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TOTAL OUTER-CONNECTED DOMINATION IN TREES

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Abstract

Let $G = (V, E)$ be a graph. Set $D \subseteq V(G)$ is a total outer-connected dominating set of G if D is a total dominating set in G and $G[V(G) - D]$ is connected. The total outer-connected domination number of G, denoted by $\gamma_{tc}(G)$, is the smallest cardinality of a total outer-connected dominating set of G. We show that if T is a tree of order n, then $\gamma_{tc}(T) \geq \lceil \frac{2n}{3} \rceil$. Moreover, we constructively characterize the family of extremal trees T of order n achieving this lower bound.

Keywords: total outer-connected domination number, domination number.

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1. Introduction

Graph theory terminology not presented here can be found in [1, 5].

Let $G = (V, E)$ be a simple graph. The *neighbourhood* of a vertex v, denoted by $N_G(v)$, is the set of all vertices adjacent to v in G and the integer $d_G(v) = |N_G(v)|$ is the *degree* of v in G. A vertex of degree one is called an *end-vertex*. A *support* is the unique neighbour of an end-vertex.

Let P_n denotes the path of order n. For a vertex v of G, we shall use the expression, *attach a P_n at v*, to refer to the operation of taking the union of G and a path P_n and joining one of the end-vertices of this path to v with an edge.

Set $D \subseteq V(G)$ is a *dominating set* in G if $N_G(v) \cap D \neq \emptyset$ for every vertex $v \in V(G) - D$. The *domination number* of G, denoted $\gamma(G)$, is the cardinality of a minimum dominating set of G.

Set $D \subseteq V(G)$ is a *total dominating set* of G if each vertex of $V(G)$ has a neighbour in D. The cardinality of a minimum total dominating set in G is the *total domination number* of G and is denoted by $\gamma_t(G)$. Total domination in graphs is currently well studied in graph theory (for examples, see [2, 6]).

Set $D \subseteq V(G)$ is said to be a *total outer-connected dominating set* of G if D is a total dominating set and $G[V(G)-D]$ is connected. The cardinality of a minimum total outer-connected dominating set in G is called the *total outer-connected domination number* of G and is denoted by $\gamma_{tc}(G)$. Observe that every graph G without isolates has a total outer-connected dominating set, since the set of all vertices of G is a total outer-connected dominating set in G.

We will show that if T is a tree of order n, then $\gamma_{tc}(T) \geq \lceil \frac{2n}{3} \rceil$. Moreover, we will constructively characterize the extremal trees T of order $n \geq 3$ achieving this lower bound.

Similar bounds for various domination numbers in trees are given in [2, 6].

2. The Lower Bound

Theorem 1. *If T is a tree of order $n \geq 2$, then*

$$\gamma_{tc}(T) \geq \left\lceil \frac{2n}{3} \right\rceil .$$

Proof. The result is obvious for $n = 2$. Assume that $n \geq 3$ and let D be a minimum total outer-connected dominating set of T. Let us denote by S any component of $T[D]$. Since T is a tree, no two vertices of $V(T)-D$ have a common neighbour in S. Hence $|N_T(S) \cap (V(T)-D)| \leq 1$. Moreover, D is dominating in T and isolate free, and thus

$$\begin{aligned} n(T) &= |V(T)-D| + |D| \\ &\geq |V(T)-D| + 2|V(T)-D| \\ &\geq n - \gamma_{tc}(T) + 2n - 2\gamma_{tc}(T). \end{aligned}$$

Finally, we have $\gamma_{tc}(T) \geq \frac{2}{3}n$, and so $\gamma_{tc}(T) \geq \lceil \frac{2n}{3} \rceil$. ∎

3. The Characterization of the Extremal Trees

For $n \geq 2$, let $\mathcal{T}_n = \{T \mid T$ is a tree of order n such that $\gamma_{tc}(T) = \lceil \frac{2n}{3} \rceil\}$, $\mathcal{T} = \bigcup_{n\geq 2} \mathcal{T}_n$. We will present a constructive characterization of the family $\mathcal{T}$. For this purpose, we define a type (1) operation on a tree T as attaching P_3 at v where v is a vertex of T not belonging to some minimum total outer-connected dominating set of T, and a type (2) operation as attaching P_1 at v where v belongs to some minimum total outer-connected dominating set of T.

We now define families of trees as follows. Let $\mathcal{C}_n = \{T \mid T$ is a tree of order n which can be obtained from the path P_3 by a finite sequence of operations of type (1) and (2), where the operation of type (2) appears in the sequence exactly $n \pmod 3$ times$\}$, $n \geq 3$, and $\mathcal{C}_2 = \{P_2\}$.

We shall establish:

Theorem 2. *For $n \geq 2$, $\mathcal{T}_n = \mathcal{C}_n$.*

We prove Theorem 2 by establishing eight lemmas.

Lemma 3. *If D is a minimum total outer-connected dominating set of a tree T of order at least 6 and $T \in \mathcal{T}$, then every end-vertex of T and every support of T belongs to D.*

Lemma 4. *If $T \in \mathcal{T}$, then $|\Omega(T)| \leq |S(T)| + 2$, where $\Omega(T)$ is the set of all end-vertices of T and $S(T)$ is the set of all supports of T.*

Proof. Let D be a minimum total outer-connected dominating set of a tree T belonging to $\mathcal{T}$. Then for some positive integer n we have $T \in \mathcal{T}_n$ and $|D| = \lceil \frac{2n}{3} \rceil$. Suppose $|\Omega(T)| = |S(T)| + t$, $t > 2$. Denote by $s_1, \ldots, s_m$ the supports of T and by $l_1, \ldots, l_m, l_{m+1}, \ldots, l_{m+t}$ the end-vertices of T, where $l_i \in N_T(s_i)$, $1 \leq i \leq m$. Notice that $D - \{l_{m+1}, l_{m+2}, l_{m+3}\}$ is a total outer-connected dominating set of a tree $T' = T - \{l_{m+1}, l_{m+2}, l_{m+3}\}$. Hence $\gamma_{tc}(T') \leq |D| - 3 = \lceil \frac{2n-9}{3} \rceil$. On the other hand, by Theorem 1, we have $\gamma_{tc}(T') \geq \left\lceil \frac{2(n-3)}{3} \right\rceil$ and consequently $\left\lceil \frac{2(n-3)}{3} \right\rceil \leq \gamma_{tc}(T') \leq \lceil \frac{2n-9}{3} \rceil$, which is impossible. ■

Thus we have what follows.

Corollary 1. *If $T \in \mathcal{T}$, then exactly one of the following conditions holds:*

(i) *every support of* T *is a neighbour of exactly one end-vertex;*

(ii) *exactly one support of* T *is a neighbour of exactly two end-vertices, while every other support is a neighbour of exactly one end-vertex;*

(iii) *exactly one support of* T *is a neighbour of three end-vertices, while evert other support is a neighbour of exactly one end-vertex or exactly two supports of* T *are the neighbours of exactly two end-vertices, while every other support is a neighbour of exactly one end-vertex.*

Lemma 5. *If* $T \in \mathcal{T}_n$, $n \geq 3$, *and* T' *is obtained from* T *by a type* (1) *operation, then* $T' \in \mathcal{T}_{n+3}$.

Proof. By definition of a type (1) operation on a tree T, there exists a minimum total outer-connected dominating set of T such that adding a new end-vertex of T' and a new support of T' to it produces a total outer-connected dominating set of T'. Hence, since $T \in \mathcal{T}_n$, $\gamma_{tc}(T') \leq \gamma_{tc}(T) + 2 = \lceil \frac{2n+6}{3} \rceil$. However, T' is a tree of order $n+3$, and so, by Theorem 1, $\gamma_{tc}(T') \geq \lceil \frac{2(n+3)}{3} \rceil$. Consequently, $\gamma_{tc}(T') = \lceil \frac{2(n+3)}{3} \rceil$, and hence $T' \in \mathcal{T}_{n+3}$. ■

Notice that $\mathcal{C}_3 = \{P_3\} = \mathcal{T}_3$. Hence an immediate consequence of Lemma 5 now follows.

Lemma 6. *If* $n \geq 3$ *and* $n \equiv 0 \,(\mathrm{mod}\ 3)$, *then* $\mathcal{C}_n \subseteq \mathcal{T}_n$.

We will now prove the inverse inclusion.

Lemma 7. *If* $n \geq 3$ *and* $n \equiv 0 \,(\mathrm{mod}\ 3)$, *then* $\mathcal{T}_n \subseteq \mathcal{C}_n$.

Proof. We proceed by induction on $n \geq 3$. Since $\mathcal{T}_3 = \{P_3\} = \mathcal{C}_3$, the result is true for $n = 3$. Let $n \geq 6$ satisfy $n \equiv 0 \,(\mathrm{mod}\ 3)$ and assume that $\mathcal{T}_k \subseteq \mathcal{C}_k$ for all integers $k \equiv 0 \,(\mathrm{mod}\ 3)$, where $3 \leq k < n$. Let $T \in \mathcal{T}_n$. We show that $T \in \mathcal{C}_n$. Let D be a minimum total outer-connected dominating set of T. Let $P = (v_1, v_2, \ldots, v_m)$ be a longest path in T. By Lemma 3, $\{v_1, v_2, v_{m-1}, v_m\} \subseteq D$.

We will show that $d_T(v_2) \equiv 2$ and $\{v_3, v_4\} \cap D = \emptyset$. Suppose that v_2 is adjacent to two end-vertices, say v_1 and l_1. Then $D' = D - \{l_1\}$ is a total outer-connected dominating set of $T' = T - l_1$. Hence, since $T \in \mathcal{T}_n$, $\gamma_{tc}(T') \leq \lceil \frac{2n}{3} \rceil - 1 = \frac{2n}{3} - 1$. However, T' is a tree of order $n - 1 \equiv 2$ (mod 3), and so, by Theorem 1, $\gamma_{tc}(T') \geq \lceil \frac{2(n-1)}{3} \rceil = \frac{2n}{3}$, a contradiction.

Suppose now $v_3 \in D$. Then the set $D' = D - \{v_1\}$ is a total outer-connected dominating set of $T' = T - v_1$ and $\frac{2n}{3} \leq \gamma_{tc}(T') \leq \frac{2n}{3} - 1$ — a contradiction. Hence $d_T(v_2) = 2$ and $v_3 \notin D$. From Lemma 3 and from the fact that $V(T) - D$ is a tree we conclude that $m \geq 6$ and $v_4 \notin D$.

We will now prove that $d_T(v_3) = 2$. Since $v_3 \notin D$, v_3 is not a support. Suppose there exists a path $P' = (u_1, u_2, v_3)$ in T such that $u_2 \notin \{v_2, v_4\}$. By Lemma 4, $\{u_1, u_2\} \subseteq D$. Moreover $D' = D - \{u_1, u_2\}$ is a total outer-connected dominating set of $T' = T - \{u_1, u_2\}$. Hence $\gamma_{tc}(T') \leq \gamma_{tc}(T) - 2 = \frac{2n}{3} - 2$, which contradicts the fact that (by Theorem 1) $\gamma_{tc}(T') \geq \lceil \frac{2(n-2)}{3} \rceil$.

Let us consider tree $T' = T - \{v_1, v_2, v_3\}$. The set $D' = D - \{v_1, v_2\}$ is a total outer-connected dominating set of T'. Hence $\gamma_{tc}(T') \leq \lceil \frac{2n}{3} \rceil - 2 = \lceil \frac{2n-6}{3} \rceil$. Moreover by Theorem 1, $\gamma_{tc}(T') \geq \lceil \frac{2(n-3)}{3} \rceil$ and so $T' \in \mathcal{T}_{n-3}$. Thus, by the inductive hypothesis, $T' \in \mathcal{C}_{n-3}$. Since v_4 does not belong to some minimum total outer-connected dominating set of T', namely D', T is constructed from T' by a type (1) operation. Hence $T \in \mathcal{C}_n$. ■

Lemma 8. *If $T \in \mathcal{T}_n$, $n \geq 3$, and $n \not\equiv 2 \pmod 3$, then a tree T' obtained from T by a type* (2) *operation belongs to $\mathcal{T}_{n+1}$.*

Proof. By definition of a type (2) operation on a tree T, there exists a minimum total outer-connected dominating set of T such that adding to it the new end-vertex of T' produces a total outer-connected dominating set of T'. Hence, since $T \in \mathcal{T}_n$ and $n \not\equiv 2 \pmod 3$, $\gamma_{tc}(T') \leq \gamma_{tc}(T) + 1 = \lceil \frac{2n+3}{3} \rceil = \lceil \frac{2n+2}{3} \rceil$. However, T' is a tree of order $n+1$, and so, by Theorem 1, $\gamma_{tc}(T') \geq \lceil \frac{2(n+1)}{3} \rceil$. Consequently, $\gamma_{tc}(T') = \lceil \frac{2n+2}{3} \rceil$ and $T' \in \mathcal{T}_{n+1}$. ■

Lemma 9. *If $n \geq 4$ and $n \not\equiv 0 \pmod 3$, then $\mathcal{C}_n \subseteq \mathcal{T}_n$.*

Proof. We proceed by induction on $n \geq 4$. The base case is true since $\mathcal{C}_4 = \{K_{1,3}, P_4\} \subseteq \mathcal{T}_4$ and $\mathcal{C}_5 = \{K_{1,4}, P_5, T_1\} \subseteq \mathcal{T}_5$, where T_1 is a tree obtained from a star $K_{1,3}$ by subdivision of exactly one of its edges.

Assume now that the result is true for $k \not\equiv 0 \pmod 3$, $4 \leq k < n$. Let T be a tree belonging to the family $\mathcal{C}_n$. Thus T can be obtained from a tree T' by either one operation of type (1) or one operation of type (2). If T is obtained from T' as a result of operation of type (1), then T' is a tree of order $n-3$ and by our induction hypothesis $T' \in \mathcal{T}_{n-3}$. Therefore, by Lemma 5, $T \in \mathcal{T}_n$.

If T is obtained from T' by one operation of type (2), then T' is a tree of order $n-1$. We consider two cases:

Case 1. If $n = 1 \pmod 3$, then the construction of T' is accomplished by using only type (1) operations starting with the path P_3 and thus $T' \in \mathcal{C}_{n-1}$. From Lemma 6 we conclude that $T' \in \mathcal{T}_{n-1}$. Hence, by Lemma 8, $T \in \mathcal{T}_n$.

Case 2. If $n \equiv 2 \pmod 3$, then $T' \in \mathcal{C}_{n-1}$ and by our induction hypothesis $T' \in \mathcal{T}_{n-1}$. Finally, by Lemma 8, $T \in \mathcal{T}_n$. ■

Lemma 10. *If* $n \geq 4$ *and* $n \not\equiv 0 \pmod 3$, *then* $\mathcal{T}_n \subseteq \mathcal{C}_n$.

Proof. We proceed by induction on $n \geq 4$. Since $\mathcal{P}_4 = \{P_4, K_{1,3}\} = \mathcal{C}_4$ and $\mathcal{P}_5 = \{K_{1,4}, P_5, T_1\} = \mathcal{C}_5$, where T_1 is a tree obtained from a star $K_{1,3}$ by subdivision of exactly one of its edges, the result is true for $n = 4$ and $n = 5$. Let $n \geq 7$ satisfy $n \not\equiv 0 \pmod 3$, and assume that $\mathcal{T}_k \subseteq \mathcal{C}_k$ for all integers $k \not\equiv 0 \pmod 3$, where $4 \leq k < n$. Let $T \in \mathcal{T}_n$ and let D be a minimum total outer-connected dominating set of T. Let $P = (v_1, v_2, \ldots, v_m)$ be the longest path in T. By Lemma 3, $\{v_1, v_2, v_{m-1}, v_m\} \subseteq D$. We consider two cases:

Case 1. One of the vertices v_2 or v_{m-1} is adjacent to at least two end-vertices. Without loss of generality, we can assume that $|N_T(v_2) \cap \Omega(T)| \geq 2$. Let $l_1 \in N_T(v_2) \cap \Omega(T)$, $l_1 \neq v_1$. In this case $D' = D - \{l_1\}$ is a total outer-connected dominating set of $T' = T - l_1$ and hence $\gamma_{tc}(T') \leq \gamma_{tc}(T) - 1 = \lceil \frac{2n-3}{3} \rceil = \lceil \frac{2n-2}{3} \rceil$. Thus, Theorem 1 implies $\gamma_{tc}(T') = \lceil \frac{2n-2}{3} \rceil$. Depending on whether $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$ we have $T' \in \mathcal{C}_{n-1}$ from Lemma 7 or by our induction hypothesis, respectively. Hence we obtain $T \in \mathcal{C}_n$.

Case 2. The vertices v_2 and v_{m-1} have degree 2. Suppose that v_3 or v_{m-2}, say v_3, belongs to D. Then for tree $T' = T - v_1$ and for $D' = D - \{v_1\}$, similarly to Case 1, we have that $T \in \mathcal{C}_n$. Hence we can assume that $\{v_3, v_{m-2}\} \cap D = \emptyset$. Thus from connectivity of $V(T) - D$ we have $\{v_4, v_{m-3}\} \cap D = \emptyset$.

We will now show that v_3 or v_{m-2} is of degree two. Suppose to the contrary, that neither v_3 nor v_{m-2} is of degree 2. Let y be the neighbour of v_3, $y \neq v_2$ and $y \neq v_4$, and let z be the neighbour of v_{m-2}, $z \neq v_{m-1}$ and $z \neq v_{m-3}$. Then neither y nor z is not an end-vertex – otherwise we would have $v_3 \in D$ or $v_{m-2} \in D$. From that and from our choice of path $(v_1, v_2, \ldots, v_m)$ it is straightforward that y and z are supports and $A = N_T(y) - \{v_3\} \subseteq \Omega(T)$, $B = N_T(z) - \{v_{m-2}\} \subseteq \Omega(T)$. We also have that $D - (A \cup B \cup \{y, z\})$ is a total outer-connected dominating set of $T' = T - (A \cup B \cup \{y, z\})$, and so $\lceil \frac{2(n-2-|A|-|B|)}{3} \rceil \leq \gamma_{tc}(T') \leq \gamma_{tc}(T) - 2 - |A| - |B| \leq \lceil \frac{2n}{3} \rceil - 2 - |A| - |B|$, which is impossible. Therefore, without the loss of generality, we may assume that $\deg_T(v_3) = 2$.

Let us consider $T' = T - \{v_1, v_2, v_3\}$. The set $D' = D - \{v_1, v_2\}$ is a total outer-connected dominating set of T', and hence $\gamma_{tc}(T') \leq \lceil \frac{2n}{3} \rceil - 2 = \lceil \frac{2n-6}{3} \rceil$. Moreover, by Theorem 1, $\gamma_{tc}(T') \geq \lceil \frac{2(n-3)}{3} \rceil$ and so $T' \in \mathcal{T}_{n-3}$. Therefore, by the inductive hypothesis, $T' \in \mathcal{C}_{n-3}$. However, T is constructed from T' by a type (1) operation. Hence $T \in \mathcal{C}_n$. ■

Theorem 2 now follows immediately from Lemmas 6, 7, 9 and 10.

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