J. Math. Anal. Appl. 386 (2012) 332–342

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On parameter derivatives of the associated Legendre function of the first kind (with applications to the construction of the associated Legendre function of the second kind of integer degree and order)

Radosław Szmytkowski

Atomic Physics Division, Department of Atomic Physics and Luminescence, Faculty of Applied Physics and Mathematics, Gdańsk University of Technology, Narutowicza 11/12, 80-233 Gdańsk, Poland

ARTICLE INFO

Article history:
Received 7 November 2009
Available online 6 August 2011
Submitted by B.C. Berndt

Keywords:
Legendre functions
Parameter derivatives
Special functions

ABSTRACT

A relationship between partial derivatives of the associated Legendre function of the first kind with respect to its degree, $[\partial P_\nu^m(z)/\partial\nu]_{\nu=n}$, and to its order, $[\partial P_n^\mu(z)/\partial\mu]_{\mu=m}$, is established for $m, n \in \mathbb{N}_0$. This relationship is used to deduce four new closed-form representations of $[\partial P_\nu^m(z)/\partial\nu]_{\nu=n}$ from those found recently for $[\partial P_n^\mu(z)/\partial\mu]_{\mu=m}$ by the author [R. Szmytkowski, On the derivative of the associated Legendre function of the first kind of integer degree with respect to its order (with applications to the construction of the associated Legendre function of the second kind of integer degree and order), J. Math. Chem. 46 (2009) 231]. Several new expressions for the associated Legendre function of the second kind of integer degree and order, $Q_n^m(z)$, suitable for numerical purposes, are also derived.

1. Introduction

Recently, interest has arisen in the derivation of closed-form expressions for parameter derivatives of the associated Legendre functions. In Ref. [1], Brychkov has arrived at several representations of $[\partial P_\nu^\mu(z)/\partial\nu]_{\nu=n}$, $[\partial Q_\nu^\mu(z)/\partial\nu]_{\nu=n}$, $[\partial P_\nu^\mu(z)/\partial\mu]_{\mu=m}$ and $[\partial Q_\nu^\mu(z)/\partial\mu]_{\mu=m}$ (here and throughout the rest of this paragraph it is understood that $m, n \in \mathbb{N}_0$), some being compact, other being quite expanded. Cohl [2,3] has ingeniously shown that from the knowledge of the parameter derivatives $[\partial K_\mu(t)/\partial\mu]_{\mu=m}$ and $[\partial I_\mu(t)/\partial\mu]_{\mu=m}$ of the modified Bessel functions, with the aid of known integral relationships between $P_\nu^\mu(z)$, $Q_\nu^\mu(z)$ and $K_\mu(t)$, $I_\mu(t)$, one may deduce elegant formulas for $[\partial P_\nu^\mu(z)/\partial\nu]_{\nu=n-1/2}$, $[\partial Q_\nu^\mu(z)/\partial\nu]_{\nu=n-1/2}$, $[\partial P_\nu^\mu(z)/\partial\mu]_{\mu=m}$ and $[\partial Q_\nu^\mu(z)/\partial\mu]_{\mu=m}$. Finally, in Refs. [4,5] the present author has extensively studied derivatives of $P_\nu^\mu(z)$ in the case when one of its parameters is a fixed integer. In particular, using finite-sum expressions for $P_n^\mu(z)$, we have derived the following two formulas for $[\partial P_n^\mu(z)/\partial\mu]_{\mu=m}$ (cf. Ref. [4, Eqs. (3.4), (3.28) and (3.29)]):

$$\begin{aligned}\left.\frac{\partial P_n^\mu(z)}{\partial\mu}\right|_{\mu=m} &= \frac{1}{2}P_n^m(z)\ln\frac{z+1}{z-1} + (-)^m\left(\frac{z+1}{z-1}\right)^{m/2}\sum_{k=0}^{m-1}(-)^k\frac{(k+n)!(m-k-1)!}{k!(n-k)!}\left(\frac{z-1}{2}\right)^k \\ &\quad + \left(\frac{z^2-1}{4}\right)^{m/2}\sum_{k=0}^{n-m}\frac{(k+n+m)!\psi(k+1)}{k!(k+m)!(n-m-k)!}\left(\frac{z-1}{2}\right)^k \quad (0 \leqslant m \leqslant n)\end{aligned} \tag{1.1}$$

and

E-mail address: radek@mif.pg.gda.pl.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.08.003

$$\left.\frac{\partial P_n^{\mu}(z)}{\partial \mu}\right|_{\mu=m} = \frac{1}{2} P_n^m(z) \ln \frac{z+1}{z-1} + \left[\psi(n+m+1) + \psi(n-m+1)\right] P_n^m(z) - (-)^n \frac{(n+m)!}{(n-m)!} \left(\frac{z+1}{z-1}\right)^{m/2} \sum_{k=0}^{n} (-)^k \frac{(k+n)!\psi(k+m+1)}{k!(k+m)!(n-k)!} \left(\frac{z+1}{2}\right)^k \quad (0 \leqslant m \leqslant n). \tag{1.2}$$

In turn, using contour-integral representations of $\partial P_\nu^m(z)/\partial \nu$, we have arrived, among others, at the following three formulas for $[\partial P_\nu^m(z)/\partial \nu]_{\nu=n}$ (cf. Ref. [5, Eqs. (5.7), (5.8) and (5.9)]):

$$\left.\frac{\partial P_\nu^m(z)}{\partial \nu}\right|_{\nu=n} = P_n^m(z) \ln \frac{z+1}{2} - \left[\psi(n+1) + \psi(n-m+1)\right] P_n^m(z) + \left(\frac{z^2-1}{4}\right)^{m/2} \sum_{k=0}^{n-m} \frac{(k+n+m)!\psi(k+n+m+1)}{k!(k+m)!(n-m-k)!} \left(\frac{z-1}{2}\right)^k + \frac{(n+m)!}{(n-m)!} \left(\frac{z-1}{z+1}\right)^{m/2} \sum_{k=0}^{n} \frac{(k+n)!\psi(k+n+1)}{k!(k+m)!(n-k)!} \left(\frac{z-1}{2}\right)^k \quad (0 \leqslant m \leqslant n), \tag{1.3}$$

$$\left.\frac{\partial P_\nu^m(z)}{\partial \nu}\right|_{\nu=n} = P_n^m(z) \ln \frac{z+1}{2} + \left[\psi(n+1) - \psi(n-m+1)\right] P_n^m(z) - (-)^n \frac{(n+m)!}{(n-m)!} \left(\frac{z^2-1}{4}\right)^{-m/2} \sum_{k=0}^{m-1} \frac{(k+n-m)!(m-k-1)!}{k!(n+m-k)!} \left(\frac{z+1}{2}\right)^k + (-)^{n+m} \left(\frac{z^2-1}{4}\right)^{m/2} \sum_{k=0}^{n-m} (-)^k \frac{(k+n+m)!}{k!(k+m)!(n-m-k)!} \times \left[\psi(k+n+m+1) - \psi(k+m+1)\right] \left(\frac{z+1}{2}\right)^k + (-)^n \frac{(n+m)!}{(n-m)!} \left(\frac{z+1}{z-1}\right)^{m/2} \sum_{k=0}^{n} (-)^k \frac{(k+n)!}{k!(k+m)!(n-k)!} \times \left[\psi(k+n+1) - \psi(k+1)\right] \left(\frac{z+1}{2}\right)^k \quad (0 \leqslant m \leqslant n) \tag{1.4}$$

and

$$\left.\frac{\partial P_\nu^m(z)}{\partial \nu}\right|_{\nu=n} = P_n^m(z) \ln \frac{z+1}{2} + \left[\psi(n+m+1) - \psi(n+1)\right] P_n^m(z) - (-)^{n+m} \left(\frac{z-1}{z+1}\right)^{m/2} \sum_{k=0}^{m-1} \frac{(k+n)!(m-k-1)!}{k!(n-k)!} \left(\frac{z+1}{2}\right)^k + (-)^{n+m} \left(\frac{z^2-1}{4}\right)^{m/2} \sum_{k=0}^{n-m} (-)^k \frac{(k+n+m)!}{k!(k+m)!(n-m-k)!} \times \left[\psi(k+n+m+1) - \psi(k+1)\right] \left(\frac{z+1}{2}\right)^k + (-)^n \frac{(n+m)!}{(n-m)!} \left(\frac{z+1}{z-1}\right)^{m/2} \sum_{k=0}^{n} (-)^k \frac{(k+n)!}{k!(k+m)!(n-k)!} \times \left[\psi(k+n+1) - \psi(k+m+1)\right] \left(\frac{z+1}{2}\right)^k \quad (0 \leqslant m \leqslant n). \tag{1.5}$$

In the above equations, and in what follows, $\psi(\zeta)$ is the digamma function defined as

$$\psi(\zeta) = \frac{1}{\Gamma(\zeta)} \frac{\mathrm{d}\Gamma(\zeta)}{\mathrm{d}\zeta}. \tag{1.6}$$

This marked effort towards extending the knowledge on parameter derivatives of the associated Legendre functions has been driven not only by a purely mathematical concern. It actually appears that such derivatives are met in solutions of a number of problems of theoretical acoustics, electromagnetism, heat conduction and other branches of theoretical physics. A very recent relevant example from the field of mathematical optics is the problem of construction of the generalized Green's function for a scalar wave in the Maxwell fish-eye medium [6]. An exhaustive list of other works where the reader will find a variety of applications of the derivatives in question is provided in Ref. [5].

In this paper, we shall pursue further the subject of derivation of closed-form expressions for parameter derivatives of the associated Legendre function of the first kind in the case when one of its parameters is a fixed integer. First, in Section 2 we shall show that there exists a simple relationship between the derivatives $[\partial P_\nu^m(z)/\partial\nu]_{\nu=n}$ and $[\partial P_n^\mu(z)/\partial\mu]_{\mu=m}$, both with $m,n\in\mathbb{N}_0$. Next, in Section 3 we shall use this relationship, in conjunction with the formulas (1.1) and (1.2), to derive four further representations of $[\partial P_\nu^m(z)/\partial\nu]_{\nu=n}$, two of them involving sums of powers of $(z+1)/2$ and the remaining two — sums of powers of $(z-1)/2$. Interestingly, each out of these four representations contains only *two* sums. Therefore, the two new expressions for $[\partial P_\nu^m(z)/\partial\nu]_{\nu=n}$ containing sums of powers of $(z+1)/2$ appear to be markedly simpler than the previously derived representations (1.4) and (1.5), while the two new expressions involving sums of powers of $(z-1)/2$ are of the same degree of complexity as the representation (1.3). In addition, as a by-product, we shall obtain in that section two useful finite-sum representations of $[\psi(n+m+1)-\psi(n+1)]P_n^m(z)$. In the final Section 4, we shall exploit the results of Section 3 to find some new representations of the associated Legendre function of the second kind of integer degree and order, $Q_n^m(z)$, suitable for use for numerical purposes in various parts of the complex z-plane.

Throughout this work, we adopt the standard convention according to which $z\in\mathbb{C}$, with the phases restricted by

$$-\pi<\arg(z)\leqslant\pi,\qquad -\pi<\arg(z\pm1)\leqslant\pi \tag{1.7}$$

(this corresponds to drawing a cut in the z-plane along the real axis from $z=-\infty$ to $z=+1$). Hence, it follows that

$$-z=\mathrm{e}^{\mp\mathrm{i}\pi}z,\qquad -z+1=\mathrm{e}^{\mp\mathrm{i}\pi}(z-1),\qquad -z-1=\mathrm{e}^{\mp\mathrm{i}\pi}(z+1)\qquad \bigl(\arg(z)\gtrless0\bigr). \tag{1.8}$$

Furthermore, we define

$$\bigl(z^2-1\bigr)^\alpha=(z-1)^\alpha(z+1)^\alpha\quad(\alpha\in\mathbb{C}) \tag{1.9}$$

(this must be remembered when implementing numerically formulas for $[\partial P_\nu^m(z)/\partial\nu]_{\nu=n}$ or $Q_n^m(z)$ derived below!). For the natural logarithm, we take the Riemann's point of view and consider $\ln\zeta$ as a single-valued function on the domain $\{0\leqslant|\zeta|<\infty,\ -\infty<\arg\zeta<\infty\}$, subject to the constraint $\ln 1=0$. Next, it will be implicit that $x\in(-1,1)$, $\mu,\nu\in\mathbb{C}$ and $k,m,n\in\mathbb{N}_0$. Finally, it will be understood that if the upper limit of a sum is less by unity than the lower one, then the sum vanishes identically.

The associated Legendre functions of the first and the second kinds used in the paper are those of Hobson [7]. Their definitions, both off and on the cut, are the same as in the standard handbooks on special functions, such as Refs. [8–12], as well as in the classic Robin's monograph [13–15].

2. A relationship between $[\partial P_\nu^m(z)/\partial\nu]_{\nu=n}$ and $[\partial P_n^\mu(z)/\partial\mu]_{\mu=m}$

The departure point for our considerations in this section is the following Rodrigues formula, due to Barnes [16, Sec. 59], for the associated Legendre function of the first kind when the sum of its degree and its order is a non-negative integer m:

$$P_\nu^{m-\nu}(z)=\frac{1}{2^\nu\Gamma(\nu+1)}\bigl(z^2-1\bigr)^{(m-\nu)/2}\frac{\mathrm{d}^m(z^2-1)^\nu}{\mathrm{d}z^m}. \tag{2.1}$$

In terms of the Jacobi polynomial

$$P_n^{(\alpha,\beta)}(z)=\frac{1}{2^n n!}(z-1)^{-\alpha}(z+1)^{-\beta}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\bigl[(z-1)^{n+\alpha}(z+1)^{n+\beta}\bigr]\quad(\alpha,\beta\in\mathbb{C}), \tag{2.2}$$

Eq. (2.1) may be rewritten as

$$P_\nu^{m-\nu}(z)=\frac{m!}{\Gamma(\nu+1)}\left(\frac{z^2-1}{4}\right)^{(\nu-m)/2}P_m^{(\nu-m,\nu-m)}(z). \tag{2.3}$$

(With no doubt, the reader has immediately realized that the Jacobi polynomial appearing on the right-hand side of Eq. (2.3) is a multiple of the Gegenbauer polynomial $C_m^{(\nu-m+1/2)}(z)$. However, we shall not make any use of this fact here.) Differentiation of Eq. (2.3) with respect to ν, followed by setting $\nu=n$, yields

$$\begin{aligned}\frac{\partial P_\nu^{m-n}(z)}{\partial\nu}\bigg|_{\nu=n}-\frac{\partial P_n^\mu(z)}{\partial\mu}\bigg|_{\mu=m-n}&=\frac12P_n^{m-n}(z)\ln\frac{z^2-1}{4}-\psi(n+1)P_n^{m-n}(z)\\&\quad+\frac{m!}{n!}\left(\frac{z^2-1}{4}\right)^{(n-m)/2}\frac{\partial P_m^{(\lambda,\lambda)}(z)}{\partial\lambda}\bigg|_{\lambda=n-m}.\end{aligned} \tag{2.4}$$

The replacement of m by $n+m$ results in the relationship

$$\left.\frac{\partial P_\nu^m(z)}{\partial\nu}\right|_{\nu=n}-\left.\frac{\partial P_n^\mu(z)}{\partial\mu}\right|_{\mu=m}=\frac{1}{2}P_n^m(z)\ln\frac{z^2-1}{4}-\psi(n+1)P_n^m(z)+\frac{(n+m)!}{n!}\left(\frac{z^2-1}{4}\right)^{-m/2}\left.\frac{\partial P_{n+m}^{(\lambda,\lambda)}(z)}{\partial\lambda}\right|_{\lambda=-m}. \tag{2.5}$$

If in Eq. (2.5) one exploits the following two explicit representations of the Jacobi polynomial $P_n^{(\alpha,\beta)}(z)$ [10, Sec. 5.2.2]:

$$P_n^{(\alpha,\beta)}(z)=\frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\beta+1)}\sum_{k=0}^{n}\frac{\Gamma(k+n+\alpha+\beta+1)}{k!(n-k)!\Gamma(k+\alpha+1)}\left(\frac{z-1}{2}\right)^k, \tag{2.6}$$

$$P_n^{(\alpha,\beta)}(z)=(-)^n\frac{\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)}\sum_{k=0}^{n}(-)^k\frac{\Gamma(k+n+\alpha+\beta+1)}{k!(n-k)!\Gamma(k+\beta+1)}\left(\frac{z+1}{2}\right)^k, \tag{2.7}$$

after making use of Eq. (2.3) and of the easily provable relation

$$\lim_{\lambda\to-m}\frac{\psi(k+\lambda+1)}{\Gamma(k+\lambda+1)}=(-)^{k+m}(m-k-1)! \quad (0\leqslant k\leqslant m-1), \tag{2.8}$$

one obtains

$$\begin{aligned}\left.\frac{\partial P_\nu^m(z)}{\partial\nu}\right|_{\nu=n}-\left.\frac{\partial P_n^\mu(z)}{\partial\mu}\right|_{\mu=m}&=\frac{1}{2}P_n^m(z)\ln\frac{z^2-1}{4}-2\psi(n-m+1)P_n^m(z)\\&\quad-(-)^m\frac{(n+m)!}{(n-m)!}\left(\frac{z^2-1}{4}\right)^{-m/2}\sum_{k=0}^{m-1}(-)^k\frac{(k+n-m)!(m-k-1)!}{k!(n+m-k)!}\left(\frac{z-1}{2}\right)^k\\&\quad+\frac{(n+m)!}{(n-m)!}\left(\frac{z-1}{z+1}\right)^{m/2}\sum_{k=0}^{n}\frac{(k+n)!}{k!(k+m)!(n-k)!}\\&\quad\times\left[2\psi(k+n+1)-\psi(k+1)\right]\left(\frac{z-1}{2}\right)^k \quad (0\leqslant m\leqslant n)\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}\left.\frac{\partial P_\nu^m(z)}{\partial\nu}\right|_{\nu=n}-\left.\frac{\partial P_n^\mu(z)}{\partial\mu}\right|_{\mu=m}&=\frac{1}{2}P_n^m(z)\ln\frac{z^2-1}{4}-2\psi(n-m+1)P_n^m(z)\\&\quad-(-)^n\frac{(n+m)!}{(n-m)!}\left(\frac{z^2-1}{4}\right)^{-m/2}\sum_{k=0}^{m-1}\frac{(k+n-m)!(m-k-1)!}{k!(n+m-k)!}\left(\frac{z+1}{2}\right)^k\\&\quad+(-)^n\frac{(n+m)!}{(n-m)!}\left(\frac{z+1}{z-1}\right)^{m/2}\sum_{k=0}^{n}(-)^k\frac{(k+n)!}{k!(k+m)!(n-k)!}\\&\quad\times\left[2\psi(k+n+1)-\psi(k+1)\right]\left(\frac{z+1}{2}\right)^k \quad (0\leqslant m\leqslant n),\end{aligned} \tag{2.10}$$

respectively.

In this work we are interested in the case when both the degree and the order of the Legendre function derivatives are integers. Therefore, above we have focused on the derivation of the relations (2.9) and (2.10) between $[\partial P_\nu^m(z)/\partial\nu]_{\nu=n}$ and $[\partial P_n^\mu(z)/\partial\mu]_{\mu=m}$. It is evident, however, that starting from Eq. (2.3) one may obtain relationships between the derivatives $[\partial P_\nu^\mu(z)/\partial\nu]_{\mu=m-\nu}$ and $[\partial P_\nu^\mu(z)/\partial\mu]_{\mu=m-\nu}$ for arbitrary complex ν.

3. Some new representations of $[\partial P_\nu^m(z)/\partial\nu]_{\nu=n}$ and $[\partial P_\nu^m(x)/\partial\nu]_{\nu=n}$ with $0\leqslant m\leqslant n$

In this section, we shall use the results of Section 2 to provide several expressions for the derivative $[\partial P_\nu^m(z)/\partial\nu]_{\nu=n}$, which supplement these given in Eqs. (1.3)–(1.5). Also, related formulas for the derivative $[\partial P_\nu^m(x)/\partial\nu]_{\nu=n}$ will be presented.

The first from among these expressions for $[\partial P_\nu^m(z)/\partial\nu]_{\nu=n}$ follows if one plugs the representation (1.2) of $[\partial P_n^\mu(z)/\partial\mu]_{\mu=m}$ into the relationship (2.10). The result,

$$\begin{aligned}\left.\frac{\partial P_\nu^m(z)}{\partial\nu}\right|_{\nu=n} &= P_n^m(z)\ln\frac{z+1}{2}+\left[\psi(n+m+1)-\psi(n-m+1)\right]P_n^m(z)\\ &\quad-(-)^n\frac{(n+m)!}{(n-m)!}\left(\frac{z^2-1}{4}\right)^{-m/2}\sum_{k=0}^{m-1}\frac{(k+n-m)!(m-k-1)!}{k!(n+m-k)!}\left(\frac{z+1}{2}\right)^k\\ &\quad+(-)^n\frac{(n+m)!}{(n-m)!}\left(\frac{z+1}{z-1}\right)^{m/2}\sum_{k=0}^{n}(-)^k\frac{(k+n)!}{k!(k+m)!(n-k)!}\\ &\quad\times\left[2\psi(k+n+1)-\psi(k+m+1)-\psi(k+1)\right]\left(\frac{z+1}{2}\right)^k\quad(0\leqslant m\leqslant n),\end{aligned}\tag{3.1}$$

is seen to be much simpler than either of the representations (1.4) or (1.5). An infinite variety of other representations of $[\partial P_\nu^m(z)/\partial\nu]_{\nu=n}$, involving sums of powers of $(z+1)/2$, may be obtained by taking linear combinations, with coefficients such that their sum is unity, of the expressions in Eqs. (1.4), (1.5) and (3.1). For instance, multiplying Eq. (3.1) by -1 and adding to the sum of Eqs. (1.4) and (1.5) leads to another remarkably simple formula

$$\begin{aligned}\left.\frac{\partial P_\nu^m(z)}{\partial\nu}\right|_{\nu=n} &= P_n^m(z)\ln\frac{z+1}{2}-(-)^{n+m}\left(\frac{z-1}{z+1}\right)^{m/2}\sum_{k=0}^{m-1}\frac{(k+n)!(m-k-1)!}{k!(n-k)!}\left(\frac{z+1}{2}\right)^k\\ &\quad+(-)^{n+m}\left(\frac{z^2-1}{4}\right)^{m/2}\sum_{k=0}^{n-m}(-)^k\frac{(k+n+m)!}{k!(k+m)!(n-m-k)!}\\ &\quad\times\left[2\psi(k+n+m+1)-\psi(k+m+1)-\psi(k+1)\right]\left(\frac{z+1}{2}\right)^k\quad(0\leqslant m\leqslant n).\end{aligned}\tag{3.2}$$

For $m=0$ both the representations (3.1) and (3.2) of $[\partial P_\nu^m(z)/\partial\nu]_{\nu=n}$ reduce to the formula

$$\left.\frac{\partial P_\nu(z)}{\partial\nu}\right|_{\nu=n}=P_n(z)\ln\frac{z+1}{2}+2\sum_{k=0}^{n}(-)^{k+n}\frac{(k+n)!}{(k!)^2(n-k)!}\left[\psi(k+n+1)-\psi(k+1)\right]\left(\frac{z+1}{2}\right)^k,\tag{3.3}$$

found by the author in Ref. [17, Sec. 5.2.7] (cf. also Ref. [18]).

From the above findings, one may deduce two interesting and, as we shall see in a moment, useful identities involving the function $P_n^m(z)$. If we equate the right-hand sides of Eqs. (1.4) and (3.1), this results in the first of these relations:

$$\begin{aligned}\left[\psi(n+m+1)-\psi(n+1)\right]P_n^m(z) &= (-)^{n+m}\left(\frac{z^2-1}{4}\right)^{m/2}\sum_{k=0}^{n-m}(-)^k\frac{(k+n+m)!}{k!(k+m)!(n-m-k)!}\\ &\quad\times\left[\psi(k+n+m+1)-\psi(k+m+1)\right]\left(\frac{z+1}{2}\right)^k\\ &\quad-(-)^n\frac{(n+m)!}{(n-m)!}\left(\frac{z+1}{z-1}\right)^{m/2}\sum_{k=0}^{n}(-)^k\frac{(k+n)!}{k!(k+m)!(n-k)!}\\ &\quad\times\left[\psi(k+n+1)-\psi(k+m+1)\right]\left(\frac{z+1}{2}\right)^k\quad(0\leqslant m\leqslant n).\end{aligned}\tag{3.4}$$

Replacement of z by $-z$ in the above equation, followed by the use of the well-known property

$$P_n^m(-z)=(-)^nP_n^m(z)\quad(0\leqslant m\leqslant n),\tag{3.5}$$

and also of Eq. (1.8), leads to the second identity:

$$\begin{aligned}\left[\psi(n+m+1)-\psi(n+1)\right]P_n^m(z) &= \left(\frac{z^2-1}{4}\right)^{m/2}\sum_{k=0}^{n-m}\frac{(k+n+m)!}{k!(k+m)!(n-m-k)!}\\ &\quad\times\left[\psi(k+n+m+1)-\psi(k+m+1)\right]\left(\frac{z-1}{2}\right)^k\end{aligned}$$

$$-\frac{(n+m)!}{(n-m)!}\left(\frac{z-1}{z+1}\right)^{m/2}\sum_{k=0}^{n}\frac{(k+n)!}{k!(k+m)!(n-k)!}$$
$$\times\left[\psi(k+n+1)-\psi(k+m+1)\right]\left(\frac{z-1}{2}\right)^{k} \quad (0 \leqslant m \leqslant n). \tag{3.6}$$

Playing with Eq. (1.3) and with the identity (3.6), one may obtain an infinite variety of representations of $[\partial P_{\nu}^{m}(z)/\partial\nu]_{\nu=n}$ containing sums of powers of $(z-1)/2$. Two examples of such representations are

$$\begin{aligned}\left.\frac{\partial P_{\nu}^{m}(z)}{\partial\nu}\right|_{\nu=n} &= P_{n}^{m}(z)\ln\frac{z+1}{2}-\left[\psi(n+m+1)+\psi(n-m+1)\right]P_{n}^{m}(z)\\ &\quad+\left(\frac{z^{2}-1}{4}\right)^{m/2}\sum_{k=0}^{n-m}\frac{(k+n+m)!}{k!(k+m)!(n-m-k)!}\\ &\quad\times\left[2\psi(k+n+m+1)-\psi(k+m+1)\right]\left(\frac{z-1}{2}\right)^{k}\\ &\quad+\frac{(n+m)!}{(n-m)!}\left(\frac{z-1}{z+1}\right)^{m/2}\sum_{k=0}^{n}\frac{(k+n)!\psi(k+m+1)}{k!(k+m)!(n-k)!}\left(\frac{z-1}{2}\right)^{k} \quad (0 \leqslant m \leqslant n)\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\left.\frac{\partial P_{\nu}^{m}(z)}{\partial\nu}\right|_{\nu=n} &= P_{n}^{m}(z)\ln\frac{z+1}{2}+\left[\psi(n+m+1)-2\psi(n+1)-\psi(n-m+1)\right]P_{n}^{m}(z)\\ &\quad+\left(\frac{z^{2}-1}{4}\right)^{m/2}\sum_{k=0}^{n-m}\frac{(k+n+m)!\psi(k+m+1)}{k!(k+m)!(n-m-k)!}\left(\frac{z-1}{2}\right)^{k}\\ &\quad+\frac{(n+m)!}{(n-m)!}\left(\frac{z-1}{z+1}\right)^{m/2}\sum_{k=0}^{n}\frac{(k+n)!}{k!(k+m)!(n-k)!}\\ &\quad\times\left[2\psi(k+n+1)-\psi(k+m+1)\right]\left(\frac{z-1}{2}\right)^{k} \quad (0 \leqslant m \leqslant n).\end{aligned} \tag{3.8}$$

For $m=0$, both Eqs. (3.7) and (3.8) reduce to the Schelkunoff's formula [19] (cf. also Ref. [17, Sec. 5.2.6])

$$\left.\frac{\partial P_{\nu}(z)}{\partial\nu}\right|_{\nu=n} = P_{n}(z)\ln\frac{z+1}{2}-2\psi(n+1)P_{n}(z)+2\sum_{k=0}^{n}\frac{(k+n)!\psi(k+n+1)}{(k!)^{2}(n-k)!}\left(\frac{z-1}{2}\right)^{k}. \tag{3.9}$$

From the representations of $[\partial P_{\nu}^{m}(z)/\partial\nu]_{\nu=n}$ found above, one may construct counterpart representations for $[\partial P_{\nu}^{-m}(z)/\partial\nu]_{\nu=n}$, using the relationship [5, Eq. (5.24)]

$$\left.\frac{\partial P_{\nu}^{-m}(z)}{\partial\nu}\right|_{\nu=n} = \frac{(n-m)!}{(n+m)!}\left.\frac{\partial P_{\nu}^{m}(z)}{\partial\nu}\right|_{\nu=n}-\left[\psi(n+m+1)-\psi(n-m+1)\right]P_{n}^{-m}(z) \quad (0 \leqslant m \leqslant n) \tag{3.10}$$

and the well-known property

$$P_{n}^{m}(z) = \frac{(n+m)!}{(n-m)!}P_{n}^{-m}(z) \quad (0 \leqslant m \leqslant n). \tag{3.11}$$

Moreover, it does not offer any difficulty to derive counterpart expressions on the cut $x \in (-1,1)$ by using the formulas

$$\begin{aligned}\left.\frac{\partial P_{\nu}^{\pm m}(x)}{\partial\nu}\right|_{\nu=n} &= \mathrm{e}^{\pm\mathrm{i}\pi m/2}\left.\frac{\partial P_{\nu}^{\pm m}(x+\mathrm{i}0)}{\partial\nu}\right|_{\nu=n} = \mathrm{e}^{\mp\mathrm{i}\pi m/2}\left.\frac{\partial P_{\nu}^{\pm m}(x-\mathrm{i}0)}{\partial\nu}\right|_{\nu=n}\\ &= \frac{1}{2}\left[\mathrm{e}^{\pm\mathrm{i}\pi m/2}\left.\frac{\partial P_{\nu}^{\pm m}(x+\mathrm{i}0)}{\partial\nu}\right|_{\nu=n}+\mathrm{e}^{\mp\mathrm{i}\pi m/2}\left.\frac{\partial P_{\nu}^{\pm m}(x-\mathrm{i}0)}{\partial\nu}\right|_{\nu=n}\right],\end{aligned} \tag{3.12}$$

together with

$$\begin{aligned}P_{n}^{\pm m}(x) &= \mathrm{e}^{\pm\mathrm{i}\pi m/2}P_{n}^{\pm m}(x+\mathrm{i}0) = \mathrm{e}^{\mp\mathrm{i}\pi m/2}P_{n}^{\pm m}(x-\mathrm{i}0)\\ &= \frac{1}{2}\left[\mathrm{e}^{\pm\mathrm{i}\pi m/2}P_{n}^{\pm m}(x+\mathrm{i}0)+\mathrm{e}^{\mp\mathrm{i}\pi m/2}P_{n}^{\pm m}(x-\mathrm{i}0)\right]\end{aligned} \tag{3.13}$$

and

$$x+1\pm \mathrm{i}0=1+x, \qquad x-1\pm \mathrm{i}0=\mathrm{e}^{\pm \mathrm{i}\pi}(1-x). \tag{3.14}$$

Proceeding accordingly, from Eqs. (3.1), (3.2), (3.7) and (3.8) one deduces that

$$\begin{aligned}\left.\frac{\partial P_\nu^m(x)}{\partial \nu}\right|_{\nu=n} &= P_n^m(x)\ln\frac{1+x}{2}+\big[\psi(n+m+1)-\psi(n-m+1)\big]P_n^m(x)\\ &\quad-(-)^n\frac{(n+m)!}{(n-m)!}\left(\frac{1-x^2}{4}\right)^{-m/2}\sum_{k=0}^{m-1}\frac{(k+n-m)!(m-k-1)!}{k!(n+m-k)!}\left(\frac{1+x}{2}\right)^k\\ &\quad+(-)^n\frac{(n+m)!}{(n-m)!}\left(\frac{1+x}{1-x}\right)^{m/2}\sum_{k=0}^{n}(-)^k\frac{(k+n)!}{k!(k+m)!(n-k)!}\\ &\quad\times\big[2\psi(k+n+1)-\psi(k+m+1)-\psi(k+1)\big]\left(\frac{1+x}{2}\right)^k \quad (0\leqslant m\leqslant n),\end{aligned} \tag{3.15}$$

$$\begin{aligned}\left.\frac{\partial P_\nu^m(x)}{\partial \nu}\right|_{\nu=n} &= P_n^m(x)\ln\frac{1+x}{2}-(-)^n\left(\frac{1-x}{1+x}\right)^{m/2}\sum_{k=0}^{m-1}\frac{(k+n)!(m-k-1)!}{k!(n-k)!}\left(\frac{1+x}{2}\right)^k\\ &\quad+(-)^n\left(\frac{1-x^2}{4}\right)^{m/2}\sum_{k=0}^{n-m}(-)^k\frac{(k+n+m)!}{k!(k+m)!(n-m-k)!}\\ &\quad\times\big[2\psi(k+n+m+1)-\psi(k+m+1)-\psi(k+1)\big]\left(\frac{1+x}{2}\right)^k \quad (0\leqslant m\leqslant n),\end{aligned} \tag{3.16}$$

$$\begin{aligned}\left.\frac{\partial P_\nu^m(x)}{\partial \nu}\right|_{\nu=n} &= P_n^m(x)\ln\frac{1+x}{2}-\big[\psi(n+m+1)+\psi(n-m+1)\big]P_n^m(x)\\ &\quad+(-)^m\left(\frac{1-x^2}{4}\right)^{m/2}\sum_{k=0}^{n-m}(-)^k\frac{(k+n+m)!}{k!(k+m)!(n-m-k)!}\\ &\quad\times\big[2\psi(k+n+m+1)-\psi(k+m+1)\big]\left(\frac{1-x}{2}\right)^k\\ &\quad+(-)^m\frac{(n+m)!}{(n-m)!}\left(\frac{1-x}{1+x}\right)^{m/2}\sum_{k=0}^{n}(-)^k\frac{(k+n)!\psi(k+m+1)}{k!(k+m)!(n-k)!}\left(\frac{1-x}{2}\right)^k \quad (0\leqslant m\leqslant n)\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}\left.\frac{\partial P_\nu^m(x)}{\partial \nu}\right|_{\nu=n} &= P_n^m(x)\ln\frac{1+x}{2}+\big[\psi(n+m+1)-2\psi(n+1)-\psi(n-m+1)\big]P_n^m(x)\\ &\quad+(-)^m\left(\frac{1-x^2}{4}\right)^{m/2}\sum_{k=0}^{n-m}(-)^k\frac{(k+n+m)!\psi(k+m+1)}{k!(k+m)!(n-m-k)!}\left(\frac{1-x}{2}\right)^k\\ &\quad+(-)^m\frac{(n+m)!}{(n-m)!}\left(\frac{1-x}{1+x}\right)^{m/2}\sum_{k=0}^{n}(-)^k\frac{(k+n)!}{k!(k+m)!(n-k)!}\\ &\quad\times\big[2\psi(k+n+1)-\psi(k+m+1)\big]\left(\frac{1-x}{2}\right)^k \quad (0\leqslant m\leqslant n).\end{aligned} \tag{3.18}$$

Counterpart expressions for $[\partial P_\nu^{-m}(x)/\partial\nu]_{\nu=n}$ are most easily obtained from Eqs. (3.15)–(3.18) and the relation

$$\left.\frac{\partial P_\nu^{-m}(x)}{\partial \nu}\right|_{\nu=n}=(-)^m\frac{(n-m)!}{(n+m)!}\left.\frac{\partial P_\nu^m(x)}{\partial \nu}\right|_{\nu=n}-\big[\psi(n+m+1)-\psi(n-m+1)\big]P_n^{-m}(x) \quad (0\leqslant m\leqslant n), \tag{3.19}$$

which follows from Eqs. (3.10), (3.12) and (3.13).

Concluding this section we note that, in principle, the relations (2.9) and (2.10) might be also used in the opposite direction, i.e., to construct representations for $[\partial P_n^\mu(z)/\partial\mu]_{\mu=m}$ from those known for $[\partial P_\nu^m(z)/\partial\nu]_{\nu=n}$. As it appears, however, that all expressions for $[\partial P_n^\mu(z)/\partial\mu]_{\mu=m}$ obtainable in this way are much more complex (and thus potentially less useful) than these in Eqs. (1.1) and (1.2), we do not present them here.

4. Some new representations of $Q_n^m(z)$ and $Q_n^m(x)$ with $0 \leqslant m \leqslant n$

The associated Legendre function of the second kind, $Q_\nu^\mu(z)$, may be defined [7] as the following linear combination of the Legendre functions of the first kind $P_\nu^\mu(z)$ and $P_\nu^\mu(-z)$:

$$Q_\nu^\mu(z) = \frac{\pi}{2} \mathrm{e}^{\mathrm{i}\pi\mu} \frac{\mathrm{e}^{\mp \mathrm{i}\pi\nu} P_\nu^\mu(z) - P_\nu^\mu(-z)}{\sin[\pi(\nu+\mu)]} \quad \big(\mathrm{Im}(z) \gtrless 0\big). \tag{4.1}$$

In the special case of $\mu = m$, Eq. (4.1) simplifies to

$$Q_\nu^m(z) = \frac{\pi}{2} \frac{\mathrm{e}^{\mp \mathrm{i}\pi\nu} P_\nu^m(z) - P_\nu^m(-z)}{\sin(\pi\nu)} \quad \big(\mathrm{Im}(z) \gtrless 0\big). \tag{4.2}$$

Hence, after exploiting the l'Hospital rule, one obtains

$$Q_n^m(z) = \mp \frac{1}{2} \mathrm{i}\pi P_n^m(z) + \frac{1}{2} \frac{\partial P_\nu^m(z)}{\partial \nu}\bigg|_{\nu=n} - \frac{(-)^n}{2} \frac{\partial P_\nu^m(-z)}{\partial \nu}\bigg|_{\nu=n} \quad \big(0 \leqslant m \leqslant n,\ \mathrm{Im}(z) \gtrless 0\big). \tag{4.3}$$

Thus, we see that the problem of evaluation of $Q_n^m(z)$ with $0 \leqslant m \leqslant n$ may be reduced to that of derivation of expressions for $[\partial P_\nu^m(\pm z)/\partial \nu]_{\nu=n}$.

Accordingly, after combining Eq. (4.3) with Eqs. (1.3) and (3.1), we obtain

$$\begin{aligned} Q_n^m(z) &= \frac{1}{2} P_n^m(z) \ln \frac{z+1}{z-1} \mp \frac{1}{2}\big[\psi(n+m+1) + \psi(n+1)\big] P_n^m(z) \\ &\quad \pm \frac{(\pm)^n (\mp)^m}{2} \frac{(n+m)!}{(n-m)!} \left(\frac{z^2-1}{4}\right)^{-m/2} \sum_{k=0}^{m-1} (\mp)^k \frac{(k+n-m)!(m-k-1)!}{k!(n+m-k)!} \left(\frac{z \mp 1}{2}\right)^k \\ &\quad \pm \frac{(\pm)^{n+m}}{2} \left(\frac{z^2-1}{4}\right)^{m/2} \sum_{k=0}^{n-m} (\pm)^k \frac{(k+n+m)!\psi(k+n+m+1)}{k!(k+m)!(n-m-k)!} \left(\frac{z \mp 1}{2}\right)^k \\ &\quad \mp \frac{(\pm)^n}{2} \frac{(n+m)!}{(n-m)!} \left(\frac{z \mp 1}{z \pm 1}\right)^{m/2} \sum_{k=0}^{n} (\pm)^k \frac{(k+n)!}{k!(k+m)!(n-k)!} \\ &\quad \times \big[\psi(k+n+1) - \psi(k+m+1) - \psi(k+1)\big] \left(\frac{z \mp 1}{2}\right)^k \quad (0 \leqslant m \leqslant n), \end{aligned} \tag{4.4}$$

where the upper signs follow if $[\partial P_\nu^m(z)/\partial \nu]_{\nu=n}$ is evaluated from Eq. (1.3) and $[\partial P_\nu^m(-z)/\partial \nu]_{\nu=n}$ from Eq. (3.1), while the lower signs result if the roles of Eqs. (1.3) and (3.1) are interchanged. The same expression for $Q_n^m(z)$ as above is obtained if Eq. (4.3) is coupled with Eqs. (1.4) and (3.7). Further, using Eqs. (1.3) and (3.2) in Eq. (4.3) leads to

$$\begin{aligned} Q_n^m(z) &= \frac{1}{2} P_n^m(z) \ln \frac{z+1}{z-1} \mp \frac{1}{2}\big[\psi(n+1) + \psi(n-m+1)\big] P_n^m(z) \\ &\quad \pm \frac{(\pm)^n (-)^m}{2} \left(\frac{z \pm 1}{z \mp 1}\right)^{m/2} \sum_{k=0}^{m-1} (\mp)^k \frac{(k+n)!(m-k-1)!}{k!(n-k)!} \left(\frac{z \mp 1}{2}\right)^k \\ &\quad \mp \frac{(\pm)^{n+m}}{2} \left(\frac{z^2-1}{4}\right)^{m/2} \sum_{k=0}^{n-m} (\pm)^k \frac{(k+n+m)!}{k!(k+m)!(n-m-k)!} \\ &\quad \times \big[\psi(k+n+m+1) - \psi(k+m+1) - \psi(k+1)\big] \left(\frac{z \mp 1}{2}\right)^k \\ &\quad \pm \frac{(\pm)^n}{2} \frac{(n+m)!}{(n-m)!} \left(\frac{z \mp 1}{z \pm 1}\right)^{m/2} \sum_{k=0}^{n} (\pm)^k \frac{(k+n)!\psi(k+n+1)}{k!(k+m)!(n-k)!} \left(\frac{z \mp 1}{2}\right)^k \quad (0 \leqslant m \leqslant n). \end{aligned} \tag{4.5}$$

Next, if $[\partial P_\nu^m(z)/\partial \nu]_{\nu=n}$ is obtained from Eq. (3.1) and $[\partial P_\nu^m(-z)/\partial \nu]_{\nu=n}$ from Eq. (3.7), or vice versa, then Eq. (4.3) yields the expressions

$$\begin{aligned} Q_n^m(z) &= \frac{1}{2} P_n^m(z) \ln \frac{z+1}{z-1} \mp \psi(n+m+1) P_n^m(z) \\ &\quad \pm \frac{(\pm)^n (\mp)^m}{2} \frac{(n+m)!}{(n-m)!} \left(\frac{z^2-1}{4}\right)^{-m/2} \sum_{k=0}^{m-1} (\mp)^k \frac{(k+n-m)!(m-k-1)!}{k!(n+m-k)!} \left(\frac{z \mp 1}{2}\right)^k \end{aligned}$$

$$\pm\frac{(\pm)^{n+m}}{2}\left(\frac{z^2-1}{4}\right)^{m/2}\sum_{k=0}^{n-m}(\pm)^k\frac{(k+n+m)!}{k!(k+m)!(n-m-k)!}$$
$$\times\left[2\psi(k+n+m+1)-\psi(k+m+1)\right]\left(\frac{z\mp 1}{2}\right)^k$$
$$\mp\frac{(\pm)^n}{2}\frac{(n+m)!}{(n-m)!}\left(\frac{z\mp 1}{z\pm 1}\right)^{m/2}\sum_{k=0}^{n}(\pm)^k\frac{(k+n)!}{k!(k+m)!(n-k)!}$$
$$\times\left[2\psi(k+n+1)-2\psi(k+m+1)-\psi(k+1)\right]\left(\frac{z\mp 1}{2}\right)^k\quad(0\leqslant m\leqslant n). \tag{4.6}$$

We are not aware of any appearance of either of the formulas (4.4)–(4.6) in the literature. Furthermore, if Eqs. (3.1) and (3.8) are used in Eq. (4.3), this results in

$$Q_n^m(z)=\frac{1}{2}P_n^m(z)\ln\frac{z+1}{z-1}\mp\psi(n+1)P_n^m(z)$$
$$\pm\frac{(\pm)^n(\mp)^m}{2}\frac{(n+m)!}{(n-m)!}\left(\frac{z^2-1}{4}\right)^{-m/2}\sum_{k=0}^{m-1}(\mp)^k\frac{(k+n-m)!(m-k-1)!}{k!(n+m-k)!}\left(\frac{z\mp 1}{2}\right)^k$$
$$\pm\frac{(\pm)^{n+m}}{2}\left(\frac{z^2-1}{4}\right)^{m/2}\sum_{k=0}^{n-m}(\pm)^k\frac{(k+n+m)!\psi(k+m+1)}{k!(k+m)!(n-m-k)!}\left(\frac{z\mp 1}{2}\right)^k$$
$$\pm\frac{(\pm)^n}{2}\frac{(n+m)!}{(n-m)!}\left(\frac{z\mp 1}{z\pm 1}\right)^{m/2}\sum_{k=0}^{n}(\pm)^k\frac{(k+n)!\psi(k+1)}{k!(k+m)!(n-k)!}\left(\frac{z\mp 1}{2}\right)^k\quad(0\leqslant m\leqslant n), \tag{4.7}$$

which is the same as what follows if Eqs. (1.3) and (1.4) are plugged into Eq. (4.3) (cf. Ref. [5]). Finally, insertion of Eqs. (3.2) and (3.7) into Eq. (4.3) leads to

$$Q_n^m(z)=\frac{1}{2}P_n^m(z)\ln\frac{z+1}{z-1}\mp\frac{1}{2}\left[\psi(n+m+1)+\psi(n-m+1)\right]P_n^m(z)$$
$$\pm\frac{(\pm)^n(-)^m}{2}\left(\frac{z\pm 1}{z\mp 1}\right)^{m/2}\sum_{k=0}^{m-1}(\mp)^k\frac{(k+n)!(m-k-1)!}{k!(n-k)!}\left(\frac{z\mp 1}{2}\right)^k$$
$$\pm\frac{(\pm)^{n+m}}{2}\left(\frac{z^2-1}{4}\right)^{m/2}\sum_{k=0}^{n-m}(\pm)^k\frac{(k+n+m)!\psi(k+1)}{k!(k+m)!(n-m-k)!}\left(\frac{z\mp 1}{2}\right)^k$$
$$\pm\frac{(\pm)^n}{2}\frac{(n+m)!}{(n-m)!}\left(\frac{z\mp 1}{z\pm 1}\right)^{m/2}\sum_{k=0}^{n}(\pm)^k\frac{(k+n)!\psi(k+m+1)}{k!(k+m)!(n-k)!}\left(\frac{z\mp 1}{2}\right)^k\quad(0\leqslant m\leqslant n), \tag{4.8}$$

which, in turn, is the same as what is obtained if Eqs. (1.3) and (1.5) are coupled with Eq. (4.3) (cf. again Ref. [5]; for alternative derivations of the above result see Ref. [14, pp. 81, 82 and 85] and Ref. [4]). Other expressions for $Q_n^m(z)$ may be obtained by combining Eqs. (4.4)–(4.8), with the possible help of the identities (3.4) and (3.6).

Once $Q_n^m(z)$ is evaluated, one may find $Q_n^{-m}(z)$ from the well-known relationship

$$Q_n^{-m}(z)=\frac{(n-m)!}{(n+m)!}Q_n^m(z)\quad(0\leqslant m\leqslant n). \tag{4.9}$$

We conclude with the observation that using the formula

$$Q_n^m(x)=\frac{(-)^m}{2}\left[\mathrm{e}^{-\mathrm{i}\pi m/2}Q_n^m(x+\mathrm{i}0)+\mathrm{e}^{\mathrm{i}\pi m/2}Q_n^m(x-\mathrm{i}0)\right]\quad(-1<x<1), \tag{4.10}$$

which follows from the Hobson's [7] definition of the associated Legendre function of the second kind on the cut, and employing the identities in Eq. (3.14), from Eqs. (4.4)–(4.8) one derives counterpart representations of $Q_n^m(x)$, with $0\leqslant m\leqslant n$, listed below:

$$Q_n^m(x)=\frac{1}{2}P_n^m(x)\ln\frac{1+x}{1-x}\mp\frac{1}{2}\left[\psi(n+m+1)+\psi(n+1)\right]P_n^m(x)$$
$$\pm\frac{(\pm)^n(\mp)^m}{2}\frac{(n+m)!}{(n-m)!}\left(\frac{1-x^2}{4}\right)^{-m/2}\sum_{k=0}^{m-1}\frac{(k+n-m)!(m-k-1)!}{k!(n+m-k)!}\left(\frac{1\mp x}{2}\right)^k$$

$$\pm \frac{(\pm)^n(\mp)^m}{2}\left(\frac{1-x^2}{4}\right)^{m/2}\sum_{k=0}^{n-m}(-)^k\frac{(k+n+m)!\psi(k+n+m+1)}{k!(k+m)!(n-m-k)!}\left(\frac{1\mp x}{2}\right)^k$$
$$\mp \frac{(\pm)^n(\mp)^m}{2}\frac{(n+m)!}{(n-m)!}\left(\frac{1\mp x}{1\pm x}\right)^{m/2}\sum_{k=0}^{n}(-)^k\frac{(k+n)!}{k!(k+m)!(n-k)!}$$
$$\times\left[\psi(k+n+1)-\psi(k+m+1)-\psi(k+1)\right]\left(\frac{1\mp x}{2}\right)^k \quad (0\leqslant m\leqslant n), \tag{4.11}$$

$$Q_n^m(x)=\frac{1}{2}P_n^m(x)\ln\frac{1+x}{1-x}\mp\frac{1}{2}\left[\psi(n+1)+\psi(n-m+1)\right]P_n^m(x)$$
$$\pm \frac{(\pm)^n(\mp)^m}{2}\left(\frac{1\pm x}{1\mp x}\right)^{m/2}\sum_{k=0}^{m-1}\frac{(k+n)!(m-k-1)!}{k!(n-k)!}\left(\frac{1\mp x}{2}\right)^k$$
$$\mp \frac{(\pm)^n(\mp)^m}{2}\left(\frac{1-x^2}{4}\right)^{m/2}\sum_{k=0}^{n-m}(-)^k\frac{(k+n+m)!}{k!(k+m)!(n-m-k)!}$$
$$\times\left[\psi(k+n+m+1)-\psi(k+m+1)-\psi(k+1)\right]\left(\frac{1\mp x}{2}\right)^k$$
$$\pm \frac{(\pm)^n(\mp)^m}{2}\frac{(n+m)!}{(n-m)!}\left(\frac{1\mp x}{1\pm x}\right)^{m/2}\sum_{k=0}^{n}(-)^k\frac{(k+n)!\psi(k+n+1)}{k!(k+m)!(n-k)!}\left(\frac{1\mp x}{2}\right)^k \quad (0\leqslant m\leqslant n), \tag{4.12}$$

$$Q_n^m(x)=\frac{1}{2}P_n^m(x)\ln\frac{1+x}{1-x}\mp\psi(n+m+1)P_n^m(x)$$
$$\pm \frac{(\pm)^n(\mp)^m}{2}\frac{(n+m)!}{(n-m)!}\left(\frac{1-x^2}{4}\right)^{-m/2}\sum_{k=0}^{m-1}\frac{(k+n-m)!(m-k-1)!}{k!(n+m-k)!}\left(\frac{1\mp x}{2}\right)^k$$
$$\pm \frac{(\pm)^n(\mp)^m}{2}\left(\frac{1-x^2}{4}\right)^{m/2}\sum_{k=0}^{n-m}(-)^k\frac{(k+n+m)!}{k!(k+m)!(n-m-k)!}$$
$$\times\left[2\psi(k+n+m+1)-\psi(k+m+1)\right]\left(\frac{1\mp x}{2}\right)^k$$
$$\mp \frac{(\pm)^n(\mp)^m}{2}\frac{(n+m)!}{(n-m)!}\left(\frac{1\mp x}{1\pm x}\right)^{m/2}\sum_{k=0}^{n}(-)^k\frac{(k+n)!}{k!(k+m)!(n-k)!}$$
$$\times\left[2\psi(k+n+1)-2\psi(k+m+1)-\psi(k+1)\right]\left(\frac{1\mp x}{2}\right)^k \quad (0\leqslant m\leqslant n), \tag{4.13}$$

$$Q_n^m(x)=\frac{1}{2}P_n^m(x)\ln\frac{1+x}{1-x}\mp\psi(n+1)P_n^m(x)$$
$$\pm \frac{(\pm)^n(\mp)^m}{2}\frac{(n+m)!}{(n-m)!}\left(\frac{1-x^2}{4}\right)^{-m/2}\sum_{k=0}^{m-1}\frac{(k+n-m)!(m-k-1)!}{k!(n+m-k)!}\left(\frac{1\mp x}{2}\right)^k$$
$$\pm \frac{(\pm)^n(\mp)^m}{2}\left(\frac{1-x^2}{4}\right)^{m/2}\sum_{k=0}^{n-m}(-)^k\frac{(k+n+m)!\psi(k+m+1)}{k!(k+m)!(n-m-k)!}\left(\frac{1\mp x}{2}\right)^k$$
$$\pm \frac{(\pm)^n(\mp)^m}{2}\frac{(n+m)!}{(n-m)!}\left(\frac{1\mp x}{1\pm x}\right)^{m/2}\sum_{k=0}^{n}(-)^k\frac{(k+n)!\psi(k+1)}{k!(k+m)!(n-k)!}\left(\frac{1\mp x}{2}\right)^k \quad (0\leqslant m\leqslant n) \tag{4.14}$$

and

$$Q_n^m(x)=\frac{1}{2}P_n^m(x)\ln\frac{1+x}{1-x}\mp\frac{1}{2}\left[\psi(n+m+1)+\psi(n-m+1)\right]P_n^m(x)$$
$$\pm \frac{(\pm)^n(\mp)^m}{2}\left(\frac{1\pm x}{1\mp x}\right)^{m/2}\sum_{k=0}^{m-1}\frac{(k+n)!(m-k-1)!}{k!(n-k)!}\left(\frac{1\mp x}{2}\right)^k$$

$$\pm \frac{(\pm)^n(\mp)^m}{2}\left(\frac{1-x^2}{4}\right)^{m/2}\sum_{k=0}^{n-m}(-)^k\frac{(k+n+m)!\psi(k+1)}{k!(k+m)!(n-m-k)!}\left(\frac{1\mp x}{2}\right)^k$$
$$\pm \frac{(\pm)^n(\mp)^m}{2}\frac{(n+m)!}{(n-m)!}\left(\frac{1\mp x}{1\pm x}\right)^{m/2}\sum_{k=0}^{n}(-)^k\frac{(k+n)!\psi(k+m+1)}{k!(k+m)!(n-k)!}\left(\frac{1\mp x}{2}\right)^k \quad (0\leqslant m\leqslant n). \tag{4.15}$$

$Q_n^{-m}(x)$ is then given by

$$Q_n^{-m}(x)=(-)^m\frac{(n-m)!}{(n+m)!}Q_n^m(x) \quad (0\leqslant m\leqslant n). \tag{4.16}$$

References

[1] Yu.A. Brychkov, On the derivatives of the Legendre functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ with respect to μ and ν, Integral Transforms Spec. Funct. 21 (2010) 175.
[2] H.S. Cohl, Derivatives with respect to the degree and order of associated Legendre functions for $|z|>1$ using modified Bessel functions, Integral Transforms Spec. Funct. 21 (2010) 581.
[3] H.S. Cohl, On parameter differentiation for integral representations of associated Legendre functions, SIGMA Symmetry Integrability Geom. Methods Appl. 7 (2011) 050.
[4] R. Szmytkowski, On the derivative of the associated Legendre function of the first kind of integer degree with respect to its order (with applications to the construction of the associated Legendre function of the second kind of integer degree and order), J. Math. Chem. 46 (2009) 231.
[5] R. Szmytkowski, On the derivative of the associated Legendre function of the first kind of integer order with respect to its degree (with applications to the construction of the associated Legendre function of the second kind of integer degree and order), J. Math. Chem. 49 (2011) 1436.
[6] R. Szmytkowski, Green's function for the wavized Maxwell fish-eye problem, J. Phys. A 44 (2011) 065203.
[7] E.W. Hobson, The Theory of Spherical and Ellipsoidal Harmonics, Cambridge University Press, Cambridge, 1931 [reprinted: Chelsea, New York, 1955].
[8] A. Erdélyi (Ed.), Higher Transcendental Functions, vol. 1, McGraw–Hill, New York, 1953.
[9] M. Abramowitz, I.A. Stegun (Eds.), Handbook of Mathematical Functions, Dover, New York, 1965.
[10] W. Magnus, F. Oberhettinger, R.P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, 3rd ed., Springer, Berlin, 1966.
[11] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, 7th ed., Elsevier, Amsterdam, 2007.
[12] F.W. Olver, D.W. Lozier, R.F. Boisvert, C.W. Clark (Eds.), NIST Handbook of Mathematical Functions, Cambridge University Press, Cambridge, 2010.
[13] L. Robin, Fonctions Sphériques de Legendre et Fonctions Sphéroïdales, vol. 1, Gauthier-Villars, Paris, 1957.
[14] L. Robin, Fonctions Sphériques de Legendre et Fonctions Sphéroïdales, vol. 2, Gauthier-Villars, Paris, 1958.
[15] L. Robin, Fonctions Sphériques de Legendre et Fonctions Sphéroïdales, vol. 3, Gauthier-Villars, Paris, 1959.
[16] E.W. Barnes, On generalized Legendre functions, Quart. J. Pure Appl. Math. 39 (1907) 97.
[17] R. Szmytkowski, On the derivative of the Legendre function of the first kind with respect to its degree, J. Phys. A 39 (2006) 15147. Erratum: J. Phys. A 40 (2007) 7819.
[18] R. Szmytkowski, Addendum to 'On the derivative of the Legendre function of the first kind with respect to its degree', J. Phys. A 40 (2007) 14887. We note parenthetically that the simplest way to arrive at Eq. (5) in that paper is to differentiate both sides of the identity $\mathrm{d}^k z^\alpha/\mathrm{d}z^k=[\Gamma(\alpha+1)/\Gamma(\alpha-k+1)]z^{\alpha-k}$ with respect to α.
[19] S.A. Schelkunoff, Theory of antennas of arbitrary size and shape, Proc. IRE 29 (1941) 493 [reprinted: Proc. IEEE 72 (1984) 1165]. Notice that Schelkunoff used a definition of the digamma function different from that adopted in Ref. [17] and in the present paper.