

## An algorithm for listing all minimal 2-dominating sets of a tree

Marcin Krzywkowski\*

Faculty of Electronics, Telecommunications and Informatics  
Gdańsk University of Technology  
Poland  
[marcin.krzywkowski@gmail.com](mailto:marcin.krzywkowski@gmail.com)

**Abstract.** We provide an algorithm for listing all minimal 2-dominating sets of a tree of order  $n$  in time  $\mathcal{O}(1.3248^n)$ . This implies that every tree has at most  $1.3248^n$  minimal 2-dominating sets. We also show that this bound is tight.

**Keywords:** domination, 2-domination, minimal 2-dominating set, tree, combinatorial bound, exponential algorithm, listing algorithm

### 1 Introduction

Let  $G = (V, E)$  be a graph. The order of a graph is the number of its vertices. By the neighborhood of a vertex  $v$  of  $G$  we mean the set  $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ . The degree of a vertex  $v$ , denoted by  $d_G(v)$ , is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. The distance between two vertices of a graph is the number of edges in a shortest path connecting them. The eccentricity of a vertex is the greatest distance between it and any other vertex. The diameter of a graph  $G$ , denoted by  $\text{diam}(G)$ , is the maximum eccentricity among all vertices of  $G$ . Denote by  $P_n$  a path on  $n$  vertices. By a star we mean a connected graph in which exactly one vertex has degree greater than one.

A subset  $D \subseteq V(G)$  is a dominating set of  $G$  if every vertex of  $V(G) \setminus D$  has a neighbor in  $D$ , while it is a 2-dominating set of  $G$  if every vertex of  $V(G) \setminus D$  has at least two neighbors in  $D$ . A dominating (2-dominating, respectively) set  $D$  is minimal if no proper subset of  $D$  is a dominating (2-dominating, respectively) set of  $G$ . A minimal 2-dominating set is abbreviated as m2ds. Note that 2-domination is a type of multiple domination in which each vertex, which is not in the dominating set, is dominated at least  $k$  times for a fixed positive integer  $k$ . Multiple domination was introduced by Fink and Jacobson [7], and further studied for example in [2, 10, 18]. For a comprehensive survey of domination in graphs, see [11, 12].

---

\* The research was supported by the Polish National Science Centre grant 2011/02/A/ST6/04404.

**Observation 1** *Every leaf of a graph  $G$  is in every 2-dominating set of  $G$ .*

One of the typical questions in graph theory is how many subgraphs of a given property can a graph on  $n$  vertices have. For example, the famous Moon and Moser theorem [17] says that every graph on  $n$  vertices has at most  $3^{n/3}$  maximal independent sets.

Combinatorial bounds are of interest not only on their own, but also because they are used for algorithm design as well. Lawler [16] used the Moon-Moser bound on the number of maximal independent sets to construct an  $(1 + \sqrt[3]{3})^n \cdot n^{\mathcal{O}(1)}$  time graph coloring algorithm, which was the fastest one known for twenty-five years. In 2003 Eppstein [6] reduced the running time of a graph coloring to  $\mathcal{O}(2.4151^n)$ . In 2006 the running time was reduced [1, 14] to  $\mathcal{O}(2^n)$ . For an overview of the field, see [9].

Fomin et al. [8] constructed an algorithm for listing all minimal dominating sets of a graph on  $n$  vertices in time  $\mathcal{O}(1.7159^n)$ . There were also given graphs ( $n/6$  disjoint copies of the octahedron) having  $15^{n/6} \approx 1.5704^n$  minimal dominating sets. This establishes a lower bound on the running time of an algorithm for listing all minimal dominating sets of a given graph.

The number of maximal independent sets in trees was investigated in [19]. Couturier et al. [5] considered minimal dominating sets in various classes of graphs. The authors of [13] investigated the enumeration of minimal dominating sets in graphs.

Bród and Skupień [3] gave bounds on the number of dominating sets of a tree. They also characterized the extremal trees. The authors of [4] investigated the number of minimal dominating sets in trees containing all leaves.

In [15] an algorithm was given for listing all minimal dominating sets of a tree of order  $n$  in time  $\mathcal{O}(1.4656^n)$ , implying that every tree has at most  $1.4656^n$  minimal dominating sets. An infinite family of trees for which the number of minimal dominating sets exceeds  $1.4167^n$  was also given. This established a lower bound on the running time of an algorithm for listing all minimal dominating sets of a given tree.

We provide an algorithm for listing all minimal 2-dominating sets of a tree of order  $n$  in time  $\mathcal{O}(1.3248^n)$ . This implies that every tree has at most  $1.3248^n$  minimal 2-dominating sets. We also show that this bound is tight.

## 2 Results

We describe an algorithm for listing all minimal 2-dominating sets of a given input tree. We prove that the running time of the algorithm is  $\mathcal{O}(1.3248^n)$ , implying that every tree has at most  $1.3248^n$  minimal 2-dominating sets.

**Theorem 2** *Every tree  $T$  of order  $n$  has at most  $\alpha^n$  minimal 2-dominating sets, where  $\alpha \approx 1.32472$  is the positive solution of the equation  $x^3 - x - 1 = 0$ .*

*Proof.* In our algorithm, the iterator of the solutions for a tree  $T$  is denoted by  $\mathcal{F}(T)$ . To obtain the upper bound on the number of minimal 2-dominating



sets of a tree, we prove that the algorithm lists these sets in time  $\mathcal{O}(1.3248^n)$ . Notice that the diameter of a tree can easily be determined in polynomial time. If  $\text{diam}(T) = 0$ , then  $T = P_1 = v_1$ . Let  $\mathcal{F}(T) = \{\{v_1\}\}$ . Obviously,  $\{v_1\}$  is the only m2ds of the path  $P_1$ . We have  $n = 1$  and  $|\mathcal{F}(T)| = 1$ . We also have  $1 < \alpha$ . If  $\text{diam}(T) = 1$ , then  $T = P_2 = v_1v_2$ . Let  $\mathcal{F}(T) = \{\{v_1, v_2\}\}$ . It is easy to observe that  $\{v_1, v_2\}$  is the only m2ds of the path  $P_2$ . We have  $n = 2$  and  $|\mathcal{F}(T)| = 1$ . Obviously,  $1 < \alpha^2$ . If  $\text{diam}(T) = 2$ , then  $T$  is a star. Denote by  $x$  the support vertex of  $T$ . Let  $\mathcal{F}(T) = \{V(T) \setminus \{x\}\}$ . It is easy to observe that  $V(T) \setminus \{x\}$  is the only m2ds of the tree  $T$ . We have  $n \geq 3$  and  $|\mathcal{F}(T)| = 1$ . Obviously,  $1 < \alpha^n$ .

Now consider trees  $T$  with  $\text{diam}(T) \geq 3$ . The results we obtain by the induction on the number  $n$ . Assume that they are true for every tree  $T'$  of order  $n' < n$ . The tree  $T$  can easily be rooted at a vertex  $r$  of maximum eccentricity  $\text{diam}(T)$  in polynomial time. A leaf, say  $t$ , at maximum distance from  $r$ , can also be easily computed in polynomial time. Let  $v$  denote the parent of  $t$  and let  $u$  denote the parent of  $v$  in the rooted tree. If  $\text{diam}(T) \geq 4$ , then let  $w$  denote the parent of  $u$ . By  $T_x$  we denote the subtree induced by a vertex  $x$  and its descendants in the rooted tree  $T$ .

If  $d_T(v) \geq 3$ , then let  $T' = T - T_v$  and let  $T''$  differ from  $T'$  only in that it has the vertex  $v$ . Let  $\mathcal{F}(T)$  be as follows,

$$\{D' \cup V(T_v) \setminus \{v\}: D' \in \mathcal{F}(T')\} \\ \cup \{D'' \cup V(T_v) \setminus \{v\}: D'' \in \mathcal{F}(T'') \text{ and } D'' \setminus \{v\} \notin \mathcal{F}(T')\}.$$

Let us observe that all elements of  $\mathcal{F}(T)$  are minimal 2-dominating sets of the tree  $T$ . Now let  $D$  be any m2ds of  $T$ . Observation 1 implies that  $V(T_v) \setminus \{v\} \subseteq D$ . If  $v \notin D$ , then observe that  $D \cap V(T')$  is an m2ds of the tree  $T'$ . By the inductive hypothesis we have  $D \cap V(T') \in \mathcal{F}(T')$ . Now assume that  $v \in D$ . It is easy to observe that  $D \cap V(T'')$  is an m2ds of the tree  $T''$ . By the inductive hypothesis we have  $D \cap V(T'') \in \mathcal{F}(T'')$ . The set  $D \cap V(T')$  is not an m2ds of the tree  $T'$ , otherwise  $D \setminus \{v\}$  is a 2-dominating set of the tree  $T$ , a contradiction to the minimality of  $D$ . By the inductive hypothesis we have  $D \cap V(T') \notin \mathcal{F}(T')$ . Therefore  $\mathcal{F}(T)$  contains all minimal 2-dominating sets of the tree  $T$ . Now we get  $|\mathcal{F}(T)| = |\mathcal{F}(T')| + |\{D'' \in \mathcal{F}(T''): D'' \setminus \{v\} \notin \mathcal{F}(T')\}| \leq |\mathcal{F}(T')| + |\mathcal{F}(T'')| \leq \alpha^{n-3} + \alpha^{n-2} = \alpha^{n-3}(\alpha + 1) = \alpha^{n-3} \cdot \alpha^3 = \alpha^n$ .

If  $d_T(v) = 2$  and  $d_T(u) \geq 3$ , then let  $T' = T - T_v$ ,  $T'' = T - T_u$ , and

$$\mathcal{F}(T) = \{D' \cup \{t\}: u \in D' \in \mathcal{F}(T')\} \cup \{D'' \cup V(T_u) \setminus \{u\}: D'' \in \mathcal{F}(T'')\}.$$

Let us observe that all elements of  $\mathcal{F}(T)$  are minimal 2-dominating sets of the tree  $T$ . Now let  $D$  be any m2ds of  $T$ . By Observation 1 we have  $t \in D$ . If  $v \notin D$ , then  $u \in D$  as the vertex  $v$  has to be dominated twice. Observe that  $D \setminus \{t\}$  is an m2ds of the tree  $T'$ . By the inductive hypothesis we have  $D \setminus \{t\} \in \mathcal{F}(T')$ . Now assume that  $v \in D$ . We have  $u \notin D$ , otherwise  $D \setminus \{v\}$  is a 2-dominating set of the tree  $T$ , a contradiction to the minimality of  $D$ . Observe that  $D \cap V(T'')$  is an m2ds of the tree  $T''$ . By the inductive hypothesis we have  $D \cap V(T'') \in \mathcal{F}(T'')$ . Therefore  $\mathcal{F}(T)$  contains all minimal 2-dominating sets of the tree  $T$ . Now we get



$$|\mathcal{F}(T)| = |\{D' \in \mathcal{F}(T') : u \in D'\}| + |\mathcal{F}(T'')| \leq |\mathcal{F}(T')| + |\mathcal{F}(T'')| \leq \alpha^{n-2} + \alpha^{n-3} = \alpha^{n-3}(\alpha + 1) = \alpha^{n-3} \cdot \alpha^3 = \alpha^n.$$

If  $d_T(v) = d_T(u) = 2$ , then let  $T' = T - T_v$ ,  $T'' = T - T_u$ , and

$$\mathcal{F}(T) = \{D' \cup \{t\} : D' \in \mathcal{F}(T')\} \cup \{D'' \cup \{v, t\} : w \in D'' \in \mathcal{F}(T'')\}.$$

Let us observe that all elements of  $\mathcal{F}(T)$  are minimal 2-dominating sets of the tree  $T$ . Now let  $D$  be any m2ds of  $T$ . By Observation 1 we have  $t \in D$ . If  $v \notin D$ , then observe that  $D \setminus \{t\}$  is an m2ds of the tree  $T'$ . By the inductive hypothesis we have  $D \setminus \{t\} \in \mathcal{F}(T')$ . Now assume that  $v \in D$ . We have  $u \notin D$ , otherwise  $D \setminus \{v\}$  is a 2-dominating set of the tree  $T$ , a contradiction to the minimality of  $D$ . Moreover, we have  $w \in D$  as the vertex  $u$  has to be dominated twice. Observe that  $D \setminus \{v, t\}$  is an m2ds of the tree  $T''$ . By the inductive hypothesis we have  $D \setminus \{v, t\} \in \mathcal{F}(T'')$ . Therefore  $\mathcal{F}(T)$  contains all minimal 2-dominating sets of the tree  $T$ . Now we get  $|\mathcal{F}(T)| = |\mathcal{F}(T')| + |\{D'' \in \mathcal{F}(T'') : w \in D''\}| \leq |\mathcal{F}(T')| + |\mathcal{F}(T'')| \leq \alpha^{n-2} + \alpha^{n-3} = \alpha^{n-3}(\alpha + 1) = \alpha^{n-3} \cdot \alpha^3 = \alpha^n$ .

We now show that paths attain the bound from the previous theorem.

**Proposition 3** *For positive integers  $n$ , let  $a_n$  denote the number of minimal 2-dominating sets of the path  $P_n$ . We have*

$$a_n = \begin{cases} 1 & \text{if } n \leq 3; \\ a_{n-3} + a_{n-2} & \text{if } n \geq 4. \end{cases}$$

*Proof.* It is easy to see that a path on at most three vertices has exactly one minimal 2-dominating set. Now assume that  $n \geq 4$ . Let  $T' = T - v_n - v_{n-1}$  and  $T'' = T' - v_{n-2}$ . It follows from the last paragraph of the proof of Theorem 2 that  $a_n = a_{n-3} + a_{n-2}$ .

Solving the recurrence  $a_n = a_{n-3} + a_{n-2}$ , we get  $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \alpha$ , where  $\alpha \approx 1.3247$  is the positive solution of the equation  $x^3 - x - 1 = 0$ . This implies that the bound from Theorem 2 is tight.

It is an open problem to prove the tightness of an upper bound on the number of minimal dominating sets of a tree. In [15] it has been proved that any tree of order  $n$  has less than  $1.4656^n$  minimal dominating sets. A family of trees having more than  $1.4167^n$  minimal dominating sets has also been given.

## References

1. A. Björklund and T. Husfeldt, *Inclusion-exclusion algorithms for counting set partitions*, Proceedings of FOCS 2006, 575–582.
2. M. Blidia, O. Favaron, and R. Lounes, *Locating-domination, 2-domination and independence in trees*, Australasian Journal of Combinatorics 42 (2008), 309–316.
3. D. Bród and Z. Skupień, *Trees with extremal numbers of dominating sets*, Australasian Journal of Combinatorics 35 (2006), 273–290.
4. D. Bród, A. Włoch, and I. Włoch, *On the number of minimal dominating sets including the set of leaves in trees*, International Journal of Contemporary Mathematical Sciences 4(2009), 1739–1748.



5. J.-F. Couturier, P. Heggernes, P. van 't Hof, and D. Kratsch, *Minimal dominating sets in graph classes: combinatorial bounds and enumeration*, Proceedings of SOFSEM 2012, 202–213, Lecture Notes in Computer Science, 7147, Springer-Verlag, Berlin, 2012.
6. D. Eppstein, *Small maximal independent sets and faster exact graph coloring*, Journal of Graph Algorithms and Applications 7 (2003), 131–140.
7. J. Fink and M. Jacobson, *n-domination in graphs*, Graph Theory with Applications to Algorithms and Computer Science, Wiley, New York, 1985, 282–300.
8. F. Fomin, F. Grandoni, A. Pyatkin, and A. Stepanov, *Combinatorial bounds via measure and conquer: bounding minimal dominating sets and applications*, ACM Transactions on Algorithms 5 (2009), article 9, 17 pp.
9. F. Fomin and D. Kratsch, *Exact Exponential Algorithms*, Springer, Berlin, 2010.
10. J. Fujisawa, A. Hansberg, T. Kubo, A. Saito, M. Sugita, and L. Volkmann, *Independence and 2-domination in bipartite graphs*, Australasian Journal of Combinatorics 40 (2008), 265–268.
11. T. Haynes, S. Hedetniemi, and P. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.
12. T. Haynes, S. Hedetniemi, and P. Slater (eds.), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.
13. M. Kanté, V. Limouzy, A. Mary, and L. Nourine, *Enumeration of minimal dominating sets and variants*, Fundamentals of computation theory, 298–309, Lecture Notes in Computer Science, 6914, Springer, Heidelberg, 2011.
14. M. Koivisto, *An  $\mathcal{O}(2^n)$  algorithm for graph coloring and other partitioning problems via inclusion-exclusion*, Proceedings of the 47th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2006), IEEE, 2006, 583–590.
15. M. Krzywkowski, *Trees having many minimal dominating sets*, Information Processing Letters 113 (2013), 276–279.
16. E. Lawler, *A note on the complexity of the chromatic number problem*, Information Processing Letters 5 (1976), 66–67.
17. J. Moon and L. Moser, *On cliques in graphs*, Israel Journal of Mathematics 3 (1965), 23–28.
18. R. Shaheen, *Bounds for the 2-domination number of toroidal grid graphs*, International Journal of Computer Mathematics 86 (2009), 584–588.
19. H. Wilf, *The number of maximal independent sets in a tree*, SIAM Journal on Algebraic and Discrete Methods 7 (1986), 125–130.

