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OPTIMAL BACKBONE COLORING OF SPLIT GRAPHS WITH MATCHING BACKBONES

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Abstract

For a graph G with a given subgraph H, the backbone coloring is defined as the mapping $c : V(G) \to \mathbb{N}_+$ such that $|c(u) - c(v)| \geq 2$ for each edge $\{u, v\} \in E(H)$ and $|c(u) - c(v)| \geq 1$ for each edge $\{u, v\} \in E(G)$. The backbone chromatic number $BBC(G, H)$ is the smallest integer k such that there exists a backbone coloring with $\max_{v \in V(G)} c(v) = k$.

In this paper, we present the algorithm for the backbone coloring of split graphs with matching backbone.

Keywords: backbone coloring, split graphs, matching.

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1. Preliminaries

The backbone coloring problem, introduced by Broersma in [4], is an example of the general framework of graph coloring problems. It is strongly related to the frequency assignment problem: given transmitters (represented by the vertices of a graph) and their adjacency (vertices are adjacent if transmitters are close enough or strong enough), assign the frequency bands so that corresponding transmitters keep interferences at a defined level, minimizing total frequency span. Furthermore, we distinguish certain substructure of the network (called the backbone) crucial for the communication and put additional restrictions on the assignment. Possible applications of backbone coloring are described e.g. in [4].

In this paper, we consider simple undirected graphs, i.e. graphs without loops or multiple edges, and digraphs, i.e. graphs with directed edges. For a graph or

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digraph G, let $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively, with cardinalities $|V(G)| = n$ and $|E(G)| = m$. Graph G is a *tree* if and only if it is simple, connected and $m = n - 1$. Graph G is *unicyclic* if and only if it is simple, connected and $m = n$. Graph is a *matching* if and only if it is simple and every its vertex is adjacent to at most one edge. We call the matching *perfect* if and only if every vertex is adjacent to exactly one edge. The size of a largest possible clique (complete subgraph K_k) in an undirected graph G is called the *clique number* of G and denoted by $\omega(G)$.

The *coloring* of a graph (or a digraph) G is defined as a function $c : V(G) \to \mathbb{N}_+$ such that $|c(u) - c(v)| \geq 1$ for each edge $\{u, v\} \in E(G)$ (or $(u, v) \in E(G)$ in case of digraphs). The *backbone coloring* of a graph (or a digraph) G and the given subgraph (subdigraph) H (called the *backbone*) of G, is a function $c : V(G) \to \mathbb{N}_+$, such that it is a coloring of G and the inequality $|c(u) - c(v)| \geq 2$ holds for each edge $\{u, v\} \in E(H)$. The minimum number k for which there exists a coloring of G with $\max_{v \in V(G)} c(v) = k$ is called the chromatic number $\chi(G)$ of G. Similarly, the *backbone chromatic number* $BBC(G, H)$ is the smallest integer k such that there exists a backbone coloring of G with backbone H and $\max_{v \in V(G)} c(v) = k$. The backbone coloring c is *optimal* if $\max_{v \in V(G)} c(v) = BBC(G, H)$. We straightforwardly obtain from these definitions that for every directed graph G with backbone H and its respective underlying undirected graphs, G' and H', the equation $BBC(G, H) = BBC(G', H')$ holds. Throughout this paper, we assume that the subgraph (subdigraph) H is spanning, i.e., $V(G) = V(H)$, and for notational simplicity we use the abbreviation „(G, H)" instead of „graph G with backbone H".

In this paper we focus on split graphs introduced by Hammer and Földes in [1]. A *split graph* is a simple graph whose vertex set can be partitioned into two sets: those of a clique C and an independent set I. Split graphs are perfect, thus satisfying the equation $\chi(G) = \omega(G)$. In this paper we consider split graphs with $\chi(G) \geq 3$, as other split graphs are empty or bipartite and thus easily colorable [3].

The remainder of the paper is organized as follows: we begin our considerations with an algorithm that produces the value of $BBC(K_n, G)$, where K_n is a complete graph of order n and G is a tree or unicyclic graph. Next, we study the backbone coloring of split graphs with perfect matching backbones and $|C| = |I|$. Finally, we obtain a polynomial algorithm that computes $BBC(G, M)$ for any split graph G and any matching backbone M.

2. Backbone Coloring of Complete Graphs with Tree or Unicyclic Backbones

Lemma 1. *Let G be a graph of order n and G' be a graph obtained from G by attaching a pendant vertex. If $BBC(K_n, G) = n$, then $BBC(K_{n+1}, G') = n + 1$.*

Proof. Let c be the optimal coloring of (K_n, G) and v be the vertex added to graph G to obtain G'. Since v is a pendant vertex, it has only one adjacent vertex $u \in V(G')$.

If $c(u) \leq n-1$, then we extend c by assigning $c(v) \leftarrow n+1$. The color assigned to v does not appear elsewhere in the coloring and $c(v) - c(u) > n + 1 - n = 1$, therefore $|c(v) - c(u)| \geq 2$ and extended c is a feasible backbone coloring of (K_{n+1}, G').

Otherwise, $c(u) \geq n$. Since $BBC(K_n, G) = n$, we know that $c(u) = n$. By assumption, since the mapping c is injective in $\{1, 2, \ldots, n\}$, there exists a vertex $w \in V(G)$ such that $c(w) = n - 1$. Let us define a function $c' : V(G') \to \mathbb{N}_+$:

$$c'(x) = \begin{cases} n-1 & \text{if } x = v, \\ n & \text{if } x = w, \\ n+1 & \text{if } x = u, \\ c(x) & \text{if } x \in V(G) - \{u, v, w\}. \end{cases}$$

It is easy to see that the mapping c' is injective. By definition, $|c'(v) - c'(u)| = 2$, $c'(u) - c'(x) > c(u) - c(x) \geq 2$ for every edge $\{u, x\} \in E(G')$ and $c'(w) - c'(x) > c(u) - c(x) \geq 2$ for every edge $\{u, w\} \in E(G')$. Finally, it suffices to note that $|c'(x) - c'(y)| = |c(x) - c(y)| \geq 2$ for every other edge $\{x, y\} \in E(G')$. Since the inequality $|c'(x) - c'(y)| \geq 2$ holds for every edge $\{x, y\} \in E(G')$ and c' is injective, c' is a backbone coloring of (K_{n+1}, G'). ■

Lemma 2. *For any $n \geq 5$, if C_n is a Hamiltonian cycle in K_n, then $BBC(K_n, C_n) = n$.*

Proof. If n is even, we color vertices on cycle sequentially with $1, 3, 5, \ldots, n-1, 2, 4, \ldots, n$. If n is odd, we use colors $1, 3, 5, \ldots, n, 2, 4, \ldots, n-1$. In any case, the mapping is injective and for every edge of the cycle the color difference between endpoints is at least 2, so $BBC(K_n, C_n) \leq n$. But on the other hand, for any graph G, $BBC(K_n, G) \geq \chi(K_n) = n$, which completes the proof. ■

A*Hamiltonian path* is a path in a graph that visits each vertex exactly once. If a graph G contains a Hamiltonian path, then we call G a *semihamiltonian* graph. Graph $\bar{G}$ is a *complement* of a graph G if and only if $V(G) = V(\bar{G})$ and for every pair of vertices $u, v \in V(G)$ it holds $uv \in E(\bar{G})$ if and only if $uv \notin E(G)$.

It can be shown that coloring of (K_n, G) using colors $\{1, 2, \ldots, n\}$ is equivalent to the semihamiltonicity of the complement of a graph G. Indeed, suppose we have an optimal coloring of (K_n, G) with colors $\{1, 2, \ldots, n\}$, then the vertex with color 1 is not connected in G to the vertex with color 2 (as it would violate the backbone condition), the vertex with color 2 is not connected to the vertex with color 3 and so on—therefore these vertices are connected in $\bar{G}$. However, if we know that the graph $\bar{G}$ is semihamiltonian, then by assigning to the vertices along the Hamiltonian path colors $1, 2, 3, \ldots$ we obtain the solution for (K_n, G), as every two vertices received distinct colors and every two vertices which received consecutive colors are adjacent in $\bar{G}$, therefore cannot be adjacent in G.

The problem of semihamiltonicity of a graph G is shown in [5] to be NP-complete in general case, but it can be solved in polynomial time for sparse graphs, therefore:

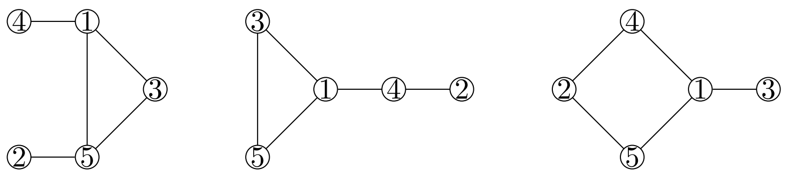
Theorem 3. *For every connected graph G of order $n \geq 5$ and with $m \leq n$ edges,*

$$BBC(K_n, G) = \begin{cases} n+1 & \textit{if } G \textit{ contains a spanning star as a subgraph,} \\ n & \textit{otherwise.} \end{cases}$$

Proof. First, we note that any connected graph with $m \leq n$ edges is either a tree or an unicyclic graph.

If G contains a spanning star as a subgraph, then it does not exists any backbone coloring function for (K_n, G) to $\{1, 2, \ldots, n\}$, because color of the root of the star must differ from colors of all leaves by at least 2. But since $m \leq n$ and the spanning star has exactly $n-1$ edges, G has at most one edge outside the spanning star. Therefore we may assign color $n+1$ to the root of the star and (if $m = n$) assign colors 1 and $n - 1 \geq 4$ to the endpoints of the edge not included in the spanning star. Finally, we assign all unused colors less than n to all uncolored vertices and obtain the backbone coloring c of (K_n, G).

If G is a tree non-isomorphic to a spanning star, it has an induced path P_4 as a subgraph. But $BBC(K_4, P_4) = 4$ and we may color G starting from the optimal coloring of P_4 and color the rest of the vertices using DFS ordering from an arbitrary vertex of P_4 and applying Lemma 1 to each step.



(a) Bull graph (b) Paw graph (c) Banner graph

Figure 1. Minimum optimally labeled supergraphs of C_3 and C_4.

In all other cases, G is unicyclic. If it contains a cycle C_k of length $k \geq 5$, then $BBC(K_k, C_k) = k$ due to Lemma 2. Otherwise, there is either C_3 or C_4 subgraph of G. But then we attach pendant vertices to this cycle until we obtain subgraph C' of G on 5 vertices (all possibilities are presented in Figure 1). It turns out that in each case $BBC(K_5, C') = 5$. Finally, as in previous case, we extend the partial coloring to the whole graph G using DFS ordering from an arbitrary vertex of the colored part of the graph and applying Lemma 1. ∎

The proof of Theorem 3 yields an algorithm for backbone coloring of (K_n, G). All operations: checking the existence of a spanning star, finding a cycle in a graph and ordering the vertices using e.g. DFS from one of the previously colored vertices can be implemented in $O(m+n)$ time and labeling each vertex can be implemented in $O(1)$. Therefore, since $m \leq n$, this algorithm runs in $O(m+n) = O(n)$ time.

We can extend Theorem 3 to forests by using the following lemma.

Lemma 4. *Let G_1 (of order n_1) and G_2 (of order n_2) be graphs non-isomorphic both to C_3, and let c_1 and c_2 be the optimal backbone colorings of (K_{n_1}, G_1) and (K_{n_2}, G_2), respectively. If $BBC(K_{n_1}, G_1) \leq n_1 + 1$ and $BBC(K_{n_2}, G_2) \leq n_2 + 1$, then $BBC(K_{n_1+n_2}, G_1 \cup G_2) \leq n_1 + n_2$.*

Proof. Since we have $n_1 \leq BBC(K_{n_1}, G_1) \leq n_1 + 1$ and $n_2 \leq BBC(K_{n_2}, G_2) \leq n_2 + 1$, we split the proof into the following possible cases:

Case 1. $[BBC(K_{n_1}, G_1) = n_1$ and $BBC(K_{n_2}, G_2) = n_2]$. In this case we define $c(v) = c_1(v)$ if $v \in V(G_1)$ and $c(v) = c_2(v) + n_1$ if $v \in V(G_2)$. The coloring function c is injective, for every $\{u, v\} \in E(G_1) \cup E(G_2)$ the inequality $|c(u) - c(v)| \geq 2$ holds and $BBC(K_{n_1+n_2}, G_1 \cup G_2) \leq n_1 + n_2$.

Case 2. $[BBC(K_{n_1}, G_1) = n_1 + 1$ and $BBC(K_{n_2}, G_2) = n_2.]$ In this case there exists color k, $2 \leq k \leq n_1$, such that for each vertex $v \in V(G_1)$, $c_1(v) \neq k$. Let us define:

$$c(v) = \begin{cases} c_1(v) & \text{if } v \in V(G_1) \text{ and } c_1(v) < k, \\ k + c_2(v) - 1 & \text{if } v \in V(G_2), \\ n_2 + c_1(v) - 1 & \text{if } v \in V(G_1) \text{ and } c_1(v) > k. \end{cases}$$

All backbone constraints of G_1 and G_2 are preserved, therefore it suffices to show that c is an injective function. If $u \in V_1$, $v \in V_2$ and $c_1(u) < k$, then $c(u) = c_1(u) < k < k + c_2(v) - 1 = c(v)$. If $u \in V_1$, $v \in V_2$ and $c_1(u) > k$, then $c(u) = n_2 + c_1(u) - 1 \geq n_2 + k > c_2(v) - 1 + k = c(v)$. Otherwise, $u, v \in V_1$ or $u, v \in V_2$. Clearly $c(u) \neq c(v)$ also holds, hence c is a backbone coloring function using exactly $n_1 + n_2 = n$ colors. The same argument applies to the case $BBC(K_{n_1}, G_1) = n_1$ and $BBC(K_{n_2}, G_2) = n_2 + 1$. In both cases, $BBC(K_{n_1+n_2}, G_1 \cup G_2) \leq n_1 + n_2$.

Case 3. $[BBC(K_{n_1}, G_1) = n_1 + 1$ and $BBC(K_{n_2}, G_2) = n_2 + 1.]$ We know that there exist colors k_1 and k_2, $2 \leq k_1 \leq n_1$, $2 \leq k_2 \leq n_2$, such that for each vertex $v \in V(G_1)$, $c_1(v) \neq k_1$ and for each vertex $v \in V(G_2)$, $c_2(v) \neq k_2$. Let us define:

$$c(v) = \begin{cases} c_1(v) & \text{if } v \in V(G_1) \text{ and } c_1(v) < k_1, \\ k_1 + c_2(v) - 1 & \text{if } v \in V(G_2) \text{ and } c_2(v) < k_2, \\ k_2 + c_1(v) - 2 & \text{if } v \in V(G_1) \text{ and } c_1(v) > k_1, \\ n_1 + c_2(v) - 1 & \text{if } v \in V(G_2) \text{ and } c_2(v) > k_2. \end{cases}$$

As in the previous case, all backbone constraints of G_1 and G_2 are preserved, therefore it suffices to show that c is an injective function. If $u \in V_1$, $v \in V_2$ and $c_1(u) < k_1$, then $c(u) = c_1(u) < k_1 < k_1 + c_2(v) - 1 \leq c(v)$. If $u \in V_1$, $v \in V_2$ and $c_1(u) > k_1$, then we have either $c_2(v) < k_2$, hence $c(u) = k_2 + c_1(u) - 2 > k_2 + k_1 - 2 \geq k_1 + c_2(v) - 1 = c(v)$, or $c_2(v) > k_2$, thus $c(u) = k_2 + c_1(u) - 2 \leq k_2 + n_1 - 1 < n_1 + c_2(v) - 1 = c(v)$.

Finally, the remaining cases are $u, v \in V_1$ or $u, v \in V_2$. Clearly, $c(u) \neq c(v)$ also holds, so c is a backbone coloring function using exactly $n_1 + n_2$ colors. Therefore $BBC(K_{n_1+n_2}, G_1 \cup G_2) \leq n_1 + n_2$. ■

Theorem 5. *Let G be a graph of order $n \geq 5$. If for each connected component G' of G the inequality $|E(G')| \leq |V(G')|$ holds, then*

$$BBC(K_n, G) = \begin{cases} n+1 & \textit{if } G \textit{ contains a spanning star as a subgraph,} \\ n & \textit{otherwise.} \end{cases}$$

Proof. If G is connected, the formula is straightforward from Theorem 3. Otherwise, we color separately all non-C_3 connected components of G using Theorem 3 and we proceed by induction using Lemma 4. Therefore, for all disconnected graphs G of order n without connected components C_3 we proved that $BBC(K_n, G) = n$.

If G contains $t \geq 2$ connected components C_3, then we number all triangles with $1, 2, \ldots, t$ and construct the backbone coloring c by assigning to the i-th C_3 colors $\{i, i+t, i+2t\}$. Such a coloring can be merged with any number of non-C_3 connected components of G using Lemma 4.

To complete the proof it suffices to show the solution for G with exactly one triangle. Then, we obtain the optimal backbone coloring c of $(K_{n-3}, G \setminus C_3)$ (nonempty, since $n \geq 5$) by combining Theorem 3 and Lemma 4. If there exists a color k, $2 \leq k \leq n-3$, such that for each vertex $v \in V(G \setminus C_3)$, $c(v) \neq k$, then we define a new backbone coloring c' as the assignment of colors $\{1, k+1, n\}$ to C_3 and $c'(v) = c(v) + 1$ for all other vertices of the graph. Clearly, all colors are distinct and for any backbone edge c' guarantees a sufficient span, based on the fact that c does.

Otherwise, $BBC(K_{n-3}, G \setminus C_3) = n - 3$. In this case, we assign the colors $\{1, 3, n\}$ to C_3. We set $c'(v) = 2$ for each vertex v with $c(v) = 1$ and $c'(v) = c(v) + 2$ for all other vertices v(for which $2 \leq c(v) \leq n-3$). Clearly, the resulting coloring is injective and uses only only colors from the set $\{1, 2, \ldots, n\}$. Moreover, the backbone conditions for C_3 and $G \setminus C_3$ remain satisfied, therefore $c'(v)$ is a backbone coloring. ■

3. Backbone Coloring of Split Graphs with Matching Backbones

Before we present the results in this section, we introduce useful concepts of conflict digraph, quasiforest and underlying undirected graph.

Digraph G' is a *conflict digraph* of (G, M) if and only if the following conditions are satisfied:

1. $V(G') = \{w_1, w_2, \ldots, w_k\}$ (where $k = \omega(G)$),
2. $E(G') = \{(w_i, w_j) : \{u_i, v_j\} \notin E(G), 1 \leq i, j \leq k\}$.

Note that G' contains no loops since $\{u_i, v_i\} \in E(M) \subset E(G)$. Each edge of the conflict digraph represents the following condition: if $c(v_j) = c(u_i)$ in a backbone coloring of (G, M) using k colors, then $|c(v_i) - c(v_j)| \geq 2$.

Digraph G is a *quasiforest* if and only if every its vertex is incident to exactly one outgoing edge. For every digraph G we define a (simple) *underlying undirected graph* obtained by replacing all directed edges of G with undirected edges. If G' is an underlying undirected graph of a quasiforest G, then it follows that G is a union of trees and unicyclic graphs.

From now on, we assume that G is a split graph with vertex set partitioned into a maximal clique C (of size $k = \chi(G) = \omega(G)$) and an independent set I. We begin with a simple observation: if a split graph G with the given matching backbone M satisfies the equality $BBC(G, M) = k$, then for each vertex $u \in I$ there exists exactly one vertex $v \in C$ such that $c(u) = c(v)$.

Theorem 6 [2]. *Every complete graph of order $n \geq 3$ with a matching backbone can be colored using n colors.*

Theorem 7. *For every split graph G with the given matching backbone M there exists a backbone $(k+1)$–coloring and it can be computed in polynomial time.*

Proof. If $k = 2$ then G is bipartite and if we color the bipartition using 1 and 3 we obtain a backbone $(k+1)$–coloring. Therefore, the case $k \geq 3$ remains.

If there is an edge $uv \in E(M)$ such that $u, v \in C$, we assign $c(u) = k$, $c(v) = k - 2$. Next, we use Theorem 6 to color the complete graph with matching backbone, induced by $C \setminus \{u, v\}$, using colors $1, 2, \ldots, k-3, k-1$. Finally, we assign to all vertices from I color $k+1$ (not used in the clique). All backbone conditions between the clique and the independent set are satisfied since for any $xy \in E(M)$ with $x \in I$, $y \in C$ we have $c(x) - c(y) \geq k + 1 - k + 1 = 2$.

Otherwise, we pick an arbitrary edge $uv \in E(M)$ (we know that $u \in I$, $v \in C$) and any vertex $w \in C$ such that $uw \notin E(G)$. The rest of the proof is similar as before: we assign $c(u) = c(w) = k - 2$, $c(v) = k$, all vertices from $C \setminus \{v, w\}$ get (different) colors $1, 2, \ldots, k-3, k-1$ and all vertices from $I \setminus \{u\}$ get color $k+1$ (not used in the clique). All backbone conditions between the clique and the independent set are satisfied since for any $xy \in E(M)$ with $x \in I \setminus \{u\}$, $y \in C$ we have $c(x) - c(y) \geq k+1-k+1 = 2$ and for $uv \in E(M)$ clearly $c(u) - c(v) = 2$. ■

3.1. Basic case: $|C| = |I|$ and perfect matching backbone

In this section we restrict our considerations only to the split graphs with matching backbones which satisfy the following requirements: $|C| = |I| = k$ and backbone matching M is perfect. It follows that every vertex in I is adjacent to exactly one other vertex in the backbone (and its neighbor is in C). But since $|C| = |I|$, there can be no backbone edges in the clique.

Let us denote the vertices of C as $v_1, v_2, \ldots, v_k$ and the vertices of I as $u_1, u_2, \ldots, u_k$ such that $E(M) = \{\{u_i, v_i\} : 1 \leq i \leq k\}$.

The following theorem establishes the relation between backbone coloring of (G, M), conflict digraphs, quasiforests and backbone coloring of complete graphs with sparse backbones.

Theorem 8. *$BBC(G, M) = BBC(K_k, F')$ for the underlying undirected graph F' of some spanning quasiforest subdigraph F of the conflict digraph D of G.*

Proof. Suppose c is an optimal backbone coloring function for (G, M). Then let $E(F) = \{(w_i, w_j) : c(u_i) = c(v_j)\}$. Of course, F is a spanning quasiforest subdigraph of D since every u_i get the same color as exactly one vertex from C. The function $f(w_i) = c(v_i)$ for $w_i \in V(F')$ is a backbone coloring of (K_k, F') since:

- $f(w_i) \neq f(w_j)$ for each $i \neq j$, because C is a clique and $c(v_i) \neq c(v_j)$,
- $|f(w_i) - f(w_j)| = |c(v_i) - c(v_j)| = |c(v_i) - c(u_i)| \geq 2$ for each $\{w_i, w_j\} \in E(F')$.

The proof in the other direction is straightforward: given conflict graph D, quasiforest F and optimal backbone coloring f of (K_k, F'), let $s(w)$ be the end vertex of an edge of the quasiforest F, which starts in w. Then, we assign $c(v_i) = f(w_i)$ for $v_i \in C$ and $c(u_i) = f(s(w_i))$ for $u_i \in I$. ■

Theorem 9. *Let $k \geq 5$, F be a spanning quasiforest subdigraph of conflict digraph D of G, and let F' be the underlying undirected graph of F. Then, $BBC(K_k, F') = k$ unless F' contains a spanning star as a subgraph.*

Proof. The underlying graph of a quasiforest is a collection of vertex-disjoint trees and unicyclic graphs so the proof follows directly from Theorem 3. ■

Unfortunately, the number of spanning quasiforests can be exponential in the size of D. But if $k \geq 5$ we can check whether a suitable quasiforest exists starting from an arbitrary one using Algorithm 1. By an 3-*allowed edge* we denote an edge $(w_i, w_j) \in E(D) \setminus E(F)$, which—substituted for the edge outgoing from w_i in F—turns C_3-free underlying undirected graph of a quasiforest into the one with an induced C_3 as a subgraph. Similarly, an 4-*allowed edge* is an edge $(w_i, w_j) \in E(D) \setminus E(F)$, which—substituted for the edge outgoing from w_i in F—turns (P_4, bull, paw, banner)-free underlying undirected graph of a quasiforest into the one with an induced P_4 or one of the graphs in Figure 1 as a subgraph.

The underlying undirected graph of a quasiforest on $n \geq 5$ vertices is a collection of vertex-disjoint trees and unicyclic graphs, therefore:

- it is isomorphic to a star if and only if it is connected and it does not contain C_3 or P_4 as an induced subgraph,
- it is isomorphic to a star with an additional edge if and only if it is connected and it contains C_3, but not P_4 or one of the graphs in Figure 1 as an induced subgraph.

Algorithm 1 The optimal coloring algorithm for the basic case (G, M)

Create conflict digraph D and empty F on vertices $w_1, w_2, \ldots, w_k$
if $k < 5$ **then**
  return the best coloring of (G, M) using Theorem 8 for all quasiforests
for all w_i, $1 \leq i \leq k$ **do**
  if $outdeg(w_i) > 0$ **then**
    Add an arbitrary outgoing edge of w_i to F
if F' is isomorphic to a star **then**
  if exists (3 or 4)-allowed edge $(w_i, w_j) \in E(D) \setminus E(F)$ **then**
    Substitute in F edge outgoing from w_i with (w_i, w_j)
  else
    return $(k+1)$-coloring of (G, M) using Theorem 7
if F' is isomorphic to a star with an additional edge **then**
  if exists 4-allowed edge $(w_i, w_j) \in E(D) \setminus E(F)$ **then**
    Substitute in F edge outgoing from w_i with (w_i, w_j)
  else
    return $(k+1)$-coloring of (G, M) using Theorem 7
Color F' with k colors using Theorem 5
return k-coloring of (G, M) using coloring of F' and Theorem 8

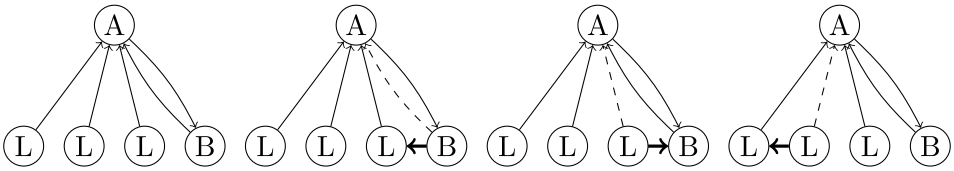
Theorem 10. *If $k < 5$, Algorithm 1 returns the optimal coloring of (G, M). In all other cases, if there exists an underlying undirected graph F' of any spanning quasiforest subdigraph F of a conflict graph of order k that $BBC(G, M) = k$, then Algorithm 1 returns a backbone k-coloring of (G, M). Otherwise, $BBC(G, M) = k + 1$ and Algorithm 1 returns a backbone $(k+1)$-coloring of (G, M).*

Proof. If $k < 5$, there are $O(1)$ of quasiforests so we can generate, color them optimally and find the one with the lowest $BBC(K_k, F')$ all in $O(1)$ time.

If the algorithm returns a backbone k-coloring, the final F' is not isomorphic to a star or a star with an additional edge (due to the allowed edge definitions). Therefore it does not contains a spanning star as a subgraph so $BBC(K_k, F') = k$ from Theorem 5 and the algorithm returns optimal backbone coloring.

If the algorithm returns backbone $(k+1)$-coloring of (G, M), both the initial and final F' are isomorphic to a star or a star with an additional edge.

If the quasiforest is isomorphic to a star, it has three distinguishable types of vertices: A—the root of the star (with ingoing edges from all other vertices and one outgoing edge), B—the non-root vertex with one ingoing edge from the root (and one outgoing edge to the root) and L—all other vertices (no ingoing edges, just one outgoing edge to the root). All edges from $B \cup L$ to A and from A to B are already included in the quasiforest F. All other edges of $E(D) \setminus E(F)$ may change the quasiforest in a following way (shown in Figure 2).



(a) Original quasitree (b) $B \to L$ (c) $L \to B$ (d) $L \to L$

Figure 2. (3 or 4)-allowed edge possibilities for a star.

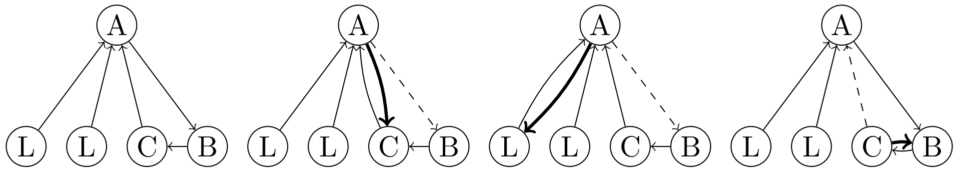
If there exists an edge from A to L, then an underlying undirected graph of a quasiforest obtained by removing an edge from A to B and replacing it by an edge from A to L is still isomorphic to a star, so the edge is not a (3 or 4)-allowed edge (by definition).

If there exists an edge from L to B, then an underlying undirected graph of a quasiforest obtained by removing an edge from L to A and replacing it by an edge from L to B is isomorphic to a star with an additional edge—so it is a 3-allowed edge (an example is presented in second picture in Figure 2).

For any other edge, a quasiforest obtained by replacing respective outgoing edge from F with a new edge has an induced a P_4 path so it is 4-allowed edge—the only two possible cases are presented in third and fourth picture in Figure 2.

If we cannot find a (3 or 4)-allowed edge while the quasiforest is isomorphic to a star, all edges from $E(D) \setminus E(F)$ are restricted to edges from A to L. But then the underlying undirected graph of any quasiforest F is isomorphic to a star and, due to Theorem 5, no (K_k, F') (and its respective (G, M)) admits backbone coloring with colors no greater than k.

If the quasiforest is isomorphic to a star with an additional edge, we distinguish four types of vertices: A—the root of the star (with ingoing edges from all but one other vertices and one outgoing edge), B—the non-root vertex with one ingoing edge from the root and outgoing edge to a non-root vertex, C—the non-root vertex with one ingoing edge from a non-root vertex and outgoing edge to the root, and L—all other vertices (no ingoing edges, just one outgoing edge to the root). All edges from $C \cup L$ to A, from A to B and from B to C are already included in the quasiforest F. All other edges of $E(D) \setminus E(F)$ may change the quasiforest in a following way (shown in Figure 3).



(a) Original quasitree (b) $A \to C$ (c) $A \to L$ (d) $C \to B$

(e) $C \to L$ (f) $L \to B$ (g) $L \to C$ (h) $L \to L$

Figure 3. 4-allowed edge possibilities for a star with an additional edge.

If there exists an edge from B to L, then an underlying undirected graph of a quasiforest obtained by removing an edge from B to C and replacing it by an edge from B to L is isomorphic to a star with an additional edge, so the edge is not a 4-allowed edge (by definition).

If there exists an edge from B to A, then an underlying undirected graph of a quasiforest obtained by removing an edge from B to C and replacing it by an edge from B to A is isomorphic to a star, so the edge is noa 4-allowed edge (by definition).

For any other edge, a quasiforest obtained by replacing respective outgoing edge from F with a new edge has an induced P_4 path or one of the graphs listed in Figure 1 (seven possible subcases are presented in Figure 3), therefore it is a 4-allowed edge.

If we cannot find a 4-allowed edge while the quasiforest is isomorphic to a star with an additional edge, all edges from $E(D) \setminus E(F)$ are restricted to edges from B to $A \cup L$. But then the underlying undirected graph of any quasiforest F is isomorphic to a star or to a star with an additional edge and, due to Theorem

5, no (K_k, F') (and its respective (G, M)) admits backbone coloring with colors no greater than k.

In both cases, $BBC(G, M) > k$ so the backbone $(k+1)$-coloring of (G, M) is guaranteed to be optimal. ■

Theorem 11. *For every split graph G of order $2k$ with maximal clique C and an independent set I both of size k and the given perfect matching backbone M with no backbone edges in the clique, $BBC(G, M)$ can be computed in polynomial time.*

Proof. This result directly follows from Theorem 10.

The Algorithm 1 can be divided into several stages with polynomial running time of every stage:

- construction of the conflict graph requires checking all pairs $\{v_i, u_j\} \in E(G)$, which can be done in $O(k^2)$ time,
- choosing the initial quasiforest F—$O(k)$ operations,
- checking and updating F using Algorithm 1—$O(k^2)$ operations,
- coloring the original graph G using the coloring of F'—$O(k)$ operations.

■

3.2. Beyond the basic case

Now we can expand our analysis to the polynomial-time algorithm for arbitrary split graphs with the given matching backbone.

Theorem 12. *For every split graph G with the given matching backbone M, $BBC(G, M)$ can be computed in polynomial time.*

Proof. There are possible types of vertices, which do not appear in Theorem 11: vertices not included in the backbone matching M and pairs of vertices from C connected with a backbone edge.

Interestingly, it turns out that if G contains in I only vertices incident to a backbone edge, we can also obtain the optimal solution by including all vertices from C in the conflict graph, e.g., adding its respective vertices w_i to the conflict graph D. If v_i is not incident to any edge of M, w_i will have no outgoing edges in D, but it will still receive unique color in the optimal coloring. If it is connected with some v_j in M, we can add new vertices w_i, w_j to the conflict graph with two directed edges between them. Because $outdeg(w_i) = outdeg(w_j) = 1$, these edges will be always included in F so they guarantee that $|c(v_i) - c(v_j)| \geq 2$ and that these colors are unique in the optimal coloring.

The rest of the proof is exactly the same as for the proof of Theorem 10 except that some original cases may be impossible, e.g. if G contains a backbone edge with both endpoints in C, then the underlying undirected graph of any quasiforest of a conflict graph cannot be isomorphic to a star.

Therefore, the whole algorithm would first find the bipartition into a maximal clique C and an independent set I. In the next step, it would find the vertices from I non-incident to the backbone edges and remove them from the graph. Then, we execute Algorithm 1 on the remaining part of (G, M); we obtain its optimal backbone coloring. Finally, we would restore all unmatched vertices from I and note that for each uncolored vertex $u \in I$ there is a vertex $v \in C$ such that $\{u, v\} \notin E(G)$, otherwise u would be in C. Therefore we extend the coloring by assigning to u the color $c(v)$ of its respective non-adjacent vertex.

Since all steps can be implemented to run in polynomial time, the whole algorithm is also polynomial-time. ■

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