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Acoustic heating produced in resonators filled by a newtonian fluid

Anna Perelomova
Gdansk University of Technology,
Faculty of Applied Physics and Mathematics,
ul. Narutowicza 11/12, 80-233 Gdansk, Poland,
anpe@mif.pg.gda.pl

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Abstract

Acoustic heating in resonators is studied. The governing equation of acoustic heating is derived by means of the special linear combination of conservation equations in differential form, allowing to reduce all acoustic terms in the linear part of the final equation, but preserving terms belonging to the thermal mode responsible for heating. This equation is instantaneous and includes nonlinear acoustic terms which form a source of acoustic heating, it is valid for weakly nonlinear flows with weak attenuation. In general, dynamics of sound in a resonator is described by coupling nonlinear equations. Though the equation for heating relates to any sound field, which may exist in a resonator, to establish sound field is a problem itself. It is well-known, that employment of method of different scales and averaging over the sound period makes it possible to consider sound waves of opposite directions separately, without account for their interaction in a volume of resonator, if they are periodic with zero mean perturbations. It allows also to add together contributions of oppositely propagating waves in production of heating. Some examples of acoustic heating in resonators, relating to periodic sound branches with zero mean perturbations, are discussed.

Keywords: Acoustic heating, Acoustic resonators. PACS no 43.35.Fj, 43.25.Gf

1 Introduction

Studies of standing waves of the finite amplitude in acoustic resonators is one of the most promising directions in the nonlinear acoustics in view of many technical and physical applications. They are hindered by the lack of the mathematical method like this which is successful in solution of the Burger's equation for the traveling waves. As usual, acoustic dispersion in resonators is weak, and all harmonic components do interact. The first investigations revealed that the sound field in a closed resonator, under some conditions, may be thought as a sum of two traveling in the opposite direction waves. That is valid with accuracy up to terms proportional to M^2 inclusively, where M is the Mach number, if perturbations in both waves are periodic in time and possess zero mean values [1, 2, 3]. The method basing on the different-scale analysis to describe sound field in resonators, was firstly applied by Kaner, Rudenko and Khokhlov in Ref. [4]. It employs variables of different scales by introducing slow and fast dependence of perturbations in a sound wave. That becomes possible if nonlinearity and attenuation are weak. Efficiency of this method in derivation of equations of nonlinear acoustics in absorbing, diffracting and dispersive media is well-established [5, 6]. Relatively to the sound field in a resonator, this method, complemented by averaging, has confirmed the important conclusion, that interaction of traveling waves of opposite directions is negligible only in the case of almost periodic high-frequency waves. In this case, waves do not interact in a volume of resonator, and they are coupled only by conditions on the walls of resonator.

In the last decades, studies in sound waves in resonators have covered arrays of Helmholtz resonators, resonators with varying cross-sections, media with hysteresis and so on [7, 8, 9, 10, 11]. Although the nonlinear distortions of sound itself are well-established, the nonlinear effects by sound, such as generation of non-wave modes in its field, remain poorly studied. It is well-known, that standard attenuation in fluids leads to a linear dissipation of sound. The acoustic heating is an increase of the ambient fluid temperature caused by *nonlinear* losses in acoustic energy. The relative increase in temperature is not an acoustic quantity but a value referred to as the entropy, or thermal mode. This excess temperature should be distinguished from an excess temperature associated with a sound wave, the latter of which is a wave quantity. In order to differ perturbations belonging to sound and non-acoustic modes, and to specify the dynamic equations for every mode, some kind of projection is required. As for unbounded volumes of a gas, Rudenko, Soluyan [5] followed Makarov, Ochmann [12] used averaging over the sound period. This allows to eliminate sound in linear parts of equations relating to entropy or vorticity modes, but does not permit to consider sound different from periodic and instantaneous (but only averaged) quantities associated with non-acoustic modes. The role of periodic sound as an origin of acoustic heating in unbounded volumes of standard thermoviscous fluids is well-studied theoretically and experimentally [5, 12, 13]. The technique which makes possible to subdivide instantaneous equations of different modes, has been worked out and applied previously by the author in some problems of nonlinear acoustics. It allows to derive weakly nonlinear equations governing both periodic and aperiodic sound, vorticity and entropy modes [14, 15, 16]. This method, complemented by method of different scales and by averaging, and results based on its application relatively to field in a resonator, are described in Sections 4,5,6. This study considers acoustic heating in one-dimensional flat resonators.

2 Dynamic equations of fluid flow

The continuity, momentum and energy equations for a thermoviscous fluid flow without external forces read [17]:

$$\begin{aligned}
&\frac{\partial\rho}{\partial t}+\overrightarrow{\nabla}\cdot(\rho\overrightarrow{v})=0\\
&\frac{\partial\overrightarrow{v}}{\partial t}+(\overrightarrow{v}\cdot\overrightarrow{\nabla})\overrightarrow{v}=\frac{1}{\rho}\left(-\overrightarrow{\nabla}p+Div\,\mathbf{P}\right),\\
&\frac{\partial e}{\partial t}+(\overrightarrow{v}\cdot\overrightarrow{\nabla})e=\frac{1}{\rho}\left(-p(\overrightarrow{\nabla}\cdot\overrightarrow{v})+\mathbf{P}:Grad\,\overrightarrow{v}\right),
\end{aligned}\qquad(1)$$

$\vec{v}$ denotes the velocity of the fluid, ρ, p are density and pressure, e marks the energy per unit mass, and x_i, t are spatial coordinates and time. The operators Div and $Grad$ denote the tensor divergence and dyad gradient, respectively. $\mathbf{P}$ is the tensor of viscous stress,

$$\mathbf{P}_{ik} = \mu\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) - \frac{2}{3}\mu(\vec{\nabla}\cdot\vec{v})\delta_{ik}, \tag{2}$$

where μ is the shear viscosity, and δ_{ik} is the Kronecker symbol, δ_{ik} equals zero if $i \neq k$ and unity otherwise. The thermodynamic function $e(p,\rho)$ complements the system (1). An excess internal energy $e' = e - e_0$ may be represented as series in powers of excess pressure and density $p' = p - p_0$, $\rho' = \rho - \rho_0$ (ambient quantities are marked by the index 0):

$$e' = \frac{E_1}{\rho_0}p' + \frac{E_2 p_0}{\rho_0^2}\rho' + \frac{E_3}{p_0\rho_0}p'^2 + \frac{E_4 p_0}{\rho_0^3}\rho'^2 + \frac{E_5}{\rho_0^2}\rho' p' + \ldots \tag{3}$$

where $E_1, \ldots E_5, \ldots$ are dimensionless coefficients. The common practice in nonlinear acoustics is to focus on the equations of the second order in the acoustic Mach number $M = v_0/c_0$, where v_0 is the magnitude of a fluid velocity, and $c_0 = \sqrt{\frac{(1-E_2)p_0}{E_1\rho_0}}$ is an infinitely-small signal velocity. This study is further constrained by considering nonlinearities of the second order, so that in the series (3) only terms up to the second order inclusively are considered. The series (3) allows to consider a wide variety of fluids in the general form. A difference in the thermodynamic properties of fluids is manifested namely by the coefficients different for different fluids. The expressions for coefficients E_1 and E_2 are as follows:

$$E_1 = \frac{\rho_0 C_V \kappa}{\beta}, E_2 = -\frac{C_p\rho_0}{\beta p_0} + 1, \tag{4}$$

where C_p denotes the heat capacity per unit mass under constant pressure, C_V marks the heat capacity per unit mass under constant volume $V = 1/\rho$, κ and β are the compressibility and thermal expansion, correspondingly:

$$\kappa = -\frac{1}{V_0}\left(\frac{\partial V}{\partial p}\right)_T = \frac{1}{\rho_0}\left(\frac{\partial \rho}{\partial p}\right)_T, \beta = \frac{1}{V_0}\left(\frac{\partial V}{\partial T}\right)_p = -\frac{1}{\rho_0}\left(\frac{\partial \rho}{\partial T}\right)_p. \tag{5}$$

3 Definition of modes in the planar flow of infinitely small amplitude

We consider the one-dimensional flow along axis Ox. Basing on the linearized version of Eq.(1), the roots of the dispersion equation may be readily derived. They determine three independent modes of infinitely small-signal disturbances in an unbounded fluid. In one dimension, there exist acoustic (two branches), and thermal (or entropy) modes. In general, every perturbation of the field variables contains contributions from each of the three modes, for example, $\rho' = \rho'_{a,1} + \rho'_{a,2} + \rho'_e$. The method developed in [14, 15] provides the possibility of consistent decoupling of the initial system (1) into specific dynamic equations for every mode by use of individual properties of each mode in a weakly nonlinear flow. Its scope is far beyond Newtonian fluids, involving flows over non-Newtonian ones, with complex dependence of shear stress on shear rate, and even flows over anomalous acoustic media where irreversible thermodynamic processes may take place [16, 18, 19]. All formulae that follow, including links of modes and governing equations, are written in the leading order. It is convenient to rearrange formulae into the dimensionless quantities in the following way:

$$p^* = \frac{p'}{c_0^2 \cdot \rho_0}, \rho^* = \frac{\rho'}{\rho_0}, v^* = \frac{v}{c_0}, x^* = \frac{\omega x}{c_0}, t^* = \omega t, \tag{6}$$

where ω is the acoustic frequency. Starting from Eq.(7), the upper indexes (asterisks) denoting dimensionless quantities will be omitted throughout the text. In the dimensionless quantities, accounting for Eqs(2, 3), Eqs(1) take the form:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial v}{\partial x} &= -v\frac{\partial \rho}{\partial x} - \rho\frac{\partial v}{\partial x}\,, \\ \frac{\partial v}{\partial t} + \frac{\partial p}{\partial x} - b\frac{\partial^2 v}{\partial x^2} &= -v\frac{\partial v}{\partial x} + \rho\frac{\partial p}{\partial x} - b\rho\frac{\partial^2 v}{\partial x^2}\,, \\ \frac{\partial p}{\partial t} + \frac{\partial v}{\partial x} &= -v\frac{\partial p}{\partial x} + (D_1 p + D_2 \rho)\frac{\partial v}{\partial x} + \frac{b}{E_1}\left(\frac{\partial v}{\partial x}\right)^2. \end{aligned} \tag{7}$$

where $b = 4\mu\omega/(3\rho_0 c_0^2)$ is the dimensionless attenuation. The terms of order M^2 form the right-hand side of the set (7). The dynamic equations in the rearranged form include the following dimensionless quantities:

$$D_1 = \frac{1}{E_1}\left(-1 + \frac{2(1-E_2)E_3}{E_1} + E_5\right),\ D_2 = \frac{1}{1-E_2}\left(1 + E_2 + 2E_4 + \frac{1-E_2}{E_1}E_5\right). \tag{8}$$

We shall consider weakly nonlinear flows discarding $O(M^3)$ terms in all expansions and confining terms to be considered to those which include b^0 and b^1. The resulting model accounts for the combined effects of nonlinearity and attenuation of one-dimensional sound and thermal modes. The linearized version of Eq.(7) describes a flow of infinitely small amplitude, when $M \to 0$:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial v}{\partial x} &= 0, \\ \frac{\partial v}{\partial t} + \frac{\partial p}{\partial x} - b\frac{\partial^2 v}{\partial x^2} &= 0, \\ \frac{\partial p}{\partial t} + \frac{\partial v}{\partial x} &= 0. \end{aligned} \tag{9}$$

The linear hydrodynamic field is represented by acoustic modes, propagating in the positive and negative directions of axis Ox, and the entropy mode. Every type of motion is determined by one of the roots of the dispersion relation of the linear flow, $\omega(k)$ (k is the wave number) [5, 12, 20] and fixes links of perturbations, which are independent on time [14, 15]. The dispersion relations for acoustic modes propagating in the positive direction of axis Ox (marked by index 1), the negative direction of axis Ox (marked by index 2), and the entropy modes (marked by index 3), are as follows [5]:

$$\omega_{a,1} = k + ib\frac{k^2}{2}, \quad \omega_{a,2} = -k + ib\frac{k^2}{2}, \quad \omega_e = 0. \tag{10}$$

They uniquely determine links of excess density inside every mode, which are valid at any time:

$$\psi_{a,1} = \begin{pmatrix} \rho_{a,1} \\ v_{a,1} \\ p_{a,1} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 - \frac{b}{2}\frac{\partial}{\partial x} \\ 1 \end{pmatrix} \rho_{a,1}, \quad \psi_{a,2} = \begin{pmatrix} 1 \\ -1 - \frac{b}{2}\frac{\partial}{\partial x} \\ 1 \end{pmatrix} \rho_{a,2},$$

$$\psi_e = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rho_e. \qquad (11)$$

The equations for every type of motion may be extracted from the system (7) in accordance with specific links inside every mode. This may be formally accomplished by means of projecting of the equations into specific sub-spaces [14, 15, 16]. Every equation includes the first-order derivative with respect to time. The linear dynamic equations are obviously independent. The equations describing an acoustic excess density in waves, propagating in the positive and negative directions of axis Ox, and the entropy mode excess density, take the forms:

$$\frac{\partial \rho_{a,1}}{\partial t} + \frac{\partial \rho_{a,1}}{\partial x} - \frac{b}{2}\frac{\partial^2 \rho_{a,1}}{\partial x^2} = 0, \quad \frac{\partial \rho_{a,2}}{\partial t} - \frac{\partial \rho_{a,2}}{\partial x} - \frac{b}{2}\frac{\partial^2 \rho_{a,2}}{\partial x^2} = 0, \quad \frac{\partial \rho_e}{\partial t} = 0. \qquad (12)$$

4 Weakly nonlinear dynamic equation for sound

Eqs (12) follow immediately from the dispersion relations (10) but may be obtained using links (11) and by means of combination of linear equations (9) in the following manner: multiplying the momentum (second) equation by 0.5, applying by $0.5 + 0.25b\partial/\partial x$ on the third one and taking their sum yields in the dynamic equation for $\rho_{a,1}$ reducing terms of two other modes in the final equality. Multiplying the momentum equation by -0.5, applying by $0.5 - 0.25b\partial/\partial x$ on the third one and taking their sum, one arrives to the dynamic equation for $\rho_{a,2}$. Finally, difference of the continuity (first) and the third equation from Eqs(9) yields the dynamic equation for the entropy excess density.

Projecting is actually a linear combination of equations in such a way as to keep the terms of the chosen mode in the linear part, and to reduce all other terms there. To describe sound itself, it is convenient to consider acoustic field in the variables reflecting slow change of the shape of progressive modes due to nonlinearity and attenuation, and its fast dependence on the retarded time (so-called method of different scales [4, 5, 6]). They are "fast" variables $\eta = t - x$ and $\xi = t + x$, correspondent to the progressive in positive and negative direction of axis OX waves, and the "slow" one, νt, where $\nu = Max(M, b)$. In the linear non-viscous flow, acoustic perturbations are functions exclusively of the retarded variables, $\rho_{a,1}(\eta)$, $\rho_{a,2}(\xi)$. We fix relations (11) in a weakly nonlinear flow and consider every variable as a sum of specific acoustic quantities, $\rho = \rho_{a,1} + \rho_{a,2}$ and so on. Hence, in this section, we consider excess density of the entropy mode much smaller than magnitudes of excess densities in both acoustic branches. It means physically, that we consider initially dominative sound, and the entropy motion is supposed to develop only slowly with time. That agrees with the idea of acoustic heating in the field of comparatively intense sound [5, 12]. We will see, that ρ_e is of order not higher that squared Mach number in the problem of acoustic heating, while magnitudes of $\rho_{a,1}$ and $\rho_{a,2}$ are of order M.

For simplicity, we will consider an ideal gas with $E_1 = -E_2 = E_4 = -E_5 = 1/(\gamma - 1)$, $E_3 = 0$, $D_1 = -\gamma$, $D_2 = 0$, where $\gamma = C_p/C_v$. Combining equations in the way pointed out in the beginning of this section, and collecting the leading-order terms, one obtains dynamic equations,

$$\frac{\partial \rho_{a,1}}{\partial t} - \frac{b}{2}\frac{\partial^2 \rho_{a,1}}{\partial \eta^2} = \frac{\gamma+1}{2}\frac{\partial \rho_{a,1}}{\partial \eta}\rho_{a,1} + \frac{\gamma+1}{2}\rho_{a,1}\frac{\partial \rho_{a,2}}{\partial \xi} - \frac{3-\gamma}{2}\frac{\partial \rho_{a,1}}{\partial \eta}\rho_{a,2} + \frac{\gamma+1}{2}\rho_{a,2}\frac{\partial \rho_{a,2}}{\partial \xi}, \qquad (13)$$

$$\frac{\partial \rho_{a,2}}{\partial t}-\frac{b}{2}\frac{\partial^2 \rho_{a,2}}{\partial \xi^2}=\frac{\gamma+1}{2}\frac{\partial \rho_{a,1}}{\partial \eta}\rho_{a,1}+\frac{\gamma+1}{2}\frac{\partial \rho_{a,1}}{\partial \eta}\rho_{a,2}-\frac{3-\gamma}{2}\rho_{a,1}\frac{\partial \rho_{a,2}}{\partial \xi}+\frac{\gamma+1}{2}\rho_{a,2}\frac{\partial \rho_{a,2}}{\partial \xi}.$$

Quantities $\frac{\partial \rho_{a,1}}{\partial \xi}$ and $\frac{\partial \rho_{a,2}}{\partial \eta}$ are of the second order in smallness. Eqs (13) are valid for periodic and aperiodic sound of opposite directions in a resonator. Both specific excess densities may be established with account for initial and boundary conditions, as the solutions of Eqs(13). Under certain conditions, the problem may be considerably simplified, as it was discovered by Kaner, Rudenko, Khokhlov [4]. Following them, we assume that $\rho_{a,1}$ and $\rho_{a,2}$ are quickly varying periodic functions of η and ξ, respectively, and their averaged over periods values are zero, $\overline{\rho_{a,1}}=0$, $\overline{\rho_{a,2}}=0$. Averaging the first equation over period in ξ, and the second one over period in η, one obtains readily the well-known Burgers equations for non-interacting acoustic modes propagating in different directions,

$$\frac{\partial \rho_{a,1}}{\partial t}-\frac{b}{2}\frac{\partial^2 \rho_{a,1}}{\partial \eta^2}-\frac{\gamma+1}{2}\rho_{a,1}\frac{\partial \rho_{a,1}}{\partial \eta}=0,\ \frac{\partial \rho_{a,2}}{\partial t}-\frac{b}{2}\frac{\partial^2 \rho_{a,2}}{\partial \xi^2}-\frac{\gamma+1}{2}\rho_{a,2}\frac{\partial \rho_{a,2}}{\partial \xi}=0. \qquad (14)$$

So that, we can consider the sound field as a simple superposition of waves propagating in opposite directions, which do not interact in a volume of resonator. They are coupled only by the conditions on the walls of resonator.

5 Acoustic heating

After establishing of the leading-order equations for both acoustic modes, let us consider the complete field of perturbations, consisting of all inputs, $\rho=\rho_{a,1}+\rho_{a,2}+\rho_e,\ \ldots$. Difference of the first and last equations from (7) is actually an equation for the entropy mode excess density,

$$\frac{\partial \rho_e}{\partial t}=-(\gamma-1)\left(\rho_{a,1}\frac{\partial \rho_{a,1}}{\partial \eta}+\rho_{a,2}\frac{\partial \rho_{a,2}}{\partial \xi}+\rho_{a,1}\frac{\partial \rho_{a,2}}{\partial \xi}+\rho_{a,2}\frac{\partial \rho_{a,1}}{\partial \eta}+\frac{b}{2}\rho_{a,1}\frac{\partial^2 \rho_{a,1}}{\partial \eta^2}+\right. \qquad (15)$$

$$\left.\frac{b}{2}\rho_{a,2}\frac{\partial^2 \rho_{a,2}}{\partial \xi^2}+b\left(\frac{\partial \rho_{a,1}}{\partial \eta}\right)^2+b\left(\frac{\partial \rho_{a,2}}{\partial \xi}\right)^2+2b\frac{\partial \rho_{a,1}}{\partial \eta}\frac{\partial \rho_{a,2}}{\partial \xi}+\frac{b}{2}\rho_{a,1}\frac{\partial^2 \rho_{a,2}}{\partial \xi^2}+\frac{b}{2}\rho_{a,2}\frac{\partial^2 \rho_{a,1}}{\partial \eta^2}\right).$$

After averaging Eq.(15) over periods in η and ξ, it may be reduced as

$$\frac{\partial \rho_e}{\partial t}=-b(\gamma-1)\left(\frac{1}{2}\overline{\rho_{a,1}\frac{\partial^2 \rho_{a,1}}{\partial \eta^2}}+\frac{1}{2}\overline{\rho_{a,2}\frac{\partial^2 \rho_{a,2}}{\partial \xi^2}}+\overline{\left(\frac{\partial \rho_{a,1}}{\partial \eta}\right)^2}+\overline{\left(\frac{\partial \rho_{a,2}}{\partial \xi}\right)^2}\right)= \qquad (16)$$

$$-\frac{b}{2}(\gamma-1)\left(\overline{\left(\frac{\partial \rho_{a,1}}{\partial \eta}\right)^2}+\overline{\left(\frac{\partial \rho_{a,2}}{\partial \xi}\right)^2}\right).$$

So that, dependence of ρ_e on time is considerably weaker than that for the sound, $\rho_e(bM^2t)$. That reflects the fact that transfer of acoustic energy into energy of the entropy mode is not only nonlinear, but viscous phenomenon. Both attenuation and nonlinearity are necessary conditions for heating to develop.

5.1 Nonlinear waves before formation of discontinuity

In a nonlinear resonator, if attenuation is ignored, the velocity field which satisfies the condition $v(x, t = 0) = 2M \sin(x)$ ($\rho(x, t = 0) = 0$) at $t = 0$, is a sum of specific parts

$$\rho_a = \rho_{a,1} + \rho_{a,2} = 2M \sum_{n=1}^{\infty} \frac{J_n(n\sigma)(\sin n(t-x) + \sin n(t+x))}{n\sigma} = 4M \sum_{n=1}^{\infty} \frac{J_n(n\sigma)\cos(nx)\sin(nt)}{n\sigma}, \tag{17}$$

where $\sigma = \frac{M(\gamma+1)t}{2}$. That is a sum of the well-known Bessel-Fubini solutions of the nonlinear Burgers equations (14) valid before forming of discontinuities [5]. Obviously, the velocity at the ends of resonator is zero at any time, $v_{a,1} + v_{a,2}|_{x=0} = v_{a,1} + v_{a,2}|_{x=L} = \rho_{a,1} - \rho_{a,2}|_{x=0} = \rho_{a,1} - \rho_{a,2}|_{x=L} \equiv 0$, so that its dimensional length L is π-fold. In accordance to (16), an increase in the entropy temperature is

$$Q = -\frac{\partial \rho_e}{\partial t} = \frac{1}{T_0}\frac{\partial T_e}{\partial t} = \frac{M^2(\gamma-1)b}{2} \sum_{n=1}^{\infty} \left(\frac{2J_n(n\sigma)}{n\sigma}\right)^2 = \frac{M^2(\gamma-1)b}{2}, \tag{18}$$

where T_0 denotes unperturbed temperature of a gas, and T_e an excess temperature in the isobaric heating. It is always positive and does not depend on time. In this example, the time required for forming of the saw-like wave equals $2/M(\gamma+1)$, so that the evaluations are not longer valid in the later stages.

5.2 Sawtooth waves

For large Reynolds numbers $b << 2\pi M$, the sawtooth wave forms after some time. Sawtooth solutions of Eqs(14), which satisfy the initial conditions as in the previous subsection, are

$$\rho_{a,1} = \frac{M\eta}{1+\sigma}, \quad -\pi < \eta \le \pi, \quad \rho_{a,2} = \frac{M\xi}{1+\sigma}, \quad -\pi < \xi \le \pi. \tag{19}$$

The shock wave may be considered as the limit to which a solution of the Burgers equation tends if b tends to zero. The solutions are periodic with period 2π, they are valid at $\sigma \ge \pi/2$ [5]. The averaged squared partial derivatives in the right-hand side of Eq.(16) may be evaluated assuming the periodicity of solutions, by average of Eqs.(14), multiplied by $\rho_{a,1}$ or $\rho_{a,2}$, respectively. It yields

$$-\frac{b}{2}\overline{\rho_{a,1}\frac{\partial^2 \rho_{a,1}}{\partial \eta^2}} = \frac{b}{2}\overline{\left(\frac{\partial \rho_{a,1}}{\partial \eta}\right)^2} = \frac{\gamma+1}{2}\overline{\rho_{a,1}^2\frac{\partial \rho_{a,1}}{\partial \eta}} = \frac{(\gamma+1)M}{2(1+\sigma)}\overline{\rho_{a,1}^2} = \frac{(\gamma+1)M^3\pi^3}{3(1+\sigma)^3}, \tag{20}$$

$$\frac{b}{2}\overline{\left(\frac{\partial \rho_{a,2}}{\partial \xi^2}\right)^2} = \frac{(\gamma+1)M^3\pi^3}{3(1+\sigma)^3}.$$

Hence, the acoustic heating is described by equation

$$Q = -\frac{\partial \rho_e}{\partial t} = \frac{2(\gamma+1)(\gamma-1)M^3\pi^3}{3(1+\sigma)^3}. \tag{21}$$

The released heat does not depend on b, its origin is pure nonlinear attenuation on the fronts of the sawtooth waves.

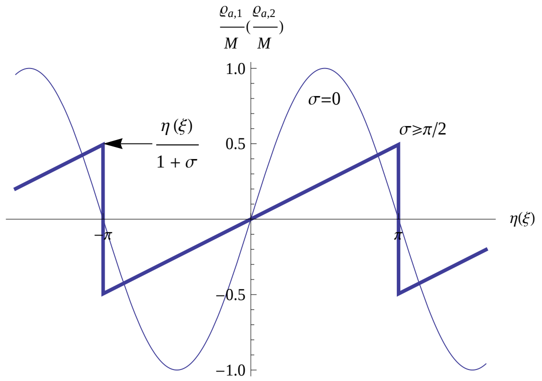


Fig.1 Acoustic excess densities in the sawtooth waves, in accordance to Eqs(19) for initially sinusoidal waveforms. Nodes in a standing wave remain in the same places.

5.3 Nonlinear waves with attenuation

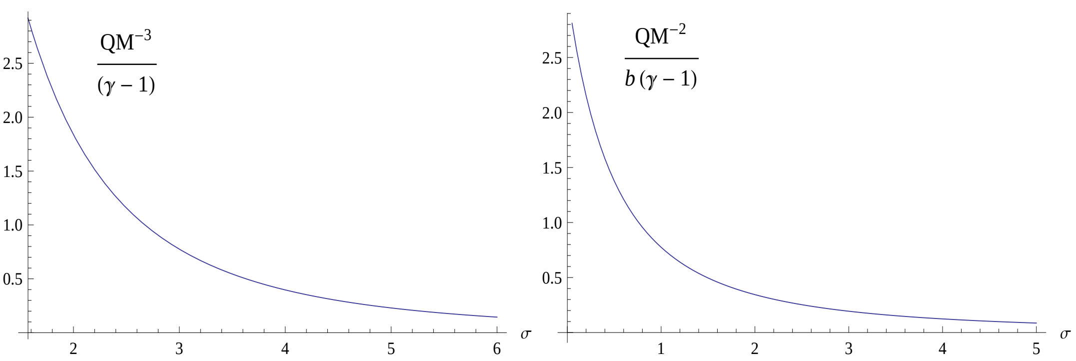
The Fay solution with the initial conditions as in the previous subsections describes the wave process in the domain of wave stabilization, where nonlinearity and absorption are balanced [5]. In resonators with π-fold length, an excess density is a sum of differently progressing parts,

$$\rho_a = \rho_{a,1} + \rho_{a,2} = \frac{2b}{(\gamma+1)} \sum_{n=1}^{\infty} \frac{\sin n(t-x) + \sin n(t+x)}{\sinh\left(\frac{n(1+\sigma)b}{M(\gamma+1)}\right)} = \frac{4b}{(\gamma+1)} \sum_{n=1}^{\infty} \frac{\cos(nx)\sin(nt)}{\sinh\left(\frac{n(1+\sigma)b}{M(\gamma+1)}\right)}. \quad (22)$$

The relative heating is

$$Q = \frac{2b^3(\gamma-1)}{(\gamma+1)^2} \sum_{n=1}^{\infty} \frac{1}{\sinh^2\left(\frac{n(1+\sigma)b}{M(\gamma+1)}\right)}. \quad (23)$$

In the evaluations of Eq.(23), only ten first summands were taken into account.



a b

Fig.2 Acoustic heating in the viscous nonlinear flow, a) sawtooth acoustic field, in accordance to Eq.(21), b) nonlinear resonator with attenuation, in accordance to Eq.(23). $QM^{-2}/(b(\gamma-1))$ is a function of b/M but almost does not depend on it.

In the evaluations of plots at Fig.2, $\gamma = 1.4$. The waveforms as follows

$$\rho_{a,1} = \rho_{a,0} \exp(-bt/2) \sin(t - x), \;\; \rho_{a,2} = \rho_{a,0} \exp(-bt/2) \sin(t + x), \tag{24}$$

are solutions of the Burgers equations (14) beyond some time where the nonlinear distortions are strong, and $\rho_{a,0} = \frac{4b}{\gamma+1}$. At this stage, the wave energy becomes enough small, and attenuation is comparatively most important. I this case, the acoustic heating in accordance to Eq.(16) is described by

$$Q = b(\gamma - 1)\rho_{a,0}^2 \exp(-bt). \tag{25}$$

6 Newtonian fluids different from ideal gases

The method may be applied to a wide variety of flows with different mechanisms of dissipation and dispersion. Examples of Sec.4,5 referred to an ideal gases for simplicity. In the fluid not rigorously being an ideal gas, both equations for sound branches take the leading-order forms:

$$\begin{aligned}
&\frac{\partial \rho_{a,1}}{\partial t} - \frac{b}{2}\frac{\partial^2 \rho_{a,1}}{\partial \eta^2} = \frac{1 - D_1 - D_2}{2}\left(\rho_{a,1}\frac{\partial \rho_{a,1}}{\partial \eta} + \rho_{a,1}\frac{\partial \rho_{a,2}}{\partial \xi} + \rho_{a,2}\frac{\partial \rho_{a,2}}{\partial \xi}\right) - \frac{3 + D_1 + D_2}{2}\rho_{a,2}\frac{\partial \rho_{a,1}}{\partial \eta}, \\
&\frac{\partial \rho_{a,2}}{\partial t} - \frac{b}{2}\frac{\partial^2 \rho_{2,1}}{\partial \eta^2} = \frac{1 - D_1 - D_2}{2}\left(\rho_{a,1}\frac{\partial \rho_{a,1}}{\partial \eta} + \rho_{a,2}\frac{\partial \rho_{a,2}}{\partial \xi} + \rho_{a,2}\frac{\partial \rho_{a,1}}{\partial \eta}\right) - \frac{3 + D_1 + D_2}{2}\rho_{a,1}\frac{\partial \rho_{a,2}}{\partial \xi}.
\end{aligned} \tag{26}$$

Averaged over ξ and η, relatively, these equations for periodic perturbations with zero mean values, are

$$\frac{\partial \rho_{a,1}}{\partial t} - \frac{b}{2}\frac{\partial^2 \rho_{a,1}}{\partial \eta^2} - \frac{1 - D_1 - D_2}{2}\rho_{a,1}\frac{\partial \rho_{a,1}}{\partial \eta} = 0, \; \frac{\partial \rho_{a,2}}{\partial t} - \frac{b}{2}\frac{\partial^2 \rho_{2,1}}{\partial \xi^2} - \frac{1 - D_1 - D_2}{2}\rho_{a,2}\frac{\partial \rho_{a,2}}{\partial \xi} = 0. \tag{27}$$

An excess density, associated with the entropy mode, is governed by equation

$$\begin{aligned}
\frac{\partial \rho_e}{\partial t} = {} & (1 + D_1 + D_2)(\rho_{a,1} + \rho_{a,2})\left(\frac{\partial \rho_{a,1}}{\partial \eta} + \frac{\partial \rho_{a,2}}{\partial \xi} + \frac{b}{2}\frac{\partial^2 \rho_{a,1}}{\partial \eta^2} + \frac{b}{2}\frac{\partial^2 \rho_{a,2}}{\partial \xi^2}\right) \\
& - \frac{b}{E_1}\left(\frac{\partial \rho_{a,1}}{\partial \eta} + \frac{\partial \rho_{a,2}}{\partial \xi}\right)^2.
\end{aligned} \tag{28}$$

It is valid for any sound branches which may exist in a resonator. The averaged over sound periods heating, if both sound waves are periodic with zero mean perturbations, is

$$\frac{\partial \rho_e}{\partial t} = -\frac{b}{2}\left(\frac{2}{E_1} + (1 + D_1 + D_2)\right)\left(\overline{\left(\frac{\partial \rho_{a,1}}{\partial \eta}\right)^2} + \overline{\left(\frac{\partial \rho_{a,2}}{\partial \xi}\right)^2}\right), \tag{29}$$

Eqs (26),(27),(28),(29) describe the field in a resonator filled by any newtonian fluid, such as van der Waals gas or a newtonian liquid.

Inclusion of bulk viscosity and thermal conductivity would yield the corrected dispersion relations and definitions of modes. It results to larger attenuation of the sound itself, i.e., to the corrected value of b in (27) and relative solutions, which will involve parts connected with the thermal conductivity and bulk viscosity [5, 12, 22]. As for the entropy mode, the dynamic equation for its specific excess density takes the form

$$\frac{\partial \rho_e}{\partial t} - \chi \frac{\partial^2 \rho_e}{\partial x^2} = -\frac{b}{2}\left(\frac{2}{E_1} + (1 + D_1 + D_2)\right)\left(\overline{\left(\frac{\partial \rho_{a,1}}{\partial \eta}\right)^2} + \overline{\left(\frac{\partial \rho_{a,2}}{\partial \xi}\right)^2}\right) = Q, \tag{30}$$

where $\chi = \frac{\lambda \omega}{\rho_0 c_0^2 C_p}$ is dimensional coefficient of thermal diffusivity with λ denoting the heat conductivity, $b = 4\mu\omega/(3\rho_0 c_0^2) + \zeta\omega/(\rho_0 c_0^2) + \delta$, ζ is the bulk viscosity. As for the periodic sound in a resonator, the right-hand side of the diffusion equation, Q, depends exclusively on time. The solution, ρ_e, can be represented as a sum of the partial solution, $\rho_e = \int_0^t Q(t)$, and the general solution of the homogeneous equation. The solution of the homogeneous equation, which satisfies zero initial condition $\rho_e(t = 0, x) = -\frac{T_e(t=0,x)}{T_0} = 0$ in a volume of a resonator, is trivial, it equals zero. So that, taking into account for thermal conductivity enlarges attenuation, and, relatively, acoustic heating, but the solution holds homogeneous over the length of resonator and depends on time exclusively. In the thermoconducting medium, there exists velocity specific for the entropy motion, $v_e = \chi \partial \rho_e / \partial x$. It is a quantity of the order $b^2 M^2$ and therefore does not belong to the domain of accuracy considered. In particular, it does not disturb zero boundary conditions of velocity at $x = 0$ and $x = L$.

The question about efficiency of heating induced by sound in resonators, is urgent and may be of importance in many medical and technical applications of ultrasound. There are cavities filled by liquids in a human body like maxillary sinus which are natural resonators. The equation governing acoustic heating, Eq.(16), is the result of decomposition of the weakly nonlinear equations for acoustic and non-acoustic kinds of motion. In its deriving, we use special linear combination of conservation equations, which allows to separate linear parts of dynamic equations. It yields instantaneous equations for both sound branches and the entropy mode. The problem itself is to establish sound field which must be a solution of coupling nonlinear equations. We give examples of heating associated with the periodic sound with zero averaged perturbations, as the most simple field, because the branches of opposite direction of propagation do not interact in the leading order in this case. That was analytically found in [4].

Attenuation, which is also a property of dispersive media [21], is the necessary condition for transfer of acoustic energy into that of the entropy motion. It may originate not only from newtonian damping, but also from molecular relaxation, or from specific thermodynamic properties of fluids different from newtonian. The special linear combination leads to instantaneous equations. In general, it requires neither the temporal averaging of the conservative equations with respect to the period of sound nor zero mean values of sound perturbations over its period. As soon as the sound field in a resonator is established, the correspondent heating may be calculated in accordance to the exact formula (28). It is local in time and may describe temporal development of heating not rigorously caused by periodic sound. The acoustic heating grows with increase of acoustic Mach number M and parameter of attenuation b. These general properties may be concluded a priori.

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