



## On the partition dimension of trees

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### ABSTRACT

Given an ordered partition  $\Pi = \{P_1, P_2, \dots, P_t\}$  of the vertex set  $V$  of a connected graph  $G = (V, E)$ , the *partition representation* of a vertex  $v \in V$  with respect to the partition  $\Pi$  is the vector  $r(v|\Pi) = (d(v, P_1), d(v, P_2), \dots, d(v, P_t))$ , where  $d(v, P_i)$  represents the distance between the vertex  $v$  and the set  $P_i$ . A partition  $\Pi$  of  $V$  is a *resolving partition* of  $G$  if different vertices of  $G$  have different partition representations, i.e., for every pair of vertices  $u, v \in V$ ,  $r(u|\Pi) \neq r(v|\Pi)$ . The *partition dimension* of  $G$  is the minimum number of sets in any resolving partition of  $G$ . In this paper we obtain several tight bounds on the partition dimension of trees.

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### 1. Introduction

The concepts of resolvability and location in graphs were described independently by Harary and Melter [9] and Slater [17]. After these papers were published several authors developed diverse theoretical works about this topic [3,2,4–10,14,19]. Slater described the usefulness of these ideas into long range aids to navigation [17]. Also, these concepts have some applications in chemistry for representing chemical compounds [12,13] or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [15]. Other applications of this concept to navigation of robots in networks and other areas appear in [5,11,14]. Some variations on resolvability or location have been appearing in the literature, like those about conditional resolvability [16], locating domination [10], resolving domination [1] and resolving partitions [4,7,8,19].

Given a graph  $G = (V, E)$  and an ordered set of vertices  $S = \{v_1, v_2, \dots, v_k\}$  of  $G$ , the *metric representation* of a vertex  $v \in V$  with respect to  $S$  is the vector  $r(v|S) = (d(v, v_1), d(v, v_2), \dots, d(v, v_k))$ , where  $d(v, v_i)$  denotes the distance between the vertices  $v$  and  $v_i$ ,  $1 \leq i \leq k$ . We say that  $S$  is a *resolving set* of  $G$  if different vertices of  $G$  have different metric representations, i.e., for every pair of distinct vertices  $u, v \in V$ ,  $r(u|S) \neq r(v|S)$ . The *metric dimension*<sup>1</sup> of  $G$  is the minimum cardinality of any resolving set of  $G$ , and it is denoted by  $\dim(G)$ . The metric dimension of graphs is studied in [3,2,4–6,18].

Given an ordered partition  $\Pi = \{P_1, P_2, \dots, P_t\}$  of the vertices of  $G$ , the *partition representation* of a vertex  $v \in V$  with respect to the partition  $\Pi$  is the vector  $r(v|\Pi) = (d(v, P_1), d(v, P_2), \dots, d(v, P_t))$ , where  $d(v, P_i)$ , with  $1 \leq i \leq t$ , represents the distance between the vertex  $v$  and the set  $P_i$ , i.e.,  $d(v, P_i) = \min_{u \in P_i} \{d(v, u)\}$ . We say that  $\Pi$  is a *resolving partition* of  $G$  if different vertices of  $G$  have different partition representations, i.e., for every pair of distinct vertices  $u, v \in V$ ,  $r(u|\Pi) \neq r(v|\Pi)$ .

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<sup>1</sup> Also called the locating number.

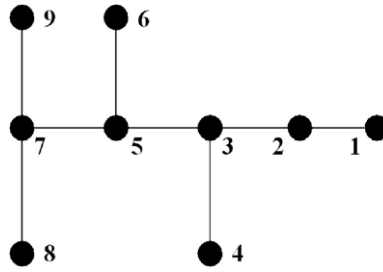


Fig. 1. In this tree the vertex 3 is an exterior major vertex of terminal degree two: 1 and 4 are terminal vertices of 3.

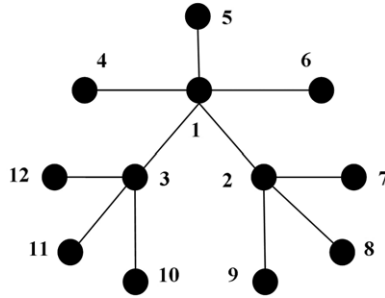


Fig. 2.  $\Pi = \{\{1, 4, 9, 12\}, \{3, 5, 8, 11\}, \{2, 6, 7, 10\}\}$  is a resolving partition.

$r(v|\Pi)$ . The *partition dimension* of  $G$  is the minimum number of sets in any resolving partition of  $G$  and it is denoted by  $pd(G)$ . The partition dimension of graphs is studied in [4,7,8,18].

## 2. The partition dimension of trees

It is natural to think that the partition dimension and metric dimension are related; in [7] it was shown that for any nontrivial connected graph  $G$  we have

$$pd(G) \leq \dim(G) + 1. \tag{1}$$

We know that the partition dimension of any path is two. That is, for any path graph  $P$ , it follows  $pd(P) = \dim(P) + 1 = 2$ . A formula for the dimension of trees that are not paths has been established in [5,9,17]. In order to present this formula, we need additional definitions. A vertex of degree at least 3 in a tree  $T$  will be called a *major vertex* of  $T$ . Any leaf  $u$  of  $T$  is said to be a *terminal vertex* of a major vertex  $v$  of  $T$  if  $d(u, v) < d(u, w)$  for every other major vertex  $w$  of  $T$ . The *terminal degree* of a major vertex  $v$  is the number of terminal vertices of  $v$ . A major vertex  $v$  of  $T$  is an *exterior major vertex* of  $T$  if it has positive terminal degree.

Let  $n_1(T)$  denote the number of leaves of  $T$ , and let  $ex(T)$  denote the number of exterior major vertices of  $T$ . We can now state the formula for the dimension of a tree [5,9,17]: if  $T$  is a tree that is not a path, then

$$\dim(T) = n_1(T) - ex(T). \tag{2}$$

As a consequence, if  $T$  is a tree that is not a path, then

$$pd(T) \leq n_1(T) - ex(T) + 1. \tag{3}$$

The above bound is tight, it is achieved for the graph in Fig. 1 where  $\Pi = \{\{8\}, \{4, 9\}, \{1, 2, 3, 5, 6, 7\}\}$  is a resolving partition and  $pd(T) = 3$ . However, there are graphs for which the following bound gives better result than bound (3), for instance, the graph in Fig. 2.

Let  $S = \{s_1, s_2, \dots, s_\kappa\}$  be the set of exterior major vertices of  $T = (V, E)$  with terminal degree greater than one; let  $\{s_{i1}, s_{i2}, \dots, s_{il_i}\}$  be the set of terminal vertices of  $s_i$  and let  $\tau = \max_{1 \leq i \leq \kappa} \{l_i\}$ . With the above notation we have the following result.

**Theorem 1.** For any tree  $T$  which is not a path,

$$pd(T) \leq \kappa + \tau - 1.$$

**Proof.** For a terminal vertex  $s_{ij}$  of a major vertex  $s_i \in S$  we denote by  $S_{ij}$  the set of vertices of  $T$ , different from  $s_i$ , belonging to the  $s_i - s_{ij}$  path. If  $l_i < \tau - 1$ , we assume  $S_{ij} = \emptyset$  for every  $j \in \{l_i + 1, \dots, \tau - 1\}$ . Now for every  $j \in \{2, \dots, \tau - 1\}$ , let

$B_j = \cup_{i=1}^{\kappa} S_{ij}$  and, for every  $i \in \{1, \dots, \kappa\}$ , let  $A_i = S_{i1}$ . Let us show that  $\Pi = \{A, A_1, A_2, \dots, A_\kappa, B_2, \dots, B_{\tau-1}\}$  is a resolving partition of  $T$ , where  $A = V - ((\cup_{i=1}^{\kappa} A_i) \cup (\cup_{j=2}^{\tau-1} B_j))$ . We consider two different vertices  $x, y \in V$ . Note that if  $x$  and  $y$  belong to different sets of  $\Pi$ , we have  $r(x|\Pi) \neq r(y|\Pi)$ .

Case 1:  $x, y \in S_{ij}$ . If  $j = \tau$ , then we have that  $x, y \in A$  and it follows that  $d(x, A_i) \neq d(y, A_i)$ . Otherwise, we obtain that  $d(x, A) = d(x, s_i) \neq d(y, s_i) = d(y, A)$ .

Case 2:  $x \in S_{ij}$  and  $y \in S_{kl}$ ,  $i \neq k$ . If  $j = 1$  or  $l = 1$ , then  $x$  and  $y$  belong to different sets of  $\Pi$ . So we suppose  $j \neq 1$  and  $l \neq 1$ . Hence, if  $d(x, A_i) = d(y, A_i)$ , then

$$\begin{aligned} d(x, A_k) &= d(x, s_i) + d(s_i, s_k) + 1 \\ &= d(x, A_i) + d(s_i, s_k) \\ &= d(y, A_i) + d(s_i, s_k) \\ &= d(y, s_k) + 2d(s_k, s_i) + 1 \\ &= d(y, A_k) + 2d(s_k, s_i) \\ &> d(y, A_k). \end{aligned}$$

Case 3:  $x \in S_{i\tau}$  and  $y \in A - \cup_{l=1}^{\kappa} S_{l\tau}$ . If  $d(x, A_i) = d(y, A_i)$ , then  $d(x, s_i) = d(y, s_i)$ . Since  $y \notin S_{l\tau}$ ,  $l \in \{1, \dots, \kappa\}$ , there exists  $A_j \in \Pi$ ,  $j \neq i$ , such that  $s_i$  does not belong to the  $y - s_j$  path. Now let  $Y$  be the set of vertices belonging to the  $y - s_j$  path, and let  $v \in Y$  such that  $d(s_i, v) = \min_{u \in Y} \{d(s_i, u)\}$ . Hence,

$$\begin{aligned} d(x, A_j) &= d(x, s_i) + d(s_i, v) + d(v, s_j) + 1 \\ &= d(y, s_i) + d(s_i, v) + d(v, s_j) + 1 \\ &= d(y, v) + 2d(v, s_i) + d(v, s_j) + 1 \\ &= d(y, A_j) + 2d(v, s_i) \\ &> d(y, A_j). \end{aligned}$$

Case 4:  $x, y \in A' = A - \cup_{l=1}^{\kappa} S_{l\tau}$ . If for some exterior major vertex  $s_i \in S$ , the vertex  $x$  belongs to the  $y - s_i$  path or the vertex  $y$  belongs to the  $x - s_i$  path, then  $d(x, A_i) \neq d(y, A_i)$ . Otherwise, there exist at least two exterior major vertices  $s_i, s_j$  such that the  $x - y$  path and the  $s_i - s_j$  path share more than one vertex (if not, then  $x, y \notin A'$ ). Let  $W$  be the set of vertices belonging to the  $s_i - s_j$  path. Let  $u, v \in W$  such that  $d(x, u) = \min_{z \in W} \{d(x, z)\}$  and  $d(y, v) = \min_{z \in W} \{d(y, z)\}$ . We suppose, without loss of generality, that  $d(s_i, u) > d(v, s_i)$ . Hence, if  $d(x, v) = d(y, v)$ , then  $d(x, u) \neq d(y, u)$ , and if  $d(x, u) = d(y, u)$ , then  $d(x, v) \neq d(y, v)$ . We have

$$\begin{aligned} d(x, A_j) &= d(x, u) + d(u, s_j) + 1 \\ &\neq d(y, u) + d(u, s_j) + 1 \\ &= d(y, A_j) \end{aligned}$$

or

$$\begin{aligned} d(x, A_i) &= d(x, v) + d(v, s_i) + 1 \\ &\neq d(y, v) + d(v, s_i) + 1 \\ &= d(y, A_i). \end{aligned}$$

Therefore, for different vertices  $x, y \in V$ , we have  $r(x|\Pi) \neq r(y|\Pi)$ .  $\square$

One example where  $pd(T) = \kappa + \tau - 1$  is the tree in Fig. 1.

Any vertex adjacent to a leaf of a tree  $T$  is called a *support* vertex. In the following result  $\xi$  denotes the number of support vertices of  $T$  and  $\theta$  denotes the maximum number of leaves adjacent to a support vertex of  $T$ .

**Corollary 2.** For any tree  $T$  of order  $n \geq 2$ ,  $pd(T) \leq \xi + \theta - 1$ .

**Proof.** If  $T$  is a path, then  $\xi = 2$  and  $\theta = 1$ , so the result follows. Now we suppose  $T$  is not a path. Let  $v$  be an exterior major vertex of terminal degree  $\tau$ . Let  $x$  be the number of leaves adjacent to  $v$  and let  $y = \tau - x$ . Since  $\kappa + y \leq \xi$  and  $x \leq \theta$ , we deduce  $\kappa + \tau \leq \xi + \theta$ .  $\square$

The above bound is achieved, for instance, for the graph of order six composed of two support vertices  $a$  and  $b$ , where  $a$  is adjacent to  $b$  and four leaves; two of them are adjacent to  $a$  and the other two leaves are adjacent to  $b$ . One example of a graph for which Theorem 1 gives a better result than Corollary 2 is the graph in Fig. 1.

Since the number of leaves,  $n_1(T)$ , of a tree  $T$  is bounded below by  $\xi + \theta - 1$ , Corollary 2 leads to the following bound.

**Remark 3.** For any tree  $T$  of order  $n \geq 2$ ,  $pd(T) \leq n_1(T)$ .

Now we are going to characterize all the trees for which  $pd(T) = n_1(T)$ . It was shown in [7] that  $pd(G) = 2$  if and only if the graph  $G$  is a path. So by the above remark we obtain the following result.

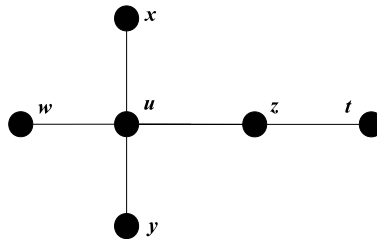


Fig. 3. A comet graph where  $3 = \theta = pd(T) < n_1(T)$ .

**Remark 4.** Let  $T$  be a tree of order  $n \geq 4$ . If  $n_1(T) = 3$ , then  $pd(T) = 3$ .

**Theorem 5.** Let  $T$  be a tree with  $n_1(T) \geq 4$ . Then  $pd(T) = n_1(T)$  if and only if  $T$  is the star graph.

**Proof.** If  $T = S_n$  is a star graph, it is clear that  $pd(T) = n_1(T)$ . Now, let  $T = (V, E) \neq S_n$ , such that  $pd(T) = n_1(T) \geq 4$ . Note that by (3) we have  $ex(T) = 1$ . Let  $t = n_1(T)$  and let  $\Omega = \{u_1, u_2, \dots, u_t\}$  be the set of leaves of  $T$ . Let  $u \in V$  be the unique exterior major vertex of  $T$ . Let us suppose, without loss of generality,  $u_t$  is a leaf of  $T$  such that  $d(u_t, u) = \max_{u_i \in \Omega} \{d(u_i, u)\}$ .

For the leaves  $u_1, u_2, u_t \in \Omega$  let the paths  $P = uu_{t_1}u_{t_2}, \dots, u_{t_r}u_t$ ,  $Q = uu_{11}u_{12}, \dots, u_{1r_1}u_1$  and  $R = uu_{21}u_{22}, \dots, u_{2r_2}u_2$ . Now, let us form the partition  $\Pi = \{A_1, A_2, \dots, A_{t-2}, A\}$ , such that  $A_1 = \{u_{11}, u_{12}, \dots, u_{1r_1}, u_1, u_{t_2}, u_{t_3}, \dots, u_{t_r}, u_t\}$ ,  $A_2 = \{u_{21}, u_{22}, \dots, u_{2r_2}, u_2, u_{t_1}\}$ ,  $A_i = \{u_i\}$ ,  $i \in \{3, \dots, t-2\}$  and  $A = V - \cup_{i=1}^{t-2} A_i$ . Let us consider two different vertices  $x, y \in V$ . Hence, we have the following cases.

Case 1:  $x, y \in A_1$ . Let us suppose  $x \in P$  and  $y \in Q$ . If  $d(x, A_2) = d(y, A_2)$ , then we have

$$\begin{aligned} d(x, A) &= d(x, u_{t_1}) + 1 \\ &= d(x, A_2) + 1 \\ &= d(y, A_2) + 1 \\ &= d(y, A) + 2 \\ &> d(y, A). \end{aligned}$$

Now, if  $x, y \in P$  or  $x, y \in Q$ , then  $d(x, A) \neq d(y, A)$ .

Case 2:  $x, y \in A_2$ . If  $x = u_{t_1}$  or  $y = u_{t_1}$ , then let us suppose for instance,  $x = u_{t_1}$ , so we have  $d(x, A_1) = 1 < 2 \leq d(y, A_1)$ . On the contrary, if  $x, y \in R$ , then  $d(x, A) \neq d(y, A)$ .

Case 3:  $x, y \in A$ . If  $d(x, A_1) = d(y, A_1)$ , then  $t \geq 5$  and there exists a leaf  $u_i$ ,  $i \neq 1, 2, t-1, t$ , such that  $d(x, A_i) = d(y, A_i) \neq d(y, u_i) = d(y, A_i)$ .

Therefore, for different vertices  $x, y \in V$  we have  $r(x|\Pi) \neq r(y|\Pi)$  and  $\Pi$  is a resolving partition in  $T$ , a contradiction.  $\square$

Let  $T$  be the comet graph shown in Fig. 3. A resolving partition for  $T$  is  $\Pi = \{A_1, A_2, A_3\}$ , where  $A_1 = \{x, t\}$ ,  $A_2 = \{y, z\}$  and  $A_3 = \{u, w\}$ . In this case,  $\theta = pd(T) = 3 < 4 = n_1(T)$ .

**Remark 6.** For any tree  $T$  of order  $n \geq 2$ ,  $pd(T) \geq \theta$ .

**Proof.** Since different leaves adjacent to the same support vertex must belong to different sets of a resolving partition, the result follows.  $\square$

Other examples where  $pd(T) = \theta$  are the star graphs and the graph in Fig. 2.

**Theorem 7.** Let  $T$  be a tree which is not a path. If every vertex belonging to the path between two exterior major vertices of terminal degree greater than one is an exterior major vertex of terminal degree greater than one, then

$$pd(T) \leq \max\{\kappa, \tau + 1\}.$$

**Proof.** We suppose  $T = (V, E)$  is not a path. Let  $S = \{s_1, s_2, \dots, s_\kappa\}$  be the set of exterior major vertices of  $T$  with terminal degree greater than one and let  $B_i = \{s_i\}$ ,  $i = 1, \dots, \kappa$ . If  $\kappa < \tau + 1$ , then for  $i \in \{\kappa + 1, \dots, \tau + 1\}$  we assume  $B_i = \emptyset$ . Let  $l_i$  be the terminal degree of  $s_i$ ,  $i \in \{1, \dots, \kappa\}$ . If  $l_i < i$ , then we denote by  $\{s_{i1}, \dots, s_{il_i}\}$  the set of terminal vertices of  $s_i$ . On the contrary, if  $l_i \geq i$ , then the set of terminal vertices of  $s_i$  is denoted by  $\{s_{i1}, \dots, s_{i(i-1)}, s_{i(i+1)}, \dots, s_{il_i+1}\}$ . Also, for a terminal vertex  $s_{ij}$  of a major vertex  $s_i$  we denote by  $S_{ij}$  the set of vertices of  $T$ , different from  $s_i$ , belonging to the  $s_i - s_{ij}$  path. Moreover, we assume  $S_{ij} = \emptyset$  for the following three cases: (1)  $i = j$ , (2)  $i \leq l_i < \tau$  and  $j \in \{l_i + 2, \dots, \tau + 1\}$ , and (3)  $i > l_i$  and  $j \in \{l_i + 1, \dots, \tau + 1\}$ . Now, let  $t = \max\{\kappa, \tau + 1\}$  and let  $\Pi = \{A_1, A_2, \dots, A_t\}$  be composed of the sets  $A_i = B_i \cup (\cup_{j=1}^{\kappa} S_{ij})$ ,  $i = 1, \dots, t$ . Since every vertex belonging to the path between two exterior major vertices of terminal degree greater than one is an exterior major vertex of terminal degree greater than one, then  $\Pi$  is a partition of  $V$ .

Let us show that  $\Pi$  is a resolving partition. Let  $x, y \in V$  be different vertices of  $T$ . If  $x, y \in A_i$ , we have the following three cases.

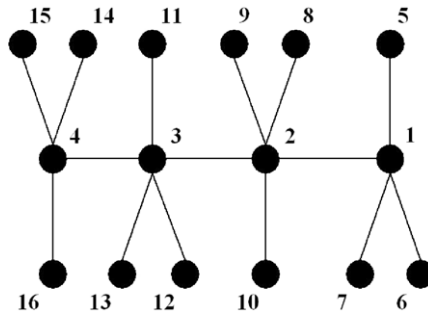


Fig. 4.  $\Pi = \{\{1, 8, 11, 14\}, \{2, 5, 12, 15\}, \{3, 6, 9, 16\}, \{4, 7, 10, 13\}\}$  is a resolving partition.

Case 1:  $x, y \in S_{ji}$ . In this case  $d(x, A_j) = d(x, s_j) \neq d(y, s_j) = d(y, A_j)$ .

Case 2:  $x \in S_{ji}$  and  $y \in S_{ki}$ ,  $j \neq k$ . If  $d(x, A_k) = d(y, A_k)$  we have  $d(y, A_j) > d(y, s_k) = d(y, A_k) = d(x, A_k) > d(x, s_j) = d(x, A_j)$ .

Case 3:  $x = s_i$  and  $y \in S_{ji}$ . As  $s_i$  has at least two terminal vertices, there exists a terminal vertex  $s_{il}$  of  $s_i$ ,  $l \neq j$ , such that  $d(x, A_l) = d(x, s_{il}) = 1$ . Hence,  $d(y, A_l) > d(y, s_j) \geq 1 = d(x, A_l)$ . Therefore, for different vertices  $x, y \in V$ , we have  $r(x|\Pi) \neq r(y|\Pi)$ .  $\square$

The above bound is achieved, for instance, for the graph in Fig. 4.

### 3. On the partition dimension of generalized trees

A *cut vertex* in a graph is a vertex whose removal increases the number of components of the graph and an *extreme vertex* is a vertex such that its closed neighborhood forms a complete graph. Also, a *block* is a maximal biconnected subgraph of the graph. Now, let  $\mathfrak{F}$  be the family of sequences of connected graphs  $G_1, G_2, \dots, G_k$ ,  $k \geq 2$ , such that  $G_1$  is a complete graph  $K_{n_1}$ ,  $n_1 \geq 2$ , and  $G_i$ ,  $i \geq 2$ , is obtained recursively from  $G_{i-1}$  by adding a complete graph  $K_{n_i}$ ,  $n_i \geq 2$ , and identifying a vertex of  $G_{i-1}$  with a vertex in  $K_{n_i}$ .

From this point we will say that a connected graph  $G$  is a *generalized tree* if and only if there exists a sequence  $\{G_1, G_2, \dots, G_k\} \in \mathfrak{F}$  such that  $G_k = G$  for some  $k \geq 2$ . Notice that in these generalized trees every vertex is either a cut vertex or an extreme vertex. Also, every complete graph used to obtain the generalized tree is a block of the graph. Note that if every  $G_i$  is isomorphic to  $K_2$ , then  $G_k$  is a tree, thus justifying the terminology used. In this section we will be centered in the study of partition dimension of generalized trees.

Let  $G = (V, E)$  be a generalized tree and let  $R_1, R_2, \dots, R_k$  be the blocks of  $G$ . A cut vertex  $v \in V$  is a *support cut vertex* if there is at least one block  $R_i$  of  $G$ , in which  $v$  is the unique cut vertex belonging to the block  $R_i$ . An extreme vertex is an *exterior extreme vertex* if it is adjacent to only one cut vertex. Let  $S = \{s_1, s_2, \dots, s_\zeta\}$  be the set of support cut vertices of  $G$  and let  $\{s_{i1}, s_{i2}, \dots, s_{it_i}\}$  be the set of exterior extreme vertices adjacent to  $s_i \in S$ . Also, let  $Q = \{Q_1, Q_2, \dots, Q_\vartheta\}$  be the set of blocks of  $G$  which contain more than one cut vertex and more than one extreme vertex and let  $\{q_{i1}, q_{i2}, \dots, q_{it_i}\}$  be the set of extreme vertices belonging to  $Q_i \in Q$ . Now, let  $\phi = \max_{1 \leq i \leq \zeta, 1 \leq j \leq \vartheta} \{l_i, t_j\}$ . With the above notation we have the following result.

**Theorem 8.** For any generalized tree  $G$ ,

$$pd(G) \leq \begin{cases} \zeta + \vartheta + \phi - 1, & \text{if } \phi \geq 3; \\ \zeta + \vartheta + 1, & \text{if } \phi \leq 2. \end{cases}$$

**Proof.** For each support cut vertex  $s_i \in S$ , let  $A_i = \{s_{i1}\}$  and for each block  $Q_j \in Q$ , let  $B_j = \{q_{j1}\}$ . Let us suppose  $\phi \geq 3$ . For every  $j \in \{2, \dots, l_i\}$  we take  $M_{ij} = \{s_{ij}\}$  and, if  $l_i < \phi - 1$ , then for every  $j \in \{l_{i+1}, \dots, \phi - 1\}$  we consider  $M_{ij} = \emptyset$ . Analogously, for every  $j \in \{2, \dots, t_i\}$  we take  $N_{ij} = \{q_{ij}\}$  and, if  $t_i < \phi - 1$ , then for every  $j \in \{t_{i+1}, \dots, \phi - 1\}$  we consider  $N_{ij} = \emptyset$ . Now, let  $C_j = \bigcup_{i=1}^{\max\{\zeta, \vartheta\}} (M_{ij} \cup N_{ij})$ , with  $j \in \{2, \dots, \phi - 1\}$ .

Let us prove that  $\Pi = \{A, A_1, A_2, \dots, A_\zeta, B_1, B_2, \dots, B_\vartheta, C_2, C_3, \dots, C_{\phi-1}\}$  is a resolving partition of  $G$ , where  $A = V - \bigcup_{i=1}^\zeta A_i - \bigcup_{i=1}^\vartheta B_i - \bigcup_{i=2}^{\phi-1} C_i$ . To begin with, let  $x, y$  be two different vertices of  $G$ . We have the following cases.

Case 1:  $x$  is a cut vertex or  $y$  is a cut vertex. Let us suppose, for instance,  $x$  is a cut vertex. So there exists an extreme vertex  $s_{i1}$  such that  $x$  belongs to a shortest  $y - s_{i1}$  path or  $y$  belongs to a shortest  $x - s_{i1}$  path. Hence, we have  $d(x, A_i) = d(x, s_{i1}) \neq d(y, s_{i1}) = d(y, A_i)$ .

Case 2:  $x, y$  are extreme vertices. If  $x, y$  belong to the same block of  $G$ , then  $x, y$  belong to different sets of  $\Pi$ . On the contrary, if  $x, y$  belong to different blocks in  $G$ , then let us suppose that there exists an extreme vertex  $c$  such that  $d(x, c) \leq 1$  or  $d(y, c) \leq 1$ . We can suppose  $c \in A_i$ , for some  $i \in \{1, \dots, \zeta\}$ , or  $c \in B_j$ , for some  $j \in \{1, \dots, \vartheta\}$ . Without the loss of generality, we suppose that  $d(x, c) \leq 1$ . Since  $x$  and  $y$  belong to different blocks of  $G$ , we have  $d(y, c) > 1$ . So we obtain either  $d(x, A_i) = d(x, c) \leq 1 < d(y, c) = d(y, A_i)$  or  $d(x, B_j) = d(x, c) \leq 1 < d(y, c) = d(y, B_j)$ .

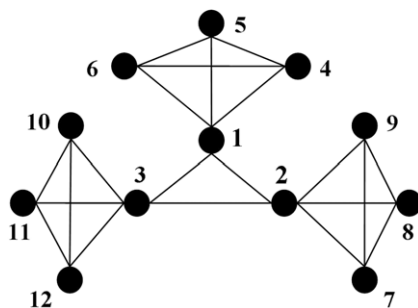


Fig. 5.  $\Pi = \{\{4\}, \{7\}, \{10\}, \{5, 8, 11\}, \{1, 2, 3, 6, 9, 12\}\}$  is a resolving partition for the generalized tree.

Now, if there exists no such a vertex  $c$ , then there exist two blocks  $H, K \notin Q$  with  $x \in H$  and  $y \in K$ , which contain more than one cut vertex and only one extreme vertex. So  $x, y \in A$ . Let  $u \in H$  be a cut vertex such that  $d(y, u) = \max_{v \in H} d(y, v)$ . Hence, there exists an extreme vertex  $s_{i1}$  such that  $u$  belongs to a shortest  $x - s_{i1}$  path and  $d(y, s_{i1}) = d(y, u) + d(u, s_{i1})$ . As  $x, y$  belong to different blocks and  $d(y, u) = \max_{v \in H} d(y, v)$  we have  $d(y, u) \geq 2$ . Thus,

$$\begin{aligned} d(y, A_i) &= d(y, s_{i1}) \\ &= d(y, u) + d(u, s_{i1}) \\ &\geq 2 + d(u, s_{i1}) \\ &> 1 + d(u, s_{i1}) \\ &= d(x, u) + d(u, s_{i1}) \\ &= d(x, A_i). \end{aligned}$$

Hence, we conclude that if  $\phi \geq 3$ , then for every  $x, y \in V$ ,  $r(x|\Pi) \neq r(y|\Pi)$ . Therefore,  $\Pi$  is a resolving partition.

On the other hand, if  $\phi \leq 2$ , then  $\Pi' = \{A, A_1, A_2, \dots, A_\zeta, B_1, B_2, \dots, B_\vartheta\}$  is a partition of  $V$ . Proceeding as above we obtain that  $\Pi'$  is a resolving partition.  $\square$

The above bound is achieved, for instance, for the graph in Fig. 5, where  $\zeta = 3$ ,  $\vartheta = 0$  and  $\phi = 3$ . Also, notice that for the particular case of trees we have  $\zeta = \xi$ ,  $\phi = \theta$  and  $\vartheta = 0$ . So the above result leads to Corollary 2.

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