Bull. Malays. Math. Sci. Soc. (2016) 39:S219–S227
DOI 10.1007/s40840-015-0290-2

BULLETIN of the MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Non-isolating Bondage in Graphs

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Received: 22 February 2014 / Revised: 18 July 2014 / Published online: 31 December 2015

Abstract A dominating set of a graph $G = (V, E)$ is a set D of vertices of G such that every vertex of $V(G)\setminus D$ has a neighbor in D. The domination number of a graph G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. The non-isolating bondage number of G, denoted by $b'(G)$, is the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\delta(G - E') \geq 1$ and $\gamma(G - E') > \gamma(G)$. If for every $E' \subseteq E$ we have $\gamma(G - E') = \gamma(G)$ or $\delta(G - E') = 0$, then we define $b'(G) = 0$, and we say that G is a γ-non-isolatingly strongly stable graph. First we discuss various properties of non-isolating bondage in graphs. We find the non-isolating bondage numbers for several classes of graphs. Next we show that for every non-negative integer, there exists a tree having such non-isolating bondage number. Finally, we characterize all γ-non-isolatingly strongly stable trees.

Keywords Domination · Bondage · Non-isolating bondage · Graph · Tree

Mathematics Subject Classification 05C05 · 05C69

Communicated by Xueliang Li.

Research partially supported by the Polish National Science Centre Grant 2011/02/A/ST6/00201. Marcin Krzywkowski—Research fellow of the Claude Leon Foundation.

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1 Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G, we mean the set $N_G(v) = \{u \in V(G) \colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. Let $\delta(G)$ mean the minimum degree among all vertices of G. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). The distance between two vertices of a graph is the number of edges in a shortest path connecting them. The eccentricity of a vertex is the greatest distance between it and any other vertex. The diameter of a graph G, denoted by $\operatorname{diam}(G)$, is the maximum eccentricity among all vertices of G. We denote the path (cycle, respectively) on n vertices by P_n (C_n, respectively). A wheel W_n, where $n \geq 4$, is a graph with n vertices, formed by connecting a vertex to all vertices of a cycle C_{n-1}. Let T be a tree, and let v be a vertex of T. We say that v is adjacent to a path P_n if there is a neighbor of v, say x, of degree two such that the tree resulting from T by removing the edge vx, and which contains the vertex x, is a path P_n. Let $K_{p,q}$ denote a complete bipartite graph the partite sets of which have cardinalities p and q. By a star we mean a connected graph in which exactly one vertex has degree greater than one.

A subset $D \subseteq V(G)$ is a dominating set, abbreviated DS, of G if every vertex of $V(G) \setminus D$ has a neighbor in D. The domination number of a graph G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. For a comprehensive survey of domination in graphs, see for example [5].

The bondage number $b(G)$ of a graph G is the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\gamma(G - E') > \gamma(G)$. The concept of bondage in graphs was introduced in [2] and further studied for example in [1,3,4,6–9].

We define the non-isolating bondage number of a graph G, denoted by $b'(G)$, to be the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\delta(G - E') \geq 1$ and $\gamma(G - E') > \gamma(G)$. Thus $b'(G)$ is the minimum number of edges of G that have to be removed in order to obtain a graph with no isolated vertices, and with the domination number greater than that of G. If for every $E' \subseteq E$ we have $\gamma(G - E') = \gamma(G)$ or $\delta(G - E') = 0$, then we define $b'(G) = 0$, and we say that G is a γ-non-isolatingly strongly stable graph.

First we discuss various properties of non-isolating bondage in graphs. We find the non-isolating bondage numbers for several classes of graphs. Next we show that for every non-negative integer, there exists a tree having such non-isolating bondage number. Finally, we characterize all γ-non-isolatingly strongly stable trees.

2 Results

We begin with the following well known observations.

For every graph G of diameter at least two there exists a $\gamma(G)$-set that contains all support vertices.

If H is a subgraph of G such that $V(H) = V(G)$, then $\gamma(H) \geq \gamma(G)$.

If n is a positive integer, then $\gamma(P_n) = \lfloor (n + 2)/3 \rfloor$.

For every integer $n \geq 3$ we have $\gamma(C_n) = \lfloor (n+2)/3 \rfloor$.

Observation 1 *If n is a positive integer, then $\gamma(K_n) = 1$.*

Observation 2 *For every integer $n \geq 4$ we have $\gamma(W_n) = 1$.*

Observation 3 *Let p and q be positive integers such that $p \leq q$. Then*

$$\gamma(K_{p,q}) = \begin{cases} 1 & \text{if } p = 1; \\ 2 & \text{otherwise.} \end{cases}$$

First we calculate the non-isolating bondage numbers of paths.

Lemma 4 *For any positive integer n we have*

$$b'(P_n) = \begin{cases} 0 & \text{if } n = 1, 2, 3, 4, 5, 7; \\ 1 & \text{if } n \geq 6 \text{ and } n \neq 3k+1; \\ 2 & \text{if } n \geq 10 \text{ and } n = 3k+1. \end{cases}$$

Proof Let us observe that if a path has at most five or exactly seven vertices, then removing any edges does not increase the domination number, or gives an isolated vertex. Assume that $n = 6$ or $n \geq 8$. First assume that $n = 3k$. We have $\gamma(P_n) = \lfloor (n+2)/3 \rfloor = \lfloor (3k+2)/3 \rfloor = k$. We also have $\gamma(P_{n-2}) + \gamma(P_2) = \lfloor n/3 \rfloor + 1 = k+1 > \gamma(P_n)$. Thus $b'(P_n) = 1$ if $n = 3k$ and $n \geq 6$. Now assume that $n = 3k+2$. We have $\gamma(P_n) = \lfloor (n+2)/3 \rfloor = \lfloor (3k+4)/3 \rfloor = k+1$. We also have $\gamma(P_{n-4}) + \gamma(P_4) = \lfloor n/3 \rfloor + 2 = k+2 > \gamma(P_n)$. Thus $b'(P_n) = 1$ if $n = 3k+2$ and $n \geq 8$. Now assume that $n = 3k+1$. We have $\gamma(P_n) = \lfloor (n+2)/3 \rfloor = \lfloor (3k+3)/3 \rfloor = k+1$. Let us observe that removing any edge does not increase the domination number. We have $\gamma(P_{n-6}) + \gamma(P_4) + \gamma(P_2) = \lfloor (n-4)/3 \rfloor + 3 = \lfloor (3k-3)/3 \rfloor + 3 = k+2 > \gamma(P_n)$. Therefore $b'(P_n) = 2$ if $n = 3k+1$ and $n \geq 10$. □

We now investigate the non-isolating bondage in cycles.

Lemma 5 *For every integer $n \geq 3$ we have*

$$b'(C_n) = \begin{cases} 0 & \text{if } b'(P_n) = 0; \\ b'(P_n) + 1 & \text{if } b'(P_n) \neq 0. \end{cases}$$

Proof We have $\gamma(P_n) = \gamma(C_n)$. Clearly, $C_n - e = P_n$. This implies that $b'(C_n) = 0$ if $b'(P_n) = 0$, while $b'(C_n) = b'(P_n) + 1$ if $b'(P_n) \neq 0$. □

We now find the non-isolating bondage numbers of complete graphs.

Proposition 6 *If n is a positive integer, then*

$$b'(K_n) = \begin{cases} 0 & \text{for } n = 1, 2, 3; \\ \lfloor (n+1)/2 \rfloor & \text{for } n \geq 4. \end{cases}$$

Proof Obviously, $b'(K_1) = 0$ and $b'(K_2) = 0$. We have $K_3 - e = C_3$ and $b'(C_3) = 0$. This implies that $b'(K_3) = 0$. Now assume that $n \geq 4$. By Observation 1 we have $\gamma(K_n) = 1$. Let us observe that the domination number of a graph equals one if and only if the graph has a universal vertex. Given a complete graph, we increase the domination number if and only if for every vertex we remove at least one incident edge. If n is even, then we remove $n/2 = \lfloor (n+1)/2 \rfloor$ edges. If n is odd, then we remove $(n-1)/2 + 1 = (n+1)/2 = \lfloor (n+1)/2 \rfloor$ edges. □

We now calculate the non-isolating bondage numbers of wheels.

Proposition 7 *For integers $n \geq 4$ we have*

$$b'(W_n) = \begin{cases} 2 & \text{if } n = 4; \\ 1 & \text{if } n \geq 5. \end{cases}$$

Proof Since $W_4 = K_4$, using Proposition 6 we get $b'(W_4) = b'(K_4) = \lfloor 5/2 \rfloor = 2$. Now assume that $n \geq 5$. By Observation 2 we have $\gamma(W_n) = 1$. The domination number of a graph equals one if and only if it has a universal vertex. Removing an edge of W_n incident to the vertex of maximum degree gives a graph without universal vertices. Therefore $b'(W_n) = 1$ for $n \geq 5$. □

We now investigate the non-isolating bondage in complete bipartite graphs.

Proposition 8 *Let p and q be positive integers such that $p \leq q$. Then*

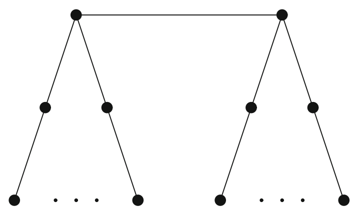
$$b'(K_{p,q}) = \begin{cases} 0 & \text{if } p = 1, 2; \\ 4 & \text{if } p = 3; \\ p & \text{otherwise.} \end{cases}$$

Proof Let $E(K_{p,q}) = \{a_i b_j : 1 \leq i \leq p \text{ and } 1 \leq j \leq q\}$. If $p = 1$, then obviously $b'(K_{p,q}) = 0$ as removing any edge produces an isolated vertex. Now assume that $p \geq 2$. By Observation 3 we have $\gamma(K_{p,q}) = 2$. Let E' be a subset of the set of edges of $K_{2,q}$ such that $\delta(K_{2,q} - E') \geq 1$. Each vertex b_i is adjacent to a_1 or a_2 in the graph $K_{2,q} - E'$. Observe that the vertices a_1 and a_2 form a dominating set of $K_{2,q} - E'$. Therefore $b'(K_{2,q}) = 0$. Now assume that $p = 3$. It is not very difficult to verify that removing any three edges does not increase the domination number while not producing an isolated vertex. We have $\gamma(K_{3,q} - a_1b_2 - a_1b_3 - a_2b_1 - a_3b_1) = 3 > 2 = \gamma(K_{3,q})$. Therefore $b'(K_{3,q}) = 4$. Now assume that $p \geq 4$. If we remove at most $p-1$ edges, then there are vertices a_i and b_j which have degrees q and p, respectively. It is easy to observe that the vertices a_i and b_j still form a dominating set. Let us observe that $\gamma(K_{p,q} - a_1b_1 - a_2b_1 - a_3b_2 - a_4b_2 - a_5b_2 - \cdots - a_pb_2) = 3 > 2 = \gamma(K_{p,q})$. Therefore $b'(K_{p,q}) = p$ if $p \geq 4$. □

The authors of [2] proved that the bondage number of any tree is either one or two.

Theorem 9 ([2]) *For every tree T we have $b(T) \in \{1, 2\}$.*

Fig. 1 A tree T_k having $4k+2$ vertices, where both central vertices are of degree $k+1$



Let us observe that for every non-negative integer there exists a tree with such non-isolating bondage number. We have $b'(P_4)=0$. For positive integers k, consider trees T_k of the form presented in Fig. 1. It is not difficult to verify that $b'(T_k)=k$.

Hartnell and Rall [3] characterized all trees with bondage number equal to two. We characterize all trees with the non-isolating bondage number equal to zero, that is, all γ-non-isolatingly strongly stable trees.

We now show that joining two γ-non-isolatingly strongly stable trees gives us also a γ-non-isolatingly strongly stable tree.

Lemma 10 *Let T_1 and T_2 be vertex-disjoint γ-non-isolatingly strongly stable trees. Let x be a support vertex of T_1 and let y be a leaf of T_2. Let T be a tree obtained by joining the vertices x and y. If $\gamma(T)=\gamma(T_1)+\gamma(T_2)$, then the tree T is also γ-non-isolatingly strongly stable.*

Proof Let E_1 be a subset of the set of edges of T such that $\delta(T-E_1)\geq 1$. If $xy\in E_1$, then we get $\gamma(T-E_1)=\gamma(T_1-E_1\cap E(T_1))+\gamma(T_2-E_1\cap E(T_2))=\gamma(T_1)+\gamma(T_2)=\gamma(T)$. Now assume that $xy\notin E_1$. Let z be the neighbor of y other than x. If $yz\notin E_1$, then let $E_2=E_1\cup\{xy\}$. Similarly as earlier we get $\gamma(T-E_2)=\gamma(T)$. We have $\gamma(T-E_1)\leq\gamma(T-E_2)$, and consequently, $\gamma(T-E_1)=\gamma(T)$. Now assume that $yz\in E_1$. Let $E_3=E_1\cup\{xy\}\backslash\{yz\}$. Similarly as earlier we get $\gamma(T-E_3)=\gamma(T)$. Let D_2 be a $\gamma(T-E_3)$-set that contains the vertices x and z. It is easy to observe that D_2 is also a DS of the graph $T-E_1$. Therefore $\gamma(T-E_1)\leq\gamma(T-E_3)$. This implies that $\gamma(T-E_1)=\gamma(T)$. We now conclude that $b'(T)=0$. □

We next show that a subtree of a γ-non-isolatingly strongly stable tree is also γ-non-isolatingly strongly stable.

Lemma 11 *Let T be a γ-non-isolatingly strongly stable tree. Assume that T' is a subtree of T such that $T-T'$ has no isolated vertices. Then $b'(T')=0$.*

Proof If T' consists of a single vertex, then obviously $b'(T')=0$. Thus assume that $T'\neq K_1$. Let E_1 be the minimum subset of $E(T)$ such that T' is a component of $T-E_1$. Now let E' be a subset of $E(T')$ such that $\delta(T'-E')\geq 1$. Notice that $\delta(T-E_1-E')\geq 1$. The assumption $b'(T)=0$ implies that $\gamma(T-E_1)=\gamma(T)$ and $\gamma(T-E_1-E')=\gamma(T)$. We have $T-E_1-E'=T'-E'\cup(T-T')$ and $T-E_1=T'\cup(T-T')$. We now get $\gamma(T'-E')=\gamma(T-E_1-E')-\gamma(T-T')$ $=\gamma(T)-\gamma(T-E_1)+\gamma(T')=\gamma(T')$. This implies that $b'(T')=0$. □

For the purpose of characterizing all γ-non-isolatingly strongly stable trees, we introduce a family $\mathcal{T}$ of trees $T=T_k$ that can be obtained as follows. Let $T_1\in\{P_1,P_2\}$.

If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a vertex by joining it to any support vertex of T_k.
- Operation $\mathcal{O}_2$: Attach a path P_2 by joining one of its vertices to a vertex of T_k, which is adjacent to a path P_1 or P_4, or is not a leaf and is adjacent to a support vertex.
- Operation $\mathcal{O}_3$: Attach a path P_3 by joining one of its leaves to a vertex of T_k adjacent to a path P_1 or P_3.
- Operation $\mathcal{O}_4$: Attach a path P_5 by joining one of its leaves to any support vertex of T_k.

We now prove that every tree of the family $\mathcal{T}$ is γ-non-isolatingly strongly stable.

Lemma 12 *If $T \in \mathcal{T}$, then $b'(T) = 0$.*

Proof We use induction on the number k of operations performed to construct the tree T. If $T = P_1$, then obviously $b'(T) = 0$. If $T = P_2$, then $b'(T) = 0$ as removing the edge gives isolated vertices. Let k be a positive integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k-1$ operations. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by k operations.

First assume that T is obtained from T' by Operation $\mathcal{O}_1$. Let x be the attached vertex, and let y be its neighbor. Let z be a leaf adjacent to y and different from x. Let D be a $\gamma(T)$-set that contains all support vertices. The set D is minimal, thus $x \notin D$. Obviously, D is a DS of the tree T'. Therefore $\gamma(T') \le \gamma(T)$. Now let E' be a subset of the set of edges of T such that $\delta(T - E') \ge 1$. Since both x and z are leaves of T, we have $xy \notin E'$ and $yz \notin E'$. The assumption $b'(T') = 0$ implies that $\gamma(T' - E') = \gamma(T')$. Let us observe that there exists a $\gamma(T' - E')$-set that contains the vertex y. Let D' be such a set. It is easy to see that D' is a DS of the graph $T - E'$. Thus $\gamma(T - E') \le \gamma(T' - E')$. We now get $\gamma(T - E') \le \gamma(T' - E') = \gamma(T') \le \gamma(T)$. On the other hand, we have $\gamma(T - E') \ge \gamma(T)$. This implies that $\gamma(T - E') = \gamma(T)$, and consequently, $b'(T) = 0$.

Now assume that T is obtained from T' by Operation $\mathcal{O}_2$. The vertex to which is attached P_2 we denote by x. Let v_1v_2 be the attached path. Let v_1 be joined to x. If x is adjacent to a leaf or a support vertex, say a, then let D be a $\gamma(T)$-set that contains all support vertices. We have $v_2 \notin D$ as the set D is minimal. It is easy to observe that $D \setminus \{v_1\}$ is a DS of the tree T'. If x is adjacent to a path P_4, then we denote it by $abcd$. Let a and x be adjacent. Let us observe that there exists a $\gamma(T)$-set that contains the vertices v_1, c, and x. Let D be such a set. It is easy to observe that $D \setminus \{v_1\}$ is a DS of the tree T'. We conclude that $\gamma(T') \le \gamma(T) - 1$. Now let E' be a subset of the set of edges of T such that $\delta(T - E') \ge 1$. Since v_2 is a leaf of T, we have $v_1v_2 \notin E'$. If $xv_1 \in E'$, then $\delta(T' - (E' \cap E(T'))) \ge 1$. We get $\gamma(T - E') = \gamma(P_2 \cup T' - (E' \setminus \{xv_1\})) = \gamma(T' - (E' \cap E(T'))) + \gamma(P_2) = \gamma(T') + 1 \le \gamma(T)$. Now assume that $xv_1 \notin E'$. By T_x (T'_x, respectively), we denote the component of $T - E'$ ($T' - E'$, respectively) which contains the vertex x. If $\delta(T' - (E' \cap E(T'))) \ge 1$, then let D'_x be any $\gamma(T'_x)$-set. It is easy to see that $D'_x \cup \{v_1\}$ is a DS of the tree T_x. Thus $\gamma(T_x) \le \gamma(T'_x) + 1$. We now get $\gamma(T - E') = \gamma(T - E' - T_x) + \gamma(T_x) \le \gamma(T - E' - T_x) + \gamma(T'_x) + 1 = \gamma(T' - E' - T'_x) + \gamma(T'_x) + 1 = \gamma(T' - E') + 1 = \gamma(T') + 1 \le \gamma(T)$. Now assume that $\delta(T' - (E' \cap E(T'))) = 0$. This implies that x is the only isolated vertex of $T' - (E' \cap E(T'))$, and so x is not adjacent to any leaf in the trees T' and T. Consequently, T'_x consists only of the vertex x, and T_x is a path P_3. Let us observe that $\delta(T' - (E' \backslash \{xa\})) \geq 1$. Let T'_a be the component of $T' - E'$, which contains the vertex a. Now let T''_a be a tree obtained from T'_a by attaching a vertex to the vertex a. We now get $\gamma(T - E') = \gamma(T - E' - T_x) + \gamma(P_3) = \gamma(T' - E' - T'_x) + 1$ $= \gamma(T' - E' - T'_x - T'_a) + \gamma(T'_a) + 1 \leq \gamma(T' - E' - T'_x - T'_a) + \gamma(T''_a) + 1 = \gamma((T' - E'$ $-T'_x - T'_a) \cup T''_a) + 1 = \gamma(T' - (E' \backslash \{xa\})) + 1 = \gamma(T' - E') + 1 = \gamma(T') + 1 \leq \gamma(T)$. We conclude that $\gamma(T - E') = \gamma(T)$, and consequently, $b'(T) = 0$.

Now assume that T is obtained from T' by Operation $\mathcal{O}_3$. The vertex to which is attached P_3 we denote by x. If x is a support vertex, then using Lemma 10, for $T_1 = T'$ and $T_2 = P_3$, we get $b'(T) = 0$. Now assume that x is adjacent to a path P_3, say abc. Let a and x be adjacent. The attached path we denote by $v_1v_2v_3$. Let v_1 be joined to x. Let us observe that there exists a $\gamma(T)$-set that contains all support vertices and does not contain the vertex v_1. Let D be such a set. We have $v_3 \notin D$ as the set D is minimal. Observe that $D \backslash \{v_2\}$ is a DS of the tree T'. Therefore $\gamma(T') \leq \gamma(T) - 1$. Now let E' be a subset of the set of edges of T such that $\delta(T - E') \geq 1$. We have $v_2v_3 \notin E'$ as the vertex v_3 is a leaf. If $xv_1 \in E'$, then $v_1v_2 \notin E'$; otherwise we get an isolated vertex. Let us observe that $\delta(T' - (E' \cap E(T'))) \geq 1$. We get $\gamma(T - E')$ $= \gamma(P_3 \cup T - (E' \backslash \{xv_1\})) = \gamma(T' - (E' \cap E(T'))) + \gamma(P_3) = \gamma(T') + 1 \leq \gamma(T)$. Now assume that $xv_1 \notin E'$. Because of the similarity between the paths abc and $v_1v_2v_3$ adjacent to the vertex x, it suffices to consider only the possibility when $xa \notin E'$. Let us observe that $\delta(T' - (E' \cap E(T'))) \geq 1$. By T_x (T'_x, respectively), we denote the component of $T - E'$ ($T' - (E' \cap E(T'))$, respectively) which contains the vertex x. If $v_1v_2 \notin E'$, then let D'_x be any $\gamma(T'_x)$-set. It is easy to see that $D'_x \cup \{v_2\}$ is a DS of the tree T_x. Thus $\gamma(T_x) \leq \gamma(T'_x) + 1$. We now get $\gamma(T - E') = \gamma(T - E' - T_x) + \gamma(T_x) \leq$ $\gamma(T - E' - T_x) + \gamma(T'_x) + 1 = \gamma(T' - E' - T'_x) + \gamma(T'_x) + 1 = \gamma(T' - E') + 1 =$ $\gamma(T') + 1 \leq \gamma(T)$. Now assume that $v_1v_2 \in E'$. Because of the similarity between the paths abc and $v_1v_2v_3$, it suffices to consider only the possibility when $ab \in E'$. Let D'_x be a $\gamma(T'_x)$-set that contains all support vertices (so $x \in D'_x$). It is easy to see that D'_x is a DS of the tree T_x. Thus $\gamma(T_x) \leq \gamma(T'_x)$. We get $\gamma(T - E') = \gamma(T - E' - T_x) + \gamma(T_x)$ $\leq \gamma(T - E' - T_x) + \gamma(T'_x) = \gamma(T' - E' - T'_x) + \gamma(T'_x) = \gamma(T' - E') = \gamma(T') \leq \gamma(T)$. We now conclude that $\gamma(T - E') = \gamma(T)$, and consequently, $b'(T) = 0$.

Now assume that T is obtained from T' by Operation $\mathcal{O}_4$. By Lemma 4 we have $b'(P_5) = 0$. Using Lemma 10, for $T_1 = T'$ and $T_2 = P_5$, we get $b'(T) = 0$. □

We now prove that if a tree is γ-non-isolatingly strongly stable, then it belongs to the family $\mathcal{T}$.

Lemma 13 *Let T be a tree. If $b'(T) = 0$, then $T \in \mathcal{T}$.*

Proof If $\mathrm{diam}(T) \in \{0, 1\}$, then $T \in \{P_1, P_2\} \subseteq \mathcal{T}$. If $\mathrm{diam}(T) = 2$, then T is a star. The tree T can be obtained from P_2 by an appropriate number of Operations $\mathcal{O}_1$. Thus $T \in \mathcal{T}$. Now assume that $\mathrm{diam}(T) \geq 3$. Thus the order n of the tree T is at least four. We obtain the result by the induction on the number n. Assume that the lemma is true for every tree T' of order $n' < n$.

First assume that some support vertex of T, say x, is strong. Let y be a leaf adjacent to x. Let $T' = T - y$. Let D' be a $\gamma(T')$-set that contains all support vertices. It is easy to see that D' is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T')$. Now let E' be a subset of the set of edges of T' such that $\delta(T' - E') \geq 1$. Since $b'(T) = 0$, we have $\gamma(T - E') = \gamma(T)$. Let us observe that there exists a $\gamma(T - E')$-set that contains the vertex x. Let D be such a set. The set D is minimal, thus $y \notin D$. Obviously, D is a DS of the graph $T' - E'$. Therefore $\gamma(T' - E') \leq \gamma(T - E')$. We now get $\gamma(T' - E') \leq \gamma(T - E') = \gamma(T) \leq \gamma(T')$. On the other hand, we have $\gamma(T' - E') \geq \gamma(T')$. This implies that $\gamma(T' - E') = \gamma(T')$, and consequently, $b'(T') = 0$. By the inductive hypothesis, we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by Operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$. Henceforth, we assume that every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity $\text{diam}(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, and u be the parent of v in the rooted tree. If $\text{diam}(T) \geq 4$, then let w be the parent of u. If $\text{diam}(T) \geq 5$, then let d be the parent of w. If $\text{diam}(T) \geq 6$, then let e be the parent of d. By T_x we denote the subtree induced by a vertex x and its descendants in the rooted tree T.

Assume that $d_T(u) \geq 3$. Thus some child of u is a leaf or a support vertex other than v. Let $T' = T - T_v$. By Lemma 11 we have $b'(T') = 0$. By the inductive hypothesis, we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by Operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$.

Now assume that $d_T(u) = 2$. Assume that $d_T(w) \geq 3$. First assume that there is a child of w other than u, say k, such that the distance of w to the most distant vertex of T_k is three. It suffices to consider only the possibility when T_k is a path P_3, say klm. Let $T' = T - T_u$. By Lemma 11 we have $b'(T') = 0$. By the inductive hypothesis, we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by Operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$.

Now assume that some child of w is a leaf. Let $T' = T - T_u$. By Lemma 11 we have $b'(T') = 0$. By the inductive hypothesis, we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by Operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$.

Thus there is a child of w, say k, such that the distance of w to the most distant vertex of T_k is two. Consequently, k is a support vertex of degree two. Due to the earlier analysis of the children of the vertex u, it suffices to consider only the possibility when $d_T(w) = 3$. Let $T' = T - T_w$. It is easy to observe that $D' \cup \{v, k\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 2$. We have $\delta(T - dw - uv - wk) \geq 1$. We now get $\gamma(T - dw - uv - wk) = \gamma(T' \cup P_2 \cup P_2 \cup P_2) = \gamma(T') + 3\gamma(P_2)$ $= \gamma(T') + 3 \geq \gamma(T) + 1 > \gamma(T)$. This implies that $b'(T) \neq 0$, a contradiction.

If $d_T(w) = 1$, then $T = P_4$. Let $T' = P_2 \in \mathcal{T}$. The tree T can be obtained from T' by Operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$. Now assume that $d_T(w) = 2$. First assume that there is a child of d other than w, say k, such that the distance of d to the most distant vertex of T_k is four or one. It suffices to consider only the possibilities when T_k is a path P_4, or k is a leaf. Let $T' = T - T_w$. Let us observe that there exists a $\gamma(T')$-set that contains the vertex d. Let D' be such a set. It is easy to observe that $D' \cup \{v\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 1$. We have $\delta(T - dw - uv) \geq 1$. We now get $\gamma(T - dw - uv) = \gamma(T' \cup P_2 \cup P_2) = \gamma(T') + 2\gamma(P_2) = \gamma(T') + 2 \geq \gamma(T) + 1 >$ $\gamma(T)$. This implies that $b'(T) \neq 0$, a contradiction.

Now assume that there is a child of d, say k, such that the distance of d to the most distant vertex of T_k is three. It suffices to consider only the possibility when T_k is a path P_3, say klm. Let $T' = T - T_l$. Due to the similarity of T' to the tree T from the previous case when d is adjacent to a leaf, we conclude that $b'(T') \neq 0$. On the other hand, by Lemma 11 we have $b'(T') = 0$, a contradiction.

Now assume that there is a child of d, say k, such that the distance of d to the most distant vertex of T_k is two. Thus k is a support vertex of degree two. Let $T' = T - T_k$. By Lemma 11 we have $b'(T') = 0$. By the inductive hypothesis, we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by Operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$.

If $d_T(d) = 1$, then $T = P_5$. Let $T' = P_2 \in \mathcal{T}$. The tree T can be obtained from T' by Operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$.

Now assume that $d_T(d) = 2$. First assume that e is adjacent to a leaf, say k. Let $T' = T - T_d$. By Lemma 11 we have $b'(T') = 0$. By the inductive hypothesis, we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by Operation $\mathcal{O}_4$. Thus $T \in \mathcal{T}$.

Now assume that e is not adjacent to any leaf. Let E' be the set of edges incident with e excluding ed. Let $G' = T - T_d - e$. Let D' be any $\gamma(G')$-set. It is easy to observe that $D' \cup \{d, v\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(G') + 2$. We have $\delta(T - (E' \cup \{dw, uv\})) \geq 1$. We now get $\gamma(T - (E' \cup \{dw, uv\})) = \gamma(G' \cup P_2 \cup P_2 \cup P_2) = \gamma(G') + 3\gamma(P_2) = \gamma(G') + 3 \geq \gamma(T) + 1 > \gamma(T)$. This implies that $b'(T) \neq 0$, a contradiction. □

As an immediate consequence of Lemmas 12 and 13, we have the following characterization of all γ-non-isolatingly strongly stable trees.

Theorem 14 *Let T be a tree. Then $b'(T) = 0$ if and only if $T \in \mathcal{T}$.*

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