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Equivariant degree of convex-valued maps applied to set-valued BVP

Research Article

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Abstract: An equivariant degree is defined for equivariant completely continuous multivalued vector fields with compact convex values. Then it is applied to obtain a result on existence of solutions to a second order BVP for differential inclusions carrying some symmetries.

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1. Introduction

The aim of this paper is to establish a Leray–Schauder equivariant degree theory for equivariant completely continuous upper semicontinuous vector fields in Banach representations of a compact Lie group. There are several versions of equivariant degree, see e.g. [3] or [2] and references therein. We have chosen a modest version without a parameter, which is the closest to the classical Leray–Schauder degree. The method is quite standard. We use equivariant approximation results obtained recently in [6], and equivariant Schauder projections. It is well known that the degree theory is a powerful tool in existence theory for differential equations and inclusions, cf. e.g. [15]. The same is with equivariant versions, see [3, 11] and references therein. We decided to present a multiplicity result for a second order multivalued Dirichlet boundary value problem (see Theorem 6.4) with some symmetries. The calculation of the equivariant degree is quite complicated in general; however, many special cases (especially for finite groups and linear maps) were treated in [3]. They may serve as a kind of a "black box", and this is the idea of our approach, cf. e.g. Example 4.2 and Remark 6.3. We do not pretend to the greatest generality of results. It is clear that the same method works for other boundary conditions and other differential inclusions with symmetric right hand side.

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The paper is organized as follows. In Section 2, basic definitions and remarks concerning group actions, equivariant maps and properties of vector Haar integral are formulated as well as some definitions concerning set-valued maps. In Section 3, we give fundamental approximation results in equivariant setting. Section 4 is devoted to the equivariant degree in a finite-dimensional orthogonal representation of a compact Lie group. An infinite-dimensional Leray–Schauder degree is defined in Section 5. The last section is devoted to second order differential inclusions with Dirichlet boundary conditions.

Notation.

In the sequel, we consider only *Hausdorff* topological spaces. If X is a topological space and $A \subset X$, then $\overline{A}$ denotes the *closure* of A. If X is a metric space (with the metric denoted by d by default), $A \subset X$ and $\varepsilon > 0$, then $B(A,\varepsilon) = \{x \in X : d(x,A) = \inf_{a\in A} d(x,a) < \varepsilon\}$; in particular $B(a,\varepsilon)$ is the open ball with radius $\varepsilon > 0$ centered at $a \in X$. Moreover, let $D(A,\varepsilon) = \{x \in X : d(x,A) \leqslant \varepsilon\}$. In what follows $\mathbb{E}$ denotes a Banach space with the norm $\|\cdot\|$ over real or complex numbers. Note that if $A \subset \mathbb{E}$ and $\varepsilon > 0$, then $B(A,\varepsilon) = A + \varepsilon B(0,1) = A + B(0,\varepsilon)$. The convex (resp. closed convex) hull of $A \subset \mathbb{E}$ is denoted by $\operatorname{conv} A$ (resp. $\overline{\operatorname{conv}}\, A$).

2. Preliminaries

We start with some notation for group actions, see [5] for more details. Let G be a group. If $H \subset G$ is a subgroup, we denote by G/H the set of left cosets gH. Two subgroups H and K of G are *conjugate* if there exists $g \in G$ such that $K = g^{-1}Hg$. The conjugacy class of H is denoted by (H). There is a natural partial order on the set $\Phi(G)$ of conjugacy classes:

$$(K) \leqslant (H) \ \text{ if there exist } \ \overline{K} \in (K) \ \text{ and } \ \overline{H} \in (H) \ \text{ such that } \ \overline{K} \subset \overline{H}.$$

Throughout the whole paper, we consider only compact Lie groups and their closed subgroups. Given a subgroup $H \subset G$, let $N(H)$ be the normalizer of H. The *Weyl group* of H is the quotient $W(H) = N(H)/H$. Let us denote $\Phi_0(G) = \{(H) \in \Phi(G) : \dim W(H) = 0\}$.

A *G-set* is a pair (X,ξ), where X is a set and $\xi\colon G \times X \to X$ is an action of G on X, i.e., a map such that

(i) $\xi(g_1, \xi(g_2, x)) = \xi(g_1 g_2, x)$ for $g_1, g_2 \in G$ and $x \in X$,

(ii) $\xi(e,x) = x$ for $x \in X$, where $e \in G$ is the group unit.

In the sequel we write gx instead of $\xi(g,x)$. For every subgroup $H \subset G$ the set G/H is a G-set by the action $g(\tilde{g}H) = g\tilde{g}H$. If ξ is continuous, we call (X,ξ) a *G-space*. We say that a real (resp. complex) Banach space $\mathbb{E}$ is a *real* (resp. *complex*) *Banach representation of G* if $\mathbb{E}$ is a G-space and, for each $g \in G$, the map $\xi_{\mathbb{E}}(g,\cdot)\colon \mathbb{E} \ni x \mapsto gx$ is linear and bounded.

For $x \in X$, the subgroup $G_x = \{g \in G : gx = x\}$ is called the *isotropy group* of X at x. The conjugacy class of an isotropy group is called an *isotropy type*. We denote by $\operatorname{Iso}(X)$ the set of all isotropy types in X. The set $Gx = \{gx : g \in G\}$ is called an *orbit* through x.

For a given subgroup $H \subset G$ we specify several subspaces of a G-space X: $X_H = \{x \in X : H = G_x\}$, $X_{(H)} = \{x \in X : (H) = (G_x)\}$, $X^H = \{x \in X : H \subset G_x\}$, $X^{(H)} = \{x \in X : (H) \leqslant (G_x)\}$.

We define the *Burnside ring* of G as follows (cf. [3] for details and examples): As a group $A(G)$ we take the free abelian group generated by $(H) \in \Phi_0(G)$. In other words, elements $a \in A(G)$ can be viewed as formal finite sums $a = n_{H_1}(H_1) + \cdots + n_{H_m}(H_m)$ with coefficients $n_{H_i} \in \mathbb{Z}$ and $(H_i) \in \Phi_0(G)$. The operation of multiplication in $A(G)$ is a bit more sophisticated. Let $(H),(K) \in \Phi_0(G)$. Consider the diagonal action of G on $(G/H) \times (G/K)$. Then for any $(L) \in \Phi_0(G)$, the spaces G/H^L and G/K^L consist of finitely many $W(L)$-orbits. Therefore, the space $\big((G/H) \times (G/K)\big)_{(L)}/G$ is finite. Let $n_L(H,K)$ denote the number of elements of this space. Define

$$(H)\cdot(K) = \sum_{(L)\in\Phi_0(G)} n_L(H,K)(L).$$

There are some useful, easy to prove, properties of open coverings of G-spaces.

Lemma 2.1 ([13]).

Let X be a G-space and $\{U_\lambda\}_{\lambda\in\Lambda}$ an open, locally finite covering of X. Then every point $x \in X$ has a G-neighborhood V such that the set $\{\lambda \in \Lambda : U_\lambda \cap V \neq \emptyset\}$ is finite, and the covering $\{GU_\lambda\}_{\lambda\in\Lambda}$ is locally finite.

Definition 2.2.

A covering $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ is *G-invariant* if each U_λ is G-invariant. A covering $\mathcal{U}$ is a *G-covering* iff

(i) $gU_\lambda \in \mathcal{U}$ for every $U_\lambda \in \mathcal{U}$ and every $g \in G$,

(ii) the index set Λ is a G-set satisfying $gU_\lambda = U_{g\lambda}$.

If $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ is a G-covering, we can produce an invariant covering called the *saturation* of $\mathcal{U}$: $\widetilde{\mathcal{U}} = \left\{\widetilde{U}_\alpha = \bigcup_{\lambda\in\alpha} U_\lambda\right\}_{\alpha\in\Lambda/G}$.

Proposition 2.3 ([13, Proposition 1.4]).

Let X be a paracompact G-space. Then

(1) *Every open G-covering $\mathcal{U}$ of X has an open refinement $\mathcal{V}$ which is a G-covering and the saturation $\widetilde{\mathcal{V}}$ of which is locally finite.*

(2) *Every open invariant covering $\mathcal{V} = \{V_\alpha\}_{\alpha\in A}$ admits an invariant partition of unity $\{p_\alpha\}_{\alpha\in A}$ such that $\overline{p_\alpha^{-1}((0,1])} \subset V_\alpha$ for every $\alpha \in A$.*

(3) *The orbit space X/G is paracompact.*

Let us denote by $C(G)$ the space of all continuous real functions on a group G. The left (right) translation of $f : G \to \mathbb{R}$ is the map $L_s f$ given by $(L_s f)(x) = f(sx)$ (resp. $(R_s f)(x) = f(xs)$).

Theorem 2.4 ([16, Theorem 5.14]).

If G is compact, then there exists a unique normalized Haar measure on G, which is a left-invariant probabilistic Borel measure:

$$\int_G f\,dm = \int_G (L_s f)\,dm \qquad s \in G, \quad f \in C(G).$$

It is also right-invariant, i.e.,

$$\int_G f\,dm = \int_G (R_s f)\,dm \qquad s \in G, \quad f \in C(G).$$

Moreover,

$$\int_G f(x)\,dm(x) = \int_G f(x^{-1})\,dm(x) \qquad f \in C(G).$$

The following proposition is formulated in a great generality in [1]. It is true in particular for Banach space-valued functions. One can find it e.g. in [10, Proposition 3.30], see also [16, Theorem 3.27] or [6].

Proposition 2.5.

Assume that V is a complete (in the sense of the natural uniformity induced from Z) convex invariant subset of a locally convex topological vector space Z on which a compact group G acts linearly. Let $C(G, V)$ denote the space of all continuous maps $f : G \to V$ endowed with the compact-open topology. Then the vector-valued Haar integral $\int : C(G, V) \to V$ is a well-defined continuous map satisfying the following properties:

(1) $\int_G f(hg)\,dg = \int_G f(g)\,dg = \int_G f(gh)\,dg$ *for any $h \in G$,*

(2) $\int_G hf(g)\,dg = h\int_G f(g)\,dg$ *for any $h \in G$,*

(3) $\int_G f(g)\,dg = v_0$, *if $f(G) = \{v_0\}$ for a point $v_0 \in V$.*

Let us note that in case of a finite group G or a finite-dimensional space Z, the completeness assumption is superfluous. If $\mathbb{E}$ is a Banach representation of G, then one can define a norm $\|\cdot\|_G$ on $\mathbb{E}$ such that the action of G on $\mathbb{E}$ is *isometric*, i.e., $\|gx\|_G = \|x\|_G$ for all $g \in G$ and $x \in \mathbb{E}$. Indeed, it is sufficient to put

$$\|x\|_G = \int_G \|gx\|\, dg, \qquad x \in \mathbb{E}.$$

This new norm is complete since it is equivalent to the original norm $\|\cdot\|$ in $\mathbb{E}$.

Now, we recall some information about set-valued maps, for more details see [4, 8]. Let X, Y be two metric spaces. By a *set-valued map* φ from X into Y (written $\varphi\colon X \multimap Y$) we mean a map that assigns to each $x \in X$ a nonempty *closed* subset $\varphi(x)$ of Y. If, for any closed (resp. open) set $U \subset Y$, the *preimage* $\varphi^{-1}(U) = \{x \in X : \varphi(x) \cap U \neq \emptyset\}$ is closed (resp. open), then we say that φ is *upper* (resp. *lower*) semicontinuous; φ is continuous if it is upper and lower semicontinuous simultaneously. The *graph* $\mathrm{Gr}(\varphi) = \{(x, y) \in X \times Y : y \in \varphi(x)\}$ of an upper semicontinuous map φ is closed. A map φ is upper semicontinuous and has compact values (i.e. for each $x \in X$, the set $\varphi(x)$ is compact) if and only if, for any sequence $(x_n, y_n) \in \mathrm{Gr}(\varphi)$ such that $x_n \to x \in X$, there is a subsequence (y_{n_k}) such that $y_{n_k} \to y \in \varphi(x)$ (in other words the restriction $p_\varphi\colon \mathrm{Gr}(\varphi) \to X$ of the projection $X \times Y \to X$ is proper). Recall that a continuous map $f\colon X \to Y$ is *proper* if, for each compact $K \subset Y$, the preimage $f^{-1}(K)$ is compact. It is worth to remind that f is proper if and only it is *perfect*, i.e., continuous, closed and such that, for any $y \in Y$, $f^{-1}(y)$ is compact (this is so since Y is, by assumption, a metric space). Observe that a continuous surjection $f\colon X \to Y$ is perfect if and only if the multivalued map $Y \ni y \multimap f^{-1}(y) \subset X$ is upper semicontinuous and has compact values. We say that a map φ is *compact* if it is upper semicontinuous and the *closure* $\mathrm{cl}\,\varphi(X)$ is compact; φ is *completely continuous* if the restriction $\varphi\restriction_B$ of φ to any bounded subset $B \subset X$ is compact.

3. Equivariant selections and approximations

Definition 3.1.
Let X and Y be G-spaces. A multivalued map $F\colon X \multimap Y$ is *G-equivariant* if $F(gx) = gF(x)$ for all $g \in G$ and $x \in X$.

The following is the classical Michael selection theorem [12].

Theorem 3.2.
Let X be a paracompact space, Y a Banach space and $F\colon X \multimap Y$ a lower semicontinuous map such that $F(x)$ is a nonempty, closed, convex set for every $x \in X$. Then there exists a continuous map $f\colon X \to Y$ such that $f(x) \in F(x)$ for all $x \in X$.

The equivariant version is easy.

Theorem 3.3 (cf. [1]).
Let X be a paracompact G-space, Y a Banach G-representation, $F\colon X \multimap Y$ an equivariant lsc map with nonempty closed convex values. Then F admits an equivariant continuous selection.

Proof. According to Theorem 3.2 there is a continuous selection $f\colon X \to Y$ of F. Let dg be the normalized Haar measure on G. We define a new map $\varphi\colon X \to Y$ by the formula

$$\varphi(x) = \int_G g^{-1} f(gx)\, dg,$$

where the integral is the vector-valued integral with respect to the Haar measure, see Proposition 2.5. Observe that $g^{-1}f(gx) \in g^{-1}(F(gx)) = g^{-1}(gF(x)) = F(x)$. Since $F(x)$ is closed and convex then $\overline{\mathrm{conv}\, A_f} \subset F(x)$, where $A_f = \{g^{-1}f(gx) : g \in G\}$. By [16, Theorem 3.27] the integral belongs to $\overline{\mathrm{conv}\, A_f}$. Continuity and equivariance follow immediately from the properties of the integral. □

Corollary 3.4.

Let X be a paracompact G-space, Y a Banach G-representation, $A \subset X$ an invariant closed subset and $F\colon A \multimap Y$ an equivariant lsc map with nonempty closed convex values. Let $f\colon A \to Y$ be a continuous equivariant selection of $F\restriction_A$. Then there exists an equivariant selection $h\colon X \to Y$ of F such that $h\restriction_A = f$.

Proof. Consider a map $\varphi\colon X \multimap Y$ given by

$$\varphi(x) = \begin{cases} f(x) & \text{if } x \in A, \\ F(x) & \text{if } x \notin A. \end{cases}$$

It is easy to verify, that φ is lower semicontinuous, equivariant and thus by Theorem 3.3 it admits an equvariant continuous selection h. □

Corollary 3.5.

If X is a paracompact space and $\mathbb{E}$ is a Banach representation of G, then any continuous G-map $f\colon A \to \mathbb{E}$ admits a continuous G-equivariant extension over X, i.e., there is a G-map $F\colon X \to \mathbb{E}$ such that $F\restriction_A = f$.

Proof. It is sufficient to take a continuous G-equivariant selection F of the lower semicontinuous G-equivariant set-valued map $\varphi\colon X \multimap \mathbb{E}$ with closed convex values defined for $x \in X$ by

$$\varphi(x) = \begin{cases} f(x) & \text{if } x \in A, \\ \mathbb{E} & \text{if } x \notin A. \end{cases}$$ □

Let X, Y be two metric spaces.

Definition 3.6.

We say that a continuous map $f\colon X \to Y$ is a graph ε-approximation of $\varphi\colon X \multimap Y$ if $f(x) \in B(\varphi(B(x,\varepsilon)),\varepsilon)$ for every $x \in X$.

The following approximation results are proved in a greater generality in [6]; we give a proof for the sake of completeness.

Theorem 3.7.

Let $\varphi\colon X \multimap \mathbb{E}$ be usc with convex compact values, where X is a metric G-space and $\mathbb{E}$ is a Banach representation of G. Then, for every $\varepsilon > 0$, there exists a G-equivariant graph ε-approximation f of φ such that $f(x) \in \overline{\operatorname{conv}\varphi(X)}$ for every $x \in X$.

Proof. Let $\varepsilon > 0$. From upper semicontinuity of φ, it follows that for every $x \in X$ there exists $0 < \delta(x) < \varepsilon$ such that $\varphi(B(x,\delta(x)) \subset B(\varphi(x),\varepsilon/2)$. Let $\mathcal{B} = \{B_j\}_{j\in J}$ be a G-covering which is a star-refinement of the open covering $U_x = \{B(x,\delta(x))\}_{x\in X}$, which exists by Proposition 2.3. That is, for every $j \in J$ there exists x_j such that $\operatorname{st}(B_j,\mathcal{B}) \subset U_{x_j}$.

We can find a continuous partition of unity $\{p_s\}_{s\in S}$ subordinate to a locally finite refinement of the G-covering $\mathcal{B}$, i.e., for each $s \in S$, there exists $j_s \in J$ such that $\operatorname{supp} p_s \subset B_{j_s}$. For every $s \in S$ we choose a point $y_s \in \varphi(B_{j_s})$. Define

$$f(x) = \sum_{s\in S} p_s(x) y_s.$$

Let us fix $x \in X$ and define $S(x) = \{s \in S : p_s(x) \neq 0\}$. Let $\bar{s} \in S(x)$. Then we have $j_{\bar{s}}$ such that $x \in B_{j_s}$ and $\operatorname{supp} p_{\bar{s}} \subset B_{j_{\bar{s}}}$. Therefore, there exists $\bar{x} \in X$ with $d(x,\bar{x}) < \varepsilon$ and such that for all $\bar{s} \in S(x)$ we have $\operatorname{st}(B_{j_{\bar{s}}},\mathcal{B}) \subset U_{\bar{x}}$. For every $g \in G$ we have $f_\varepsilon(gx) = \sum_{s\in S(gx)} p_s(gx) y_s$.

Let now $s \in S(gx)$. Then $gx \in \operatorname{supp} p_s \subset B_{j_s}$. Thus $x \in g^{-1}(B_{j_s}) = B_{g^{-1}j_s}$. However, $B_{g^{-1}j_s} \subset \operatorname{st}(B_{j_{\overline{s}}}, \mathcal{B}) \subset U_{\overline{x}}$. Since $y_s \in \varphi(B_{j_s})$, then

$$g^{-1}y_s \in g^{-1}(\varphi(B_{j_s})) = \varphi(g^{-1}(B_{j_s})) = \varphi(B_{g^{-1}j_s}) \subset \varphi(U_{\overline{x}}) \subset B\left(\varphi(\overline{x}), \frac{\varepsilon}{2}\right),$$

where $\overline{x}$ has been chosen as above (with $d(x, \overline{x}) < \varepsilon$).

Now, put

$$F_\varepsilon(x) = \int_G g^{-1} f_\varepsilon(gx)\, dg,$$

where dg denotes the unique normalized Haar measure and the integral is the vector-valued Haar integral from Proposition 2.5. Given $x \in X$, for every $g \in G$ we have

$$g^{-1} f_\varepsilon(gx) = g^{-1} \sum_{s \in S(gx)} p_s(gx) y_s = \sum_{s \in S(gx)} p_s(gx) g^{-1} y_s \;\in\; B\left(\varphi(\overline{x}), \frac{\varepsilon}{2}\right).$$

Therefore,

$$\int_G g^{-1} f_\varepsilon(gx)\, dg \;\in\; \overline{\operatorname{conv}}\, B\left(\varphi(\overline{x}), \frac{\varepsilon}{2}\right) \;\subset\; B(\varphi(\overline{x}), \varepsilon)$$

by the properties of the integral. Thus we have verified that F_ε is an ε-approximation of φ. It is G-equivariant by definition, cf. (2). of Proposition 2.5. □

The following is an immediate consequence of [6, Theorem 5.1].

Proposition 3.8.
Let X be a compact metric G-space, $A \subset X$ closed and G-invariant, $\mathbb{E}$ a Banach representation of G and $\varphi\colon X \multimap \mathbb{E}$ an upper semicontinuous compact convex valued and G-equivariant map. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that any G-equivariant continuous δ-approximation $f\colon A \to \mathbb{E}$ of $\varphi\restriction_A$ has a G-equivariant continuous extension $F\colon X \to \mathbb{E}$ which is an ε-approximation of φ.

Corollary 3.9.
Given a compact metric G-space X, $\mathbb{E}$ a Banach G-representation, $\varphi\colon X \multimap \mathbb{E}$ an upper semicontinuous compact convex valued and G-equivariant map, and $\varepsilon > 0$, there exists $\delta > 0$ such that any two equivariant δ-approximations of φ are joined by a homotopy $h\colon X \times [0,1] \to \mathbb{E}$ such that for each $t \in [0,1]$ the map $h_t\colon X \to \mathbb{E}$ is a G-equivariant ε-approximation of φ.

Proof. The space $X \times [0,1]$ is a G-space with an action $g(x,t) = (gx, t)$. We apply Proposition 3.8 to the map $\varphi'\colon X \times [0,1] \multimap \mathbb{E}$, $\varphi'(x,t) = \varphi(x)$ and $A = X \times \{0,1\}$. □

4. G-degree

Let V be a finite-dimensional orthogonal representation of the compact Lie group G. Let $\Omega \subset V$ be a G-invariant open subset. A G-equivariant map $f\colon (\overline{\Omega}) \to V$ is *admissible* provided $f(x) \neq 0$ for all $x \in \partial\Omega$. In fact, one can always assume that f is defined on the whole V because of the equivariant Dugundji theorem, Corollary 3.5.

Then one can define a *G-equivariant degree* $\deg_G(f, \Omega) \in A(G)$ of the form

$$\deg_G(f, \Omega) \;=\; \sum_{(H_i) \in \Phi_0(G)} n_{H_i}(H_i), \tag{1}$$

where n_{H_i} are integer coefficients, which satisfy the following recurrence formula:

$$n_H = \frac{1}{|W(H)|}\left(\deg(f^H,\Omega^H) - \sum_{(K)>(H)} n_K \cdot n(H,K)\cdot|W(K)|\right). \tag{2}$$

Here $|X|$ denotes the number of elements in X and $\deg(f^H,\Omega^H)$ is the Brouwer degree.

This G-degree is uniquely determined by the above formulae (1)–(2) and the following properties, see [2, Theorem 3.4].

Theorem 4.1.

(a, **Existence**) *If some coefficient n_H is different from 0 in the above formula, then there exists $x\in\Omega$ such that $f(x)=0$ and $(G_x)\geqslant(H)$.*

(b, **Additivity**) *Let Ω_1,Ω_2 be two disjoint, open, G-invariant subsets of Ω and $f^{-1}(0)\cap(\overline{\Omega})\subset\Omega_1\cup\Omega_2$. Then*

$$\deg_G(f,\Omega)=\deg_G(f,\Omega_1)+\deg_G(f,\Omega_2).$$

(c, **Homotopy**) *If $h\colon[0,1]\times\overline{\Omega}\to V$ is an admissible G-equivariant homotopy (i.e., h_t is equivariant and admissible for each $t\in[0,1]$), then*

$$\deg_G(h_0,\Omega)=\deg_G(h_1,\Omega).$$

(d, **Normalization**) *If Ω is a G-invariant, open, bounded neighborhood of 0 in V, then*

$$\deg_G(\mathrm{Id},\Omega)=1\cdot(G).$$

(e, **Multiplicativity**) *Given two representations V_1,V_2, let $f_i\colon\overline{\Omega}_i\to V_i$, $i=1,2$, be admissible. Then*

$$\deg_G(f_1\times f_2,\Omega_1\times\Omega_2)=\deg_G(f_1,\Omega_1)\cdot\deg_G(f_2,\Omega_2),$$

where the multiplication is in the Burnside ring $A(G)$.

(f, **Suspension**) *If $f\colon\Omega\to V$ is admissible and $B\in W$ is an open G-invariant bounded neighborhood of 0 in W, then*

$$\deg_G(f\times\mathrm{Id}_W,\Omega\times B)=\deg_G(f,\Omega).$$

(g, **Hopf property**) *Let $B(V)$ be the unit ball of the representation V and for two admissible maps $f_1,f_2\colon\overline{B(V)}\to V$ one has $\deg_G(f_1,B(V))=\deg_G(f_2,B(V))$. Then f_1 and f_2 are homotopic by admissible G-equivariant homotopy.*

Example 4.2.

Consider the dihedral group $D_4=\{1,i,-1,i,\kappa,\kappa i,-\kappa,-\kappa i\}$, where the first four elements represent rotations of the complex plane C and $\kappa(z)=\bar{z}$. Denote the subgroups of D_4 as $\mathbb{Z}_1=\{1\}$, $\mathbb{Z}_2=\{1,-1\}$, $\mathbb{Z}_4=\{1,i,-1,-i\}$, $D_1=\{1,\kappa\}$, $\widetilde{D}_1=\{1,\kappa i\}$, $D_2=\{1,-1,\kappa,-\kappa\}$, $\widetilde{D}_2=\{1,-1,\kappa i,-\kappa i\}$, D_4. There are five irreducible representations of D_4: $\mathcal{V}_0=\mathbb{R}$ — the trivial one, $\mathcal{V}_1=\mathbb{C}$ – the natural one, $\mathcal{V}_2=\mathcal{V}_3=\mathcal{V}_4=\mathbb{R}$ — given by the homomorphisms $h\colon D_4\to O(1)$ with the kernels $\mathbb{Z}_4,D_2,\widetilde{D}_2$, respectively. Then the "basic" degrees $\deg_{\mathcal{V}_i}=\deg_G(-\mathrm{Id},B_i)$ can be calculated by use of multiplication table of generators in $A(D_4)$, see [3] or [2]; here $B_i\subset\mathcal{V}_i$ are the unit balls. The degrees are as follows:

$$\deg_{\mathcal{V}_0}=(D_4),\qquad \deg_{\mathcal{V}_1}=(D_4)-(D_1)-(\widetilde{D}_1)+(\mathbb{Z}_1),\qquad \deg_{\mathcal{V}_2}=(D_4)-(\mathbb{Z}_4),$$
$$\deg_{\mathcal{V}_3}=(D_4)-(D_2),\qquad \deg_{\mathcal{V}_4}=(D_4)-(\widetilde{D}_2).$$

It follows, in particular, that in $V = \mathcal{V}_1$ the maps Id and $-$Id are not D_4-homotopic since their D_4-degrees are different. These maps have their Brouwer degrees equal to 1. Thus, they are homotopic by virtue of Hopf theorem. The extension of this degree theory to multivalued mappings is now quite standard (similar to the non-equivariant case).

Let $\Omega \subset \mathcal{V}$ be an open bounded G-invariant subset.

Definition 4.3.

An upper semicontinuous compact convex valued mapping $\varphi\colon \overline{\Omega} \multimap \mathcal{V}$ is *admissible*, if $0 \notin \varphi(x)$ for all $x \in \partial\Omega$.

Observe that if $\varphi\colon \overline{\Omega} \multimap \mathcal{V}$ is an equivariant admissible map, then by compactness of the domain there exists $\varepsilon > 0$ such that $0 \notin \varphi(x)$ for all $x \in B(\partial\Omega, \varepsilon)$. Therefore every ε-approximation of φ is admissible. Moreover, we can choose a $\delta < \varepsilon$ from Corollary 3.9.

Definition 4.4.

For an admissible G-equivariant upper semicontinuos map $\varphi\colon \overline{\Omega} \multimap V$ with compact convex values we choose $\delta > 0$ as above and define

$$\deg_G(\varphi, \Omega) = \deg_G(f_\delta, \Omega),$$

where f_δ is any continuous G-equivariant δ-approximation of φ.

Our choice of δ assures the existence of f_δ by Theorem 3.7 and independence of the approximation follows from Corollary 3.9 and Theorem 4.1 (c). We can formulate an obvious consequence of Theorem 4.1.

Theorem 4.5.

The equivariant degree defined for convex-valued upper semicontinuous maps satisfies all the properties from Theorem 4.1 with a natural modification in Existence*: If some coefficient n_H in the formula of the degree (1) is different from 0, then there exists $x \in \Omega$ such that $0 \in \varphi(x)$ and $(G_x) \geqslant (H)$.*

Proof. All the properties are straightforward consequences of the single-valued case. We prove the existence. Suppose that some coefficient n_H is non-zero in formula (1). Then taking n and $\delta < 1/n$ in Definition 4.4 we obtain by Theorem 4.1 an infinite sequence of points $x_n \in \Omega$ such that $(G_{x_n}) \geqslant (H)$ and $f_n(x_n) = 0$, where $f_n\colon \overline{\Omega} \to V$ is a continuous G-equivariant $(1/n)$-approximation of φ. Taking a subsequence we may assume that x_n is convergent to some $x_0 \in \Omega$. One easily verifies that $(G_{x_0}) \geqslant (H)$. Moreover, we have $(x_n, 0) \in \overline{B(\mathrm{Gr}(\varphi), 1/k)}$ for $n > k$. Hence $(x_0, 0) \in \bigcap_{n\in\mathbb{N}} \overline{B(\mathrm{Gr}(\varphi), 1/n)} = \mathrm{Gr}(\varphi)$. This means that $0 \in \varphi(x_0)$. □

5. Leray–Schauder equivariant degree

Let $\mathbb{E}$ be a real isometric Banach representation of a compact Lie group G. Denote by $\{\mathcal{V}_k : k = 0, 1, 2, \dots\}$ the sequence of all orthogonal irreducible representations of G and $\chi_k\colon G \to \mathbb{R}$ the corresponding characters. Define the *intrinsic dimension* of $\mathcal{V}_k$ to be the number

$$n(\mathcal{V}_k) = \begin{cases} \dim_{\mathbb{R}} \mathcal{V}_k & \text{if } \mathcal{V}_k \text{ is of real type,} \\ \dfrac{\dim_{\mathbb{R}} \mathcal{V}_k}{2} & \text{if } \mathcal{V}_k \text{ is of complex type,} \\ \dfrac{\dim_{\mathbb{R}} \mathcal{V}_k}{4} & \text{if } \mathcal{V}_k \text{ is of quaternionic type.} \end{cases}$$

The linear mappings

$$P_k x = n(\mathcal{V}_k) \cdot \int_G \chi_k(g) g(x)\, dg$$

are then G-equivariant and bounded projections onto G-invariant subspaces $V_k = P_k(V)$ of $\mathbb{E}$, called *isotypical components* of $\mathbb{E}$. Then one proves that for every G-equivariant linear operator $A\colon \mathbb{E} \to \mathbb{E}$ and each $k = 0, 1, 2, \dots$ the subspace V_k is A-invariant: $A(V_k) \subseteq V_k$. We also have a decomposition $\mathbb{E} = \overline{\bigoplus_k V_k}$. Therefore, for any finite subset $X \subset \bigoplus_k V_k$ the subspace $\operatorname{span}(G(X))$ spanned by the orbits from X is finite-dimensional and G-invariant, see [3, Corollary 2.17]. Hence, we can produce equivariant Schauder projections.

Proposition 5.1.
Let $Y \subset \mathbb{E}$ be a compact G-invariant subset of an isometric Banach representation $\mathbb{E}$ of G. Then for every $\varepsilon > 0$ there exists a finite set $N = \{v_1, v_2, \dots, v_n\}$ and a continuous G-equivariant projection $P_\varepsilon\colon \mathbb{E} \to \operatorname{conv} G(N) \subset \mathbb{E}$ such that $P_\varepsilon(K)$ is in a finite-dimensional subrepresentation of $\mathbb{E}$ and for all $x \in Y$ we have $\|P_\varepsilon(x) - x\| \leqslant \varepsilon$.

Proof. There exists a finite set $N = \{v_1, v_2, \dots, v_n\} \subset \bigoplus_k V_k$ such that $Y \subset \bigcup_{j=1}^n B(v_j, \varepsilon)$. Define functions $\nu_k\colon \mathbb{E} \to \mathbb{R}$ by

$$\nu_k(v) = \max\{0, \varepsilon - \|v - v_k\|\},$$

and define the (non-equivariant) Schauder projection

$$\widetilde{P}_\varepsilon(v) = \frac{1}{\sum_{j=1}^n \nu_j(v)} \sum_{j=1}^n \nu_j(v) v_j \;\in\; \operatorname{conv}\{v_1, \dots, v_n\}.$$

Since ν_k are G-invariant, we obtain that for every $g \in G$,

$$g\widetilde{P}_\varepsilon(g^{-1}v) \in \operatorname{span}\{G(v_1), G(v_2), \dots, G(v_n)\} = A.$$

Thus, averaging our projection,

$$P_\varepsilon(v) = \int_G g\widetilde{P}_\varepsilon(g^{-1}v)\, dg$$

we obtain the desired equivariant one. □

Corollary 5.2.
Let $X \subset \mathbb{E}$ be a G-invariant bounded subset and $\varphi\colon X \multimap \mathbb{E}$ a G-equivariant upper semicontinuous compact closed valued and compact map (i.e., $\overline{\Phi(X)}$ is compact in $\mathbb{E}$). Then for every $\varepsilon > 0$ there exists a G-equivariant upper semicontinuous compact closed valued map $\Phi_\varepsilon\colon X \to \mathbb{E}$ such that $\operatorname{Gr}(\Phi_\varepsilon) \subset B(\operatorname{Gr}(\Phi), \varepsilon)$ and $\Phi_\varepsilon(X)$ is contained in a finite-dimensional subrepresentation of $\mathbb{E}$.

Proof. It is enough to define $\Phi_\varepsilon(x) = (P_\varepsilon \circ \phi)(x)$, where P_ε is the equivariant Schauder projection for $Y = \overline{\Phi(X)}$. In fact, we have $\Phi_\varepsilon(x) \subset B(\Phi(x), \varepsilon)$ for every $x \in X$. □

Now, we are ready to define a Leray–Schauder equivariant degree. Let $\Omega \subset \mathbb{E}$ be an open bounded G-invariant subset and $\varphi\colon \overline{\Omega} \multimap \mathbb{E}$ a *compact equivariant vector field*, i.e., map of the form $\varphi(x) = x - \Phi(x)$, where Φ is compact convex valued, upper semicontinuous G-equivariant and compact.

Assume that φ is *admissible*, i.e., $0 \notin \varphi(x)$ for $x \in \partial\Omega$. Then $\varepsilon_0 = \operatorname{dist}(\overline{\Phi(\partial\Omega)}, \partial\Omega) > 0$ and for $\varepsilon < \varepsilon_0$ we find a G-equivariant Schauder projection P_ε into a finite-dimensional subrepresentation $W \subset \mathbb{E}$ (with $Y = \overline{\varphi(\overline{\Omega})}$). It follows that the vector field $\varphi_\varepsilon(x) = x - P_\varepsilon(\Phi(x))$ is admissible on Ω, as well as its restriction $\varphi_W\colon \overline{\Omega} \cap W \multimap W$. Thus we define

$$\deg_G(\varphi, \Omega) = \deg_G(\varphi_W, \Omega \cap W),$$

where the latter map is defined as above. This definition does not depend on the choice of the projection and a finite-dimensional subrepresentation. It also has properties similar to the finite-dimensional degree. We formulate them with the notation as above. The proofs are standard and we omit the details.

Proposition 5.3.
The Leray–Schauder equivariant degree satisfies all the properties of Theorem 4.1, when restricted to the class of admissible compact vector fields with compact convex values:

(a, **Existence**) *If some coefficient n_H is different from 0 in the degree formula (1), then there exists $x \in \Omega^{(H)}$ such that $0 \in \varphi(x)$, i.e., $(G_x) \geqslant (H)$.*

(b, **Additivity**) *Let $\Omega_1, \Omega_2 \subset \Omega$ be two disjoint, open, G-invariant subsets of Ω and $\varphi^{-1}(0) \cap (\overline{\Omega}) \subset \Omega_1 \cup \Omega_2$. Then*

$$\deg_G(\varphi, \Omega) = \deg_G(\varphi, \Omega_1) + \deg_G(\varphi, \Omega_2).$$

(c, **Homotopy**) *If $\varphi\colon [0,1] \times \overline{\Omega} \multimap \mathbb{E}$ is an admissible G-equivariant homotopy (i.e., φ_t is equviariant and admissible for each $t \in [0,1]$), then*

$$\deg_G(\varphi_0, \Omega) = \deg_G(\varphi_1, \Omega).$$

(d, **Normalization**) *If Ω is a G-invariant, open, bounded neighborhood of 0 in $\mathbb{E}$, then*

$$\deg_G(\mathrm{Id}, \Omega) = 1 \cdot (G).$$

(e, **Suspension**) *If $\varphi\colon \Omega \multimap \mathbb{E}$ is admissible and $B \subset W$ is an open G-invariant bounded neighborhood of 0 in another isometric Banach representation W of G, then*

$$\deg_G(\varphi \times \mathrm{Id}_W, \Omega \times B) = \deg_G(\varphi, \Omega).$$

(f, **Hopf property**) *Let $B(\mathbb{E})$ be the unit ball of the representation $\mathbb{E}$ and for two admissible vector fields $\varphi_1, \varphi_2\colon \overline{B(\mathbb{E})} \multimap V$ one has $\deg_G(\varphi_1, B(\mathbb{E})) = \deg_G(\varphi_2, B(\mathbb{E}))$. Then φ_1 and φ_2 are homotopic by an admissible G-equivariant homotopy.*

6. Second order differential inclusions

In this section, we describe an application of the G-degree to a second-order differential boundary value problem of the form

$$\begin{cases} y'' \in F(t, y, y') & \text{for a.e.} \quad t \in [0,1], \\ y(0) = 0 = y(1), \end{cases} \tag{3}$$

where $F\colon [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a compact convex valued map satisfying (*upper*) *Carathéodory conditions*:

1. the map $t \mapsto F(t, u)$ is Lebesgue measurable for each $u \in \mathbb{R}^{2n}$;
2. the map $u \mapsto F(t, u)$ is upper semicontinuous for each $t \in [0,1]$;
3. for any $r \geqslant 0$ there is a function $\psi_r \in L^2[0,1]$ such that for all $t \in [0,1], u \in \mathbb{R}^{2n}$ with $\|u\| \leqslant r$ and $y \in F(t,u)$ we have $\|y\| \leqslant \psi_r(t)$.

In order to apply a degree theory, we transform the problem (3) into a fixed point problem in some function spaces. We use the following standard notation: $C = C([0,1], \mathbb{R}^{2n}) = \{u\colon [0,1] \to \mathbb{R}^{2n} : u \text{ is continuous}\}$ with the norm $\|u\|_\infty = \sup_{t \in [0,1]} \|u(t)\|$, $L^2 = L^2([0,1], \mathbb{R}^n) = \{u\colon [0,1] \to \mathbb{R}^n : \|u(\cdot)\| \text{ is } L^2\text{-integrable}\}$ with the norm

$$\|u\|_2 = \left(\int_0^1 \|u(t)\|^2 dt \right)^{1/2},$$

$H^2 = H^2([0,1], \mathbb{R}^n) = \{u\colon [0,1] \to \mathbb{R}^n : u \text{ has weak derivatives } u^{(i)} \in L^2 \text{ for } i \leqslant 2\}$ with the norm $\|u\|_{2,2} = \max\{\|u^{(i)}\|_2 : 0 \leqslant i \leqslant 2\}$.

Recall that under the above Carathéodory conditions the associated Nemytskii (superposition) operator $N_F\colon C \multimap L^2$, given by

$$N_F(u) = \left\{ w \in L^2 : w(t) \in F(t, u(t)) \text{ for a.e. } t \in [0,1] \right\},$$

is well defined with nonempty closed convex values and is such that the composed map $(J \circ N_F)(u) = J(N_F(u))$ is completely continuous for any completely continuous linear operator $J\colon L^2 \to C$, see e.g. [14, Proposition 1.7] or [15, Proposition 3.6]. In particular, we know by the Ascoli theorem that $j\colon H^2([0,1],\mathbb{R}^n) \to C([0,1],\mathbb{R}^{2n})$, $j(u) = (u,u')$ is a completely continuous linear operator (it sends bounded sets into relatively compact ones). Denote by H^2_0 the subspace $H^2_0 = \{y \in H^2 : y(0) = y(1) = 0\}$. Then the linear operator $L\colon H^2_0 \to L^2$, $L(y) = y''$ is invertible. Define $\Phi\colon C \multimap C$ as a composition $j \circ L^{-1} \circ N_F$, which is upper semicontinuous with compact convex values and is compact on bounded sets (i.e, completely continuous). One easily sees, cf. [7], that the problem of finding solutions to (3) in H^2 is equivalent to the fixed point problem

$$w \in \Phi(w), \qquad W = (u,v) \in C,$$

or in other words $0 \in \varphi(w)$, where $\varphi(w) = w - \Phi(w)$. Indeed, if $w \in \Phi(w) = j \circ L^{-1} \circ N_F(w)$, then w being in the image of j is of the form (y,y'), where $y \in H^2_0$ and $L(y) \in N_F(y,y')$. This means exactly that y is a solution to the problem (3). One needs some *a priori* bounds in order to apply a degree theory. The following conditions proposed in [7] are related to Bernstein–Nagumo growth conditions and reduce to them in the single-valued case, cf. [7] and also [9].

(H1) There exists a constant $R > 0$ such that if $\|y_0\| > R$ and $y_0 \cdot y_0' = 0$ then there is $\delta > 0$ such that

$$\operatorname*{ess\,inf}_{t\in[0,1]} \Big(\inf\, \{ y \cdot w + \|y'\|^2 : w \in F(t,y,y'),\ (y,y') \in D_\delta \} \Big) > 0,$$

where $D_\delta = \{(y,y') \in \mathbb{R}^{2n} : \|y - y_0\| + \|y' - y_0'\| < \delta\}$.

(H2) There is a function $\beta\colon [0,\infty) \to [0,\infty)$ such that $s/\beta(s) \in L^\infty_{\mathrm{loc}}[0,\infty)$, $\int_0^\infty s/\beta(s)\,ds = \infty$, and $\|F(t,y,y')\| \leqslant \beta(\|y'\|)$ for a.e. $t \in [0,1]$ and all $(y,y') \in \mathbb{R}^{2n}$ with $\|y\| \leqslant R$ (R is the same as in (H1)).

(H3) There exist constants $k, \alpha > 0$ such that

$$\|F(t,y,y')\| \leqslant \alpha(y \cdot w + \|y'\|^2) + k$$

for a.e. $t \in [0,1]$, all $(y,y') \in \mathbb{R}^{2n}$ with $\|y\| \leqslant R$ and $w \in F(t,y,y')$. Here $\|F(t,y,y')\| = \sup\{\|w\| : w \in F(t,y,y')\}$.

Together with the problem (3) one can consider a family of BVP

$$\begin{cases} y'' \in \lambda F(t,y,y') & \text{for a.e.} \quad t \in [0,1], \\ y(0) = 0 = y(1), \end{cases} \tag{4}$$

where $\lambda \in [0,1]$.

Using the above assumptions, the idea of proof of main results in [7] may be read as follows.

Proposition 6.1.

Let F satisfy (H1)–(H3). Then there exists sufficiently large $R > 0$ such that the map $\varphi_\lambda\colon C \multimap C$, $\varphi_\lambda(w) = w - j \circ L^{-1} \circ N_{\lambda F}(w)$ is admissible with $\Omega = B(0,R)$ for each $\lambda \in [0,1]$. Therefore the Leray–Schauder degree $\deg(\varphi_1, B(0,R)) = 1$, and thus the problem (3) has a solution in H^2.

Remark 6.2.

In fact, in [7] much more general boundary value problems were treated and the method used there was topological transversality instead of degree theory.

Let us consider the space $V = \mathbb{R}^4$ as a linear representation of the dihedral group $D_4 = \{1, i, -1, i, \kappa, \kappa i, -\kappa, -\kappa i\}$, which permutes the coordinates, i.e., on generators $i(y_1, y_2, y_3, y_4) = (y_2, y_3, y_4, y_1)$ and $\kappa(y_1, y_2, y_3, y_4) = (y_3, y_2, y_1, y_4)$. It gives a natural action on function spaces $C^k([0, 1], V)$, $H^2([0, 1], V)$, etc. by $((gu)(t) = gu(t)$ for $t \in [0, 1])$.

Given a positive constant $a > 0$ we define a symmetric matrix

$$C = \begin{bmatrix} -2a & a & 0 & a \\ a & -2a & a & 0 \\ 0 & a & -2a & a \\ a & 0 & a & -2a \end{bmatrix}$$

with the eigenvalues $\lambda_0 = 0$, $\lambda_1 = -2a$, $\lambda_3 = -4a$. Their eigenspaces are $\mathcal{V}_0, V_1, \mathcal{V}_3$, respectively, where the notation comes from Example 4.2. This implies that $H_0^2 = H^2([0, 1], \mathcal{V}_0) \oplus H^2([0, 1], \mathcal{V}_1) \oplus H^2([0, 1], \mathcal{V}_3)$. The linear BVP

$$y'' = Cy, \qquad y(0) = 0 = y(1), \tag{5}$$

can be translated as above to an operator form in a function space. The resulting linear operator $\mathcal{A}\colon H_0^2 \to H_0^2$ is given by the formula $\mathcal{A}(y) = y - L^{-1}(Cy)$. The operator $\mathcal{A}$ is D_4-equivariant and its spectrum is of the form

$$\sigma(\mathcal{A}) = \left\{1, 1 - \frac{2a}{\pi^2 k^2}, 1 - \frac{4a}{\pi^2 k^2} \ : \ k \in \mathbb{N}\right\}.$$

Let us make additional simplifying assumptions on the constant a in the language of $\sigma(\mathcal{A})$:

(A1) $\quad 0 \notin \sigma(\mathcal{A})$;

(A2) $\quad \sigma_-(\mathcal{A}) = \{\lambda \in \sigma(\mathcal{A}) : \lambda < 0\} = \{1 - 2a/\pi^2, 1 - 4a/\pi^2\}$.

Remark 6.3.

If (A1)–(A2) are satisfied, then $\deg_{D_4}(\mathcal{A}, B(0, r)) = (D_4) - (D_2) + (D_1) - (\widetilde{D}_1)$, as it is calculated in [2, p. 34]. In our case, we consider $\widetilde{\mathcal{A}}\colon C([0, 1], \mathbb{R}^8) \to C([0, 1], \mathbb{R}^8)$, $\widetilde{\mathcal{A}}(u, v) = (u, v) - (j \circ L^{-1})(Cu)$. It is also D_4-equivariant, if we define the action $g(u, v) = (gu, gv)$ in $C([0, 1], \mathbb{R}^8)$. Note that $\sigma(\widetilde{\mathcal{A}}) = \sigma(\mathcal{A})$ and, since $\widetilde{\mathcal{A}}(0, v) = (0, v)$, the degree is the same as for $\mathcal{A}$.

Now, we consider a symmetric set-valued perturbation of (5)

$$\begin{cases} y'' \in Cy + F(t, y, y') & \text{for a.e.} \quad t \in [0, 1], \\ y(0) = 0 = y(1), \end{cases} \tag{6}$$

where $F\colon [0, 1] \times V \times V \multimap V$ is a compact convex valued map satisfying (upper) Carathéodory conditions, which is also D_4-equivariant ($V = \mathbb{R}^4$ with the action described above).

Theorem 6.4.

Let $\widetilde{F}(t, y, y') = Cy + F(t, y, y')$ satisfy conditions (H1)–(H3) and also (A1)–(A2). Assume further that

$$\lim_{(y,y') \to 0} \frac{\|F(t, y, y')\|}{\|(y, y')\|^2} = 0. \tag{7}$$

Then the problem (6) has at least one nontrivial solution in addition to the trivial one $y = 0$. More precisely, we have the following alternative:

- *either there exists a nonzero solution with D_4-symmetry, i.e., $y \in (H^2)^{D_4} = H^2([0,1],\mathcal{V}_0)$ (in other words $y_1 = y_2 = y_3 = y_4$);*
- *or there are at least two different nonzero solutions: one with D_2-symmetry, and the other one with $\widetilde{D}_1$-symmetry.*

Proof. Rewrite (6) as an operator inclusion

$$0 \in \varphi(y), \tag{8}$$

where $\varphi\colon C([0,1],\mathbb{R}^8) \multimap C([0,1],\mathbb{R}^8)$, given by $\varphi(w) = w - j \circ L^{-1} \circ N_{\widetilde{F}}(w)$, is a completely continuous vector field with compact convex values, which is D_4-equivariant. Then conditions (H1)–(H3) assure that there exists sufficiently large $R > 0$ such that the family of operators $\varphi_\lambda(w) = w - j \circ L^{-1} \circ N_{\lambda\widetilde{F}}(w)$ for $\lambda \in [0,1]$ define a homotopy which is admissible on $B(0,R) \subset C([0,1],\mathbb{R}^8)$. It is obviously D_4-equivariant. We will not repeat the proof of *a priori* bounds from [7], which give the admissibility. Therefore, we have $\deg_{D_4}(\varphi, B(0,R)) = (D_4)$.

On the other hand, for sufficiently small $r > 0$ the formula $\varphi_\mu(w) = w - j \circ L^{-1} \circ N_{C+\lambda F}(w)$ gives a D_4-equivariant homotopy, which is admissible on $B(0,r)$. This is assured by the condition (7), as one easily verifies. Therefore,

$$\deg_{D_4}(\varphi, B(0,r)) = \deg_{D_4}(\widetilde{\mathcal{A}}, B(0,r)) = (D_4) - (D_2) + (D_1) - (\widetilde{D}_1),$$

because of our assumptions (A1)–(A2). Now, using the additivity property of the equivariant degree, Proposition 5.3, we obtain

$$\deg_{D_4}\big((\varphi, B(0,R)) \setminus \overline{B(0,r)}\big) = (D_2) - (D_1) + (\widetilde{D}_1).$$

Notice that we have the following hierarchy of the orbit types in our representation:

$$(\mathbb{Z}_1) \leqslant (D_1) \leqslant (D_2) \leqslant (D_4), \qquad (\mathbb{Z}_1) \leqslant (\widetilde{D}_1) \leqslant (D_4).$$

Since the coefficients n_{D_2} and $n_{\widetilde{D}_1}$ are nonzero, there exists $w \neq 0$ such that $0 \in \varphi(w)$ and $(G_w) \geqslant (D_2)$. This gives $y \in (H_0^2)^{D_2}$ such that $L(y) \in \widetilde{F}(y,y')$, i.e., it is a solution to (6). Similarly, we find a solution $z \in (H_0^2)^{\widetilde{D}_1}$. Since both orbit types are submaximal, one possibility is that $y, z \in (H_0^2)^{D_4}$ and they may be the same functions. If there are no solutions in the fixed point subspace $(H_0^2)^{D_4}$, then y, z are forced to be different and they have different symmetries:

$$y_1 = y_3 \neq y_2 = y_4, \qquad z_1 = z_4 \neq z_2 = z_3.$$

Note that in the latter case we obtain in fact two orbits of solutions (two and four solutions, respectively). □

Our application was strongly influenced by the following single-valued example from [2]:

Example 6.5.

$$\begin{cases} y_1'' = -2ay_1 + ay_2 + ay_4 + y_1' e^{y_1 y_2 y_3 y_4} + y_1^3 + y_1 y_2^2 y_4^2, \\ y_2'' = -2ay_2 + ay_1 + ay_3 + y_2' e^{y_1 y_2 y_3 y_4} + y_2^3 + y_2 y_1^2 y_3^2, \\ y_3'' = -2ay_3 + ay_2 + ay_4 + y_3' e^{y_1 y_2 y_3 y_4} + y_2^3 + y_3 y_2^2 y_4^2, \\ y_4'' = -2ay_4 + ay_1 + ay_3 + y_4' e^{y_1 y_2 y_3 y_4} + y_4^3 + y_4 y_1^2 y_3^2, \\ y_k(0) = 0 = y_k(1), \qquad k = 1,2,3,4. \end{cases}$$

When you consider a bounded upper-semicontinuous D_4-equivariant convex-valued perturbation of the system in Example 6.5, which vanishes in a neighborhood of 0 and in the fixed point subspace $\{(y_1,y_1,y_1,y_1) \in \mathbb{R}^4\}$, then the assumptions of Theorem 6.4 are fulfilled.

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