1}{2}|\Psi^+\rangle\langle\Psi^+|_{12} \otimes |0\rangle\langle0|_3 + \frac{1}{2}|\Psi^-\rangle\langle\Psi^-|_{12} \otimes |1\rangle\langle1|_3. \quad (38)$$

If three parties cooperate they can extract one bit of work. However if only the first and the second party cooperate they can also extract one bit of work. On the other hand the state is nonproduct across any bipartite cut and according to Definition 1 it has genuine tripartite classical correlations.

The previous two examples show that on the one hand δW can indicate multipartite correlations; on the other hand it may vanish, even though the state is nonproduct against any cut. This is analogous to the behavior of some entanglement measures; e.g., distillable entanglement can vanish for states despite that they are entangled. Thus δW quantifies some particular type of genuine multipartite correlations, which may be absent in some states even though they contain genuinely multipartite correlations.

Let us note that the aforementioned property of δW is similar to covariance, which disappears for state (19), even though it has genuine multipartite correlations with respect to some other criteria. One basic difference is, however, that covariance can be positive even for states such as the product of Einstein-Podolsky-Rosen (EPR) pairs $|\Psi^+\rangle_{AB} \otimes |\Psi^+\rangle_{CD}$, which quite obviously do not represent fourpartite correlations, as follows easily from Postulates 1–3 (an example of non zero covariance is $\text{Cov}(\sigma_Z^A, \sigma_Z^B, \sigma_Z^C, \sigma_Z^D) = 1$). Moreover, we believe (though we have not proven) that δW would satisfy our postulates; i.e., having been zero for some state, it will not go up under the operations described in the formulation of the postulates, while covariance, as we have shown in previous sections, violates two postulates.

V. CONCLUSIONS

In conclusion we have proposed reasonable postulates which each measure or indicator of genuine multipartite correlations (or genuine multipartite entanglement) should satisfy. We also introduced the concept of degree of correlations, which gives partial characterization of multipartite correlations. We have shown that covariance does not satisfy the proposed postulates and it cannot be used as an indicator of genuine multipartite classical correlations. In particular, our postulates show that the claim that there exist genuine n-partite quantum correlations without genuine n-partite classical correlations is not justified. As a by-product we obtained a protocol of distillation of W states from a wide class of states. Finally, we propose a candidate for a measure of genuine multipartite correlations based on work that can be drawn from local environments by means a multipartite state. We hope that our results, especially the proposed postulates, will allow us to develop understanding and a quantitative description of genuine multipartite correlations.

APPENDIX: CALCULATION OF COVARIANCE

We calculate all covariances for an equal mixture of two W states consisting of an odd number of qubits:

$$\rho = \tfrac{1}{2}|W\rangle\langle W| + \tfrac{1}{2}|\overline{W}\rangle\langle\overline{W}|. \qquad (A1)$$

We can write it as

$$\begin{aligned}
\mathrm{Cov}(&X_1,\ldots,X_n)\\
&= \mathrm{Tr}((X_1 - \langle X_1\rangle)\ldots(X_n - \langle X_n\rangle)\rho)\\
&= F\langle W|(X_1 - \langle X_1\rangle)\ldots(X_n - \langle X_n\rangle)|W\rangle\\
&\quad + (1-F)\langle\overline{W}|(X_1 - \langle X_1\rangle)\ldots(X_n - \langle X_n\rangle)|\overline{W}\rangle, \quad (A2)
\end{aligned}$$

where X_i denotes the Pauli matrix acting on the ith qubit. Since $\langle X_i\rangle = 0$ we have

$$\begin{aligned}
\mathrm{Cov}(&X_1,\ldots,X_n)\\
&= \tfrac{1}{2}\langle W|X_1\ldots X_n|W\rangle + \tfrac{1}{2}\langle\overline{W}|X_1\ldots X_n|\overline{W}\rangle, \quad (A3)
\end{aligned}$$

and we only need to calculate $\langle W|X_1\ldots X_n|W\rangle$ and $\langle\overline{W}|X_1\ldots X_n|\overline{W}\rangle$. Moreover, since σ_x and σ_y exchange $|0\rangle$ and $|1\rangle$ the only nonvanishing terms are those which contain

  (1) n times σ_z,
  (2) 2 times σ_x and $n-2$ times σ_z,
  (3) 2 times σ_y and $n-2$ times σ_z,
  (4) 1 times σ_x, 1 times σ_y, and $n-2$ times σ_z.

The other products of Pauli matrices when acting on the W or $\overline{W}$ state transform it into some orthogonal state, and hence their expectation value in the W or $\overline{W}$ state is equal to zero. We do not consider in detail the preceding four cases as they are are similar and we restrict our attention to only the second case. Using the relations

$$\sigma_x^1\sigma_x^2\sigma_z^3\ldots\sigma_z^n|100\ldots0\rangle = |010\ldots0\rangle, \qquad (A4)$$

$$\sigma_x^1\sigma_x^2\sigma_z^3\ldots\sigma_z^n|010\ldots0\rangle = |100\ldots0\rangle, \qquad (A5)$$

$$\sigma_x^1\sigma_x^2\sigma_z^3\ldots\sigma_z^n|001\ldots0\rangle = -|111\ldots0\rangle, \qquad (A6)$$

$$\sigma_x^1\sigma_x^2\sigma_z^3\ldots\sigma_z^n|011\ldots1\rangle = (-1)^n|101\ldots1\rangle, \quad (A7)$$

$$\sigma_x^1\sigma_x^2\sigma_z^3\ldots\sigma_z^n|101\ldots1\rangle = (-1)^n|011\ldots1\rangle, \quad (A8)$$

$$\sigma_x^1\sigma_x^2\sigma_z^3\ldots\sigma_z^n|110\ldots1\rangle = -(-1)^n|000\ldots1\rangle, \quad (A9)$$

and so on, we obtain

$$\langle W|\sigma_x^1\sigma_x^2\sigma_z^3\ldots\sigma_z^n|W\rangle = \frac{2}{n}, \qquad (A10)$$

$$\langle\overline{W}|\sigma_x^1\sigma_x^2\sigma_z^3\ldots\sigma_z^n|\overline{W}\rangle = (-1)^n\frac{2}{n}, \qquad (A11)$$

which leads to

$$\mathrm{Tr}(\sigma_x^1\sigma_x^2\sigma_z^3\ldots\sigma_z^n\rho) = 0 \qquad (A12)$$

and similarly for other products of n Pauli matrices.

We also calculate the covariance $\mathrm{Cov}(\sigma_z^1,\ldots,\sigma_z^n)$ for an arbitrary mixture of two W states consisting of an odd number of qubits,

$$\rho = F|W\rangle\langle W| + (1-F)|\overline{W}\rangle\langle\overline{W}|. \qquad (A13)$$

We can write it as

$$\begin{aligned}
\mathrm{Cov}(&\sigma_z^1,\ldots\sigma_z^n)\\
&= \mathrm{Tr}[(\sigma_z - \langle\sigma_z\rangle)^{\otimes n}\rho]\\
&= F\mathrm{Tr}[(\sigma_z - \langle\sigma_z\rangle)^{\otimes n}|W\rangle\langle W|]\\
&\quad + (1-F)\mathrm{Tr}[(\sigma_z - \langle\sigma_z\rangle)^{\otimes n}|\overline{W}\rangle\langle\overline{W}|]. \quad (A14)
\end{aligned}$$

Using identities

$$(\sigma_z - \langle\sigma_z\rangle)^{\otimes n}|W\rangle = (1 - \langle\sigma_z\rangle)^{n-1}(-1 - \langle\sigma_z\rangle)|W\rangle \quad (A15)$$

and

$$(\sigma_z - \langle\sigma_z\rangle)^{\otimes n}|\overline{W}\rangle = (-1 - \langle\sigma_z\rangle)^{n-1}(1 - \langle\sigma_z\rangle)|\overline{W}\rangle, \quad (A16)$$

we obtain

$$\begin{aligned}
\mathrm{Cov}(&\sigma_z^1,\ldots\sigma_z^n)\\
&= \mathrm{Tr}[(\sigma_z - \langle\sigma_z\rangle)^{\otimes n}\rho]\\
&= -F(1 - \langle\sigma_z\rangle)^{n-1}(1 + \langle\sigma_z\rangle)\\
&\quad + (1-F)(-1)^{n-1}(1 + \langle\sigma_z\rangle)^{n-1}(1 - \langle\sigma_z\rangle). \quad (A17)
\end{aligned}$$

The average value of σ_z for W and $\overline{W}$ states is $\frac{n-2}{n}$ and $-\frac{n-2}{n}$, respectively. Hence, the average value of σ_z for a mixture of W and $\overline{W}$ states is

$$\begin{aligned}
\langle\sigma_z\rangle &= \mathrm{Tr}(\sigma_z\rho)\\
&= F\mathrm{Tr}(\sigma_z|W\rangle\langle W|) + (1-F)\mathrm{Tr}(\sigma_z|\overline{W}\rangle\langle\overline{W}|)\\
&= (2F-1)\frac{n-2}{n}. \qquad (A18)
\end{aligned}$$

[1] L. Henderson and V. Vedral, J. Phys. A **34**, 6899 (2001).

[2] H. Ollivier and W. H. Żurek, Phys. Rev. Lett. **88**, 017901 (2001).

[3] A. SaiToh, R. Rahimi, and M. Nakahara, Quantum Inf. Comput. **11**, 0167 (2011).

[4] K. Modi, T. Paterek, W. Son, V. Vedral, and M. Williamson, Phys. Rev. Lett. **104**, 080501 (2010).

[5] P. Hayden, D. W. Leung, and A. Winter, Commun. Math. Phys. **265**, 95 (2006).

[6] J. Oppenheim, M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **89**, 180402 (2002).

[7] W. H. Żurek, Phys. Rev. A **67**, 012320 (2003).

[8] M. Horodecki, P. Horodecki, R. Horodecki, J. Oppenheim, A. Sen, U. Sen, and B. Synak-Radtke, Phys. Rev. A **71**, 062307 (2005).

[9] C. H. Bennett, Int. J. Theor. Phys. **21**, 905 (1982).

[10] Ł. Pankowski and B. Synak-Radtke, J. Phys. A: Math. Theor. **41**, 075308 (2008).

[11] D. Kaszlikowski, A. Sen, U. Sen, V. Vedral, and A. Winter, Phys. Rev. Lett. **101**, 070502 (2008).

[12] A. Miyake and H. J. Briegel, Phys. Rev. Lett. **95**, 220501 (2005).

[13] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).

[14] M. Seevinck and J. Uffink, Phys. Rev. A **78**, 032101 (2008).

[15] N. Gisin, Phys. Lett. A **210**, 151 (1996).

[16] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A **53**, 2046 (1996).

[17] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **78**, 574 (1997).

[18] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A **60**, 1888 (1999).

[19] G. Brassard, P. Horodecki, and T. Mor, IBM J. Res. Dev. **48**, 87 (2004).

[20] R. Alicki, M. Horodecki, P. Horodecki, and R. Horodecki, OSID **11**, 205 (2004).

[21] I. Devetak, Phys. Rev. A **71**, 062303 (2005).