C. R. Acad. Sci. Paris, Ser. I 352 (2014) 363–366

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Combinatorics

# Bounds on the vertex–edge domination number of a tree

## *Bornes sur le nombre de domination sommet–arête d'un arbre*

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ARTICLE INFO

*Article history:*
Received 22 November 2013
Accepted after revision 18 March 2014
Available online 3 April 2014

Presented by the Editorial Board

ABSTRACT

A vertex–edge dominating set of a graph $G$ is a set $D$ of vertices of $G$ such that every edge of $G$ is incident with a vertex of $D$ or a vertex adjacent to a vertex of $D$. The vertex–edge domination number of a graph $G$, denoted by $\gamma_{ve}(T)$, is the minimum cardinality of a vertex–edge dominating set of $G$. We prove that for every tree $T$ of order $n \geqslant 3$ with $l$ leaves and $s$ support vertices, we have $(n-l-s+3)/4 \leqslant \gamma_{ve}(T) \leqslant n/3$, and we characterize the trees attaining each of the bounds.



RÉSUMÉ

Un ensemble sommet–arête dominant d'un graphe $G$ est un ensemble $D$ de sommets de $G$ tel que chaque arête de $G$ soit incidente à un sommet de $D$ ou à un sommet adjacent à un sommet de $D$. Le nombre de domination sommet–arête d'un graphe $G$, noté $\gamma_{ve}(T)$, est le cardinal minimum d'un ensemble sommet–arête dominant de $G$. Nous prouvons que, pour chaque arbre $T$ d'ordre $n \geqslant 3$ avec $l$ feuilles et des sommets $s$ de soutien, que nous avons $(n-l-s+3)/4 \leqslant \gamma_{ve}(T) \leqslant n/3$, et nous caractérisons les arbres atteignant chacune des limites.



### 1. Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex $v$ of $G$ we mean the set $N_G(v) = \{u \in V(G)\colon\ uv \in E(G)\}$. The degree of a vertex $v$, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). We denote by $P_n$ the path on $n$ vertices. Let $T$ be a tree, and let $v$ be a vertex of $T$. We say that $v$ is adjacent to a path $P_n$ if there is a neighbor of $v$, say $x$, such that the subtree

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[2] Research partially supported by the Polish National Science Centre grant 2011/02/A/ST6/00201.

http://dx.doi.org/10.1016/j.crma.2014.03.017
1631-073X/

resulting from $T$ by removing the edge $vx$ and which contains the vertex $x$ as a leaf, is a path $P_n$. By a star we mean a connected graph in which exactly one vertex has degree greater than one.

A subset $D \subseteq V(G)$ is a dominating set of $G$ if every vertex of $V(G) \setminus D$ has a neighbor in $D$. The domination number of $G$ is the minimum cardinality of a dominating set of $G$. For a comprehensive survey of domination in graphs, see [2,3].

An edge $e \in E(G)$ is vertex–edge dominated by a vertex $v \in V(G)$ if $e$ is incident to $v$, or $e$ is adjacent to an edge incident to $v$. A subset $D \subseteq V(G)$ is a vertex–edge dominating set, abbreviated VEDS, of $G$ if every edge of $G$ is vertex–edge dominated by a vertex of $D$. The vertex–edge domination number of $G$, denoted by $\gamma_{ve}(G)$, is the minimum cardinality of a vertex–edge dominating set of $G$. A vertex–edge dominating set of $G$ of minimum cardinality is called a $\gamma_{ve}(G)$-set. Vertex–edge domination in graphs was introduced in [7], and further studied in [6].

Chellali and Haynes [1] established the following lower bound on the total domination number of a tree. For every tree $T$ of order $n$ with $l$ leaves, we have $\gamma_t(T) \geqslant (n-l+2)/2$. They also characterized the extremal trees. In [4] a lower bound on the total outer-independent domination number of a tree was given together with the characterization of the extremal trees. Lemańska [5] proved that the domination number of a tree is bounded below by $(n-l+2)/3$.

We prove the following bounds on the vertex–edge domination number of a tree $T$ of order $n \geqslant 3$ with $l$ leaves and $s$ support vertices, $(n-l-s+3)/4 \leqslant \gamma_{ve}(T) \leqslant n/3$. We also characterize the trees attaining each of the bounds.

## 2. Results

We begin with the following straightforward observation.

**Observation 1.** *For every connected graph $G$ of diameter at least two, there exists a $\gamma_{ve}(G)$-set that contains no leaf.*

First we show that if $T$ is a nontrivial tree of order $n$ with $l$ leaves and $s$ support vertices, then $\gamma_{ve}(T)$ is bounded below by $(n-l-s+3)/4$. For the purpose of characterizing the trees attaining this bound, we introduce a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let $T_1$ be a path $P_5$. If $k$ is a positive integer, then $T_{k+1}$ can be obtained recursively from $T_k$ by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a vertex by joining it to any support vertex of $T_k$.
- Operation $\mathcal{O}_2$: Attach a path $P_2$ by joining one of its vertices to a vertex of $T_k$, which is not a leaf and is adjacent to a support vertex of degree two.
- Operation $\mathcal{O}_3$: Attach a path $P_4$ by joining one of its leaves to a leaf of $T_k$ adjacent to a weak support vertex.

We now prove that for every tree $T$ of the family $\mathcal{T}$ we have $\gamma_{ve}(T) = (n-l-s+3)/4$.

**Lemma 2.** *If $T \in \mathcal{T}$, then $\gamma_{ve}(T) = (n-l-s+3)/4$.*

**Proof.** We use the induction on the number $k$ of operations performed to construct the tree $T$. If $T = T_1 = P_5$, then $(n-l-s+3)/4 = (5-2-2+3)/4 = 1 = \gamma_{ve}(T)$. Let $k$ be a positive integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k-1$ operations. Let $n'$ be the order of the tree $T'$, $l'$ the number of its leaves, and $s'$ the number of support vertices. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by $k$ operations.

First assume that $T$ is obtained from $T'$ by operation $\mathcal{O}_1$. We have $n = n'+1$, $l = l'+1$ and $s = s'$. It is straightforward to see that any $\gamma_{ve}(T')$-set is a VEDS of the tree $T$. Thus $\gamma_{ve}(T) \leqslant \gamma_{ve}(T')$. Obviously, $\gamma_{ve}(T') \leqslant \gamma_{ve}(T)$. This implies that $\gamma_{ve}(T) = \gamma_{ve}(T')$. We now get $\gamma_{ve}(T) = \gamma_{ve}(T') = (n'-l'-s'+3)/4 = (n-1-l+1-s+3)/4 = (n-l-s+3)/4$.

Now assume that $T$ is obtained from $T'$ by operation $\mathcal{O}_2$. We have $n = n'+2$, $l = l'+1$ and $s = s'+1$. It is straightforward to see that any $\gamma_{ve}(T')$-set is a VEDS of the tree $T$. Thus $\gamma_{ve}(T) \leqslant \gamma_{ve}(T')$. This implies that $\gamma_{ve}(T) = \gamma_{ve}(T')$. We now get $\gamma_{ve}(T) = \gamma_{ve}(T') = (n'-l'-s'+3)/4 = (n-2-l+1-s+1+3)/4 = (n-l-s+3)/4$.

Now assume that $T$ is obtained from $T'$ by operation $\mathcal{O}_3$. We have $n = n'+4$, $l = l'$ and $s = s'$. We denote by $x$ the leaf to which $P_4$ is attached. Let $v_1v_2v_3v_4$ be the attached path. Let $v_1$ be joined to $x$. Let $D'$ be any $\gamma_{ve}(T')$-set. It is easy to see that $D' \cup \{v_2\}$ is a VEDS of the tree $T$. Thus $\gamma_{ve}(T) \leqslant \gamma_{ve}(T')+1$. Now let us observe that there exists a $\gamma_{ve}(T)$-set that does not contain the vertices $v_4$, $v_3$, and $v_1$. Let $D$ be such a set. To dominate the edge $v_3v_4$, we have $v_2 \in D$. Observe that $D \setminus \{v_2\}$ is a VEDS of the tree $T'$. Therefore $\gamma_{ve}(T') \leqslant \gamma_{ve}(T)-1$. We now conclude that $\gamma_{ve}(T) = \gamma_{ve}(T')+1$. We get $\gamma_{ve}(T) = \gamma_{ve}(T')+1 = (n'-l'-s'+3)/4+1 = (n-4-l-s+7)/4 = (n-l-s+3)/4$. □

We now give a lower bound on the vertex–edge domination number of a tree together with the characterization of the extremal trees.

**Theorem 3.** *If $T$ is a nontrivial tree of order $n$ with $l$ leaves and $s$ support vertices, then $\gamma_{ve}(T) \geqslant (n-l-s+3)/4$ with equality if and only if $T \in \mathcal{T}$.*

**Proof.** If $\mathrm{diam}(T) = 1$, then $T = P_2$. We have $(n-l-s+3)/4 = (2-2-2+3)/4 < 1 = \gamma_{ve}(T)$. If $\mathrm{diam}(T) = 2$, then $T$ is a star. We have $l = n-1$ and $s = 1$. Consequently, $(n-l-s+3)/4 = (n-n+1-1+3)/4 = 3/4 < 1 = \gamma_{ve}(T)$.

Now assume that $\mathrm{diam}(T) \geqslant 3$. Thus the order $n$ of the tree $T$ is at least four. We obtain the result by the induction on the number $n$. Assume that the theorem is true for every tree $T'$ of order $n' < n$ with $l'$ leaves and $s'$ support vertices.

First assume that some support vertex of $T$, say $x$, is strong. Let $y$ be a leaf adjacent to $x$. Let $T' = T - y$. We have $n' = n - 1$, $l' = l - 1$ and $s' = s$. Obviously, $\gamma_{\mathrm{ve}}(T') \leqslant \gamma_{\mathrm{ve}}(T)$. We get $\gamma_{\mathrm{ve}}(T) \geqslant \gamma_{\mathrm{ve}}(T') \geqslant (n' - l' - s' + 3)/4 = (n - 1 - l + 1 - s + 3)/4 = (n - l - s + 3)/4$. If $\gamma_{\mathrm{ve}}(T) = (n - l - s + 3)/4$, then obviously $\gamma_{\mathrm{ve}}(T') = (n' - l' - s' + 3)/4$. By the inductive hypothesis, we have $T' \in \mathcal{T}$. The tree $T$ can be obtained from $T'$ by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$. Henceforth, we can assume that every support vertex of $T$ is weak.

We now root $T$ at a vertex $r$ of maximum eccentricity $\mathrm{diam}(T)$. Let $t$ be a leaf at maximum distance from $r$, $v$ be the parent of $t$, and $u$ be the parent of $v$ in the rooted tree. If $\mathrm{diam}(T) \geqslant 4$, then let $w$ be the parent of $u$. If $\mathrm{diam}(T) \geqslant 5$, then let $d$ be the parent of $w$. If $\mathrm{diam}(T) \geqslant 6$, then let $e$ be the parent of $d$. By $T_x$ we denote the subtree induced by a vertex $x$ and its descendants in the rooted tree $T$.

Assume that some child of $u$, say $x$, is a leaf. Let $T' = T - x$. We have $n' = n - 1$, $l' = l - 1$ and $s' = s - 1$. We get $\gamma_{\mathrm{ve}}(T) \geqslant \gamma_{\mathrm{ve}}(T') \geqslant (n' - l' - s' + 3)/4 = (n - 1 - l + 1 - s + 1 + 3)/4 > (n - l - s + 3)/4$.

Now assume that among the children of $u$ there is a support vertex other than $v$. Let $T' = T - T_v$. We have $n' = n - 2$, $l' = l - 1$ and $s' = s - 1$. We get $\gamma_{\mathrm{ve}}(T) \geqslant \gamma_{\mathrm{ve}}(T') \geqslant (n' - l' - s' + 3)/4 = (n - 2 - l + 1 - s + 1 + 3)/4 = (n - l - s + 3)/4$. If $\gamma_{\mathrm{ve}}(T) = (n - l - s + 3)/4$, then obviously $\gamma_{\mathrm{ve}}(T) = (n' - l' - s' + 3)/4$. By the inductive hypothesis, we have $T' \in \mathcal{T}$. The tree $T$ can be obtained from $T'$ by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$.

Now assume that $d_T(u) = 2$. Assume that $d_T(w) \geqslant 3$. First assume that some child of $w$, say $x$, is a leaf. Let $T' = T - x$. We have $n' = n - 1$, $l' = l - 1$ and $s' = s - 1$. We get $\gamma_{\mathrm{ve}}(T) \geqslant \gamma_{\mathrm{ve}}(T') \geqslant (n' - l' - s' + 3)/4 = (n - 1 - l + 1 - s + 1 + 3)/4 > (n - l - s + 3)/4$.

Now assume that no child of $w$ is a leaf. Let $T' = T - T_u$. We have $n' = n - 3$, $l' = l - 1$ and $s' = s - 1$. Let us observe that there exists a $\gamma_{\mathrm{ve}}(T)$-set that does not contain the vertices $t$ and $v$. Let $D$ be such a set. To dominate the edge $vt$, we have $u \in D$. Let us observe that $D \setminus \{u\}$ is a VEDS of the tree $T'$. Therefore $\gamma_{\mathrm{ve}}(T') \leqslant \gamma_{\mathrm{ve}}(T) - 1$. We now get $\gamma_{\mathrm{ve}}(T) \geqslant \gamma_{\mathrm{ve}}(T') + 1 \geqslant (n' - l' - s' + 3)/4 + 1 = (n - 3 - l + 1 - s + 1 + 7)/4 > (n - l - s + 3)/4$.

If $d_T(w) = 1$, then $T = P_4$. We have $(n - l - s + 3)/4 = (4 - 2 - 2 + 3)/4 < 1 = \gamma_{\mathrm{ve}}(T)$. Now assume that $d_T(w) = 2$. First assume that $d_T(d) \geqslant 3$. Let $T' = T - T_w$. We have $n' = n - 4$, $l' = l - 1$ and $s' = s - 1$. Let us observe that there exists a $\gamma_{\mathrm{ve}}(T)$-set that does not contain the vertices $t$, $v$ and $w$. Let $D$ be such a set. To dominate the edge $vt$, we have $u \in D$. Observe that $D \setminus \{u\}$ is a VEDS of the tree $T'$. Therefore $\gamma_{\mathrm{ve}}(T') \leqslant \gamma_{\mathrm{ve}}(T) - 1$. We now get $\gamma_{\mathrm{ve}}(T) \geqslant \gamma_{\mathrm{ve}}(T') + 1 \geqslant (n' - l' - s' + 3)/4 + 1 = (n - 4 - l + 1 - s + 1 + 7)/4 > (n - l - s + 3)/4$.

Now assume that $d_T(d) = 2$. First assume that some child of $e$ is a leaf. Let $T' = T - T_w$. We have $n' = n - 4$, $l' = l$ and $s' = s - 1$. Similarly as in the previous possibility, we conclude that $\gamma_{\mathrm{ve}}(T') \leqslant \gamma_{\mathrm{ve}}(T) - 1$. We get $\gamma_{\mathrm{ve}}(T) \geqslant \gamma_{\mathrm{ve}}(T') + 1 \geqslant (n' - l' - s' + 3)/4 + 1 = (n - 4 - l - s + 1 + 7)/4 > (n - l - s + 3)/4$.

Now assume that no child of $e$ is a leaf. Let $T' = T - T_w$. We have $n' = n - 4$, $l' = l$ and $s' = s$. If $n' = 1$, then $T = P_5 = T_1 \in \mathcal{T}$. Assume that $n' \geqslant 2$. Similarly as earlier, we conclude that $\gamma_{\mathrm{ve}}(T') \leqslant \gamma_{\mathrm{ve}}(T) - 1$. We now get $\gamma_{\mathrm{ve}}(T) \geqslant \gamma_{\mathrm{ve}}(T') + 1 \geqslant (n' - l' - s' + 3)/4 + 1 = (n - 4 - l - s + 7)/4 = (n - l - s + 3)/4$. If $\gamma_{\mathrm{ve}}(T) = (n - l - s + 3)/4$, then obviously $\gamma_{\mathrm{ve}}(T') = (n' - l' - s' + 3)/4$. By the inductive hypothesis, we have $T' \in \mathcal{T}$. The tree $T$ can be obtained from $T'$ by operation $\mathcal{O}_3$. $\square$

Next we show that if $T$ is a tree of order $n \geqslant 3$, then $\gamma_{\mathrm{ve}}(T)$ is bounded above by $n/3$. For the purpose of characterizing the trees attaining this bound, we introduce a family $\mathcal{F}$ of trees $T = T_k$ that can be obtained as follows. Let $T_1$ be a path $P_3$. If $k$ is a positive integer, then $T_{k+1}$ can be obtained recursively from $T_k$ by attaching a path $P_3$ by joining one of its leaves to a vertex of $T_k$ adjacent to a path $P_2$ or $P_3$.

We now prove that for every tree $T$ of the family $\mathcal{F}$ we have $\gamma_{\mathrm{ve}}(T) = n/3$.

**Lemma 4.** *If $T \in \mathcal{F}$, then $\gamma_{\mathrm{ve}}(T) = n/3$.*

**Proof.** We use the induction on the number $k$ of operations performed to construct the tree $T$. If $T = T_1 = P_3$, then $\gamma_{\mathrm{ve}}(T) = 1 = n/3$. Let $k$ be a positive integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{F}$ constructed by $k-1$ operations. Let $n'$ be the order of the tree $T'$. Let $T = T_{k+1}$ be a tree of the family $\mathcal{F}$ constructed by $k$ operations. We have $n = n' + 3$. We denote by $x$ the vertex to which is attached $P_3$. Let $v_1 v_2 v_3$ be the attached path. Let $v_1$ be adjacent to $x$. Let $D'$ be any $\gamma_{\mathrm{ve}}(T')$-set. It is easy to see that $D' \cup \{v_1\}$ is a VEDS of the tree $T$. Thus $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T') + 1$. If $x$ is adjacent to a path $P_2$, then let us observe that there exists a $\gamma_{\mathrm{ve}}(T)$-set that contains the vertices $v_1$ and $x$. Let $D$ be such a set. The set $D$ is minimal, thus $v_2, v_3 \notin D$. It is easy to observe that $D \setminus \{v_1\}$ is a VEDS of the tree $T'$. If $x$ is adjacent to a path $P_3$ different from $v_1 v_2 v_3$, say $abc$, then let $a$ and $x$ be adjacent. Let us observe that there exists a $\gamma_{\mathrm{ve}}(T)$-set that contains the vertices $v_1$ and $a$. Let $D$ be such a set. The set $D$ is minimal, thus $v_2, v_3 \notin D$. Let us observe that $D \setminus \{v_1\}$ is a VEDS of the tree $T'$. We now conclude that $\gamma_{\mathrm{ve}}(T') \leqslant \gamma_{\mathrm{ve}}(T) - 1$, and consequently, $\gamma_{\mathrm{ve}}(T) = \gamma_{\mathrm{ve}}(T') + 1$. We get $\gamma_{\mathrm{ve}}(T) = \gamma_{\mathrm{ve}}(T') + 1 = n'/3 + 1 = (n - 3)/3 + 1 = n/3$. $\square$

We now give an upper bound on the vertex–edge domination number of a tree together with the characterization of the extremal trees.

**Theorem 5.** *If $T$ is a tree of order $n \geqslant 3$, then $\gamma_{\mathrm{ve}}(T) \leqslant n/3$ with equality if and only if $T \in \mathcal{F}$.*

**Proof.** First assume that $\mathrm{diam}(T) = 2$. Thus $T$ is a star. If $T = P_3$, then $T = T_1 \in \mathcal{F}$. If $T$ is a star different from $P_3$, then we get $n/3 > 1 = \gamma_{\mathrm{ve}}(T)$.

Now assume that $\mathrm{diam}(T) \geqslant 3$. Thus the order $n$ of the tree $T$ is at least four. We obtain the result by the induction on the number $n$. Assume that the theorem is true for every tree $T'$ of order $n' < n$.

First assume that some support vertex of $T$, say $x$, is strong. Let $y$ be a leaf adjacent to $x$. Let $T' = T - y$. It is straightforward to see that any $\gamma_{\mathrm{ve}}(T')$-set is a VEDS of the tree $T$. Thus $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T')$. We now get $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T') \leqslant n'/3 < n/3$.

We now root $T$ at a vertex $r$ of maximum eccentricity $\mathrm{diam}(T)$. Let $t$ be a leaf at maximum distance from $r$, $v$ be the parent of $t$, and $u$ be the parent of $v$ in the rooted tree. If $\mathrm{diam}(T) \geqslant 4$, then let $w$ be the parent of $u$. By $T_x$ we denote the subtree induced by a vertex $x$ and its descendants in the rooted tree $T$.

Assume that some child of $u$, say $x$, is a leaf. Let $T' = T - x$. Let $D'$ be a $\gamma_{\mathrm{ve}}(T')$-set that contains no leaf. It is easy to observe that $D'$ is a VEDS of the tree $T$. Thus $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T')$. We now get $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T') \leqslant n'/3 < n/3$.

Now assume that among the children of $u$ there is a support vertex other than $v$. Let $T' = T - T_v$. Let us observe that there exists a $\gamma_{\mathrm{ve}}(T')$-set that contains the vertex $u$. Let $D'$ be such a set. It is easy to see that $D'$ is a VEDS of the tree $T$. Thus $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T')$. We now get $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T') \leqslant n'/3 < n/3$.

Now assume that $d_T(u) = 2$. First assume that $w$ is adjacent to a leaf, say $x$. Let $T' = T - x$. Let us observe that there exists a $\gamma_{\mathrm{ve}}(T')$-set that contains the vertex $u$. Let $D'$ be such a set. It is easy to see that $D'$ is a VEDS of the tree $T$. Thus $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T')$. We now get $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T') \leqslant n'/3 < n/3$.

Now assume that there is a child of $w$ other than $u$, say $x$, such that the distance of $w$ to the most distant vertex of $T_x$ is two or three. It suffices to consider only the possibilities when $T_x$ is a path $P_2$ or $P_3$. Let $T' = T - T_u$. We have $n' = n - 3$. Let $D'$ be any $\gamma_{\mathrm{ve}}(T')$-set. It is easy to observe that $D' \cup \{u\}$ is a VEDS of the tree $T$. Thus $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T') + 1$. We now get $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T') + 1 \leqslant n'/3 + 1 = n/3$. If $\gamma_{\mathrm{ve}}(T) = n/3$, then obviously $\gamma_{\mathrm{ve}}(T') = n'/3$. By the inductive hypothesis we have $T' \in \mathcal{F}$. The tree $T$ can be obtained from $T'$ by attaching a path $P_3$ by joining one of its leaves to the vertex $u$. Thus $T \in \mathcal{F}$.

Now assume that $d_T(w) = 2$. Let $T' = T - T_w$. We have $n' = n - 4$. If $n' = 1$, then $T = P_5$. We have $\gamma_{\mathrm{ve}}(P_5) = 1 < 5/3$. If $n' = 2$, then $T = P_6$. The path $P_6$ can be obtained from two paths $P_3$ by joining them through leaves. Thus $T \in \mathcal{F}$. Now assume that $n' \geqslant 3$. Let $D'$ be any $\gamma_{\mathrm{ve}}(T')$-set. Let us observe that $D' \cup \{u\}$ is a VEDS of the tree $T$. Thus $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T') + 1$. We now get $\gamma_{\mathrm{ve}}(T) \leqslant \gamma_{\mathrm{ve}}(T') + 1 \leqslant n'/3 + 1 = (n-1)/3 < n/3$. $\square$

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