



# Boundary problems for fractional differential equations

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## ABSTRACT

In this paper, the existence of solutions of fractional differential equations with nonlinear boundary conditions is investigated. The monotone iterative method combined with lower and upper solutions is applied. Fractional differential inequalities are also discussed. Two examples are added to illustrate the results.

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## 1. Introduction

This paper discusses the boundary value problem

$$\begin{cases} (D_T^q x)(t) = f(t, x(t)) \equiv (Fx)(t), & t \in J_0 = [0, T], \quad T > 0, \\ 0 = g(\bar{x}(0), \bar{x}(T)), \end{cases} \quad (1)$$

where  $f \in C(J \times \mathbb{R}, \mathbb{R})$ ,  $J = [0, T]$ ,  $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ ,  $\bar{x}(0) = (T-t)^{1-q}x(t)|_{t=0}$ ,  $\bar{x}(T) = (T-t)^{1-q}x(t)|_{t=T}$  and  $D_T^q x$  denotes the right-handed Riemann–Liouville fractional derivative of  $x$  with  $q \in (0, 1)$  defined by

$$(D_T^q x)(t) = -\frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_t^T (s-t)^{-q} x(s) ds, \quad t \in J_0,$$

and if  $q = 1$ , then  $D_T^1 x(t) = -x'(t)$ , see [1].

Let us introduce the right-sided fractional integral  $I_T^q$  of order  $q > 0$  by

$$(I_T^q x)(t) = \frac{1}{\Gamma(q)} \int_t^T (s-t)^{q-1} x(s) ds, \quad t \in J_0,$$

see [1].

Function  $x$  is a solution of problem (1) if:

- (i)  $x \in C(J_0, \mathbb{R})$ ,  $(T-t)^{1-q}u \in C(J, \mathbb{R})$  and its fractional integral  $I_T^{1-q}x$  is continuously on  $J_0$ ,
- (ii)  $x$  satisfies problem (1).

Recently, much attention has been paid to study fractional differential problems with fractional derivatives  $D^q x$ , using the monotone iterative method, see for example, [1–11]. In this paper, we study fractional differential problems with derivatives

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$D_T^q x$  of order  $q \in (0, 1]$ . As we see later, we will discuss fractional differential equations with initial conditions at the endpoint of interval  $J$ . Therefore, we have to introduce the space  $C_{1-q}$  by

$$C_{1-q}(J, \mathbb{R}) = \{u \in C([0, T], \mathbb{R}) : (T - t)^{1-q}u \in C(J, \mathbb{R})\}, \quad q \in (0, 1)$$

and  $C_0(J, \mathbb{R}) = C(J, \mathbb{R})$  if  $q = 1$ . Our first result concerns the existence of a unique solution of nonlinear fractional differential equations with initial condition at the point  $T$ , see [Theorem 1](#). [Theorem 2](#) gives the unique solution of linear fractional differential problems in terms of the Mittag-Leffler function. In Section 4, we discuss problem (1) giving sufficient conditions which guarantee that problem (1) has extremal solutions, [Theorem 3](#). Two examples illustrate the results.

## 2. Nonlinear fractional differential equations

Let us consider the following problem

$$(D_T^q x)(t) = (Fx)(t), \quad t \in J_0, \quad \bar{x}(T) = k. \tag{2}$$

For  $u \in C_{1-q}(J, \mathbb{R})$ , we define a weighted norm:

$$\|u\|_* = \max_{[0, T]} (T - t)^{1-q} e^{\lambda(t-T)} |x(t)|$$

with a corresponding fixed positive constant  $\lambda$ .

**Theorem 1.** Let  $q \in (0, 1)$ ,  $f \in C(J \times \mathbb{R}, \mathbb{R})$ . In addition, we assume that:

$H_1$ : there exists a nonnegative constant  $K$  such that

$$|f(t, u_1) - f(t, v_1)| \leq K|v_1 - u_1|.$$

Then problem (2) has a unique solution.

**Proof.** Using Lemma 2.6 [1], it is easy to show that problem (2) is equivalent to the following integral equation:

$$x(t) = k(T - t)^{q-1} + \frac{1}{\Gamma(q)} \int_t^T (s - t)^{q-1} Fx(s) ds. \tag{3}$$

We write Eq. (3) in the form  $u = Au$ , where  $A$  is defined by the right-hand-side of (3). Now, we have to show that operator  $A$  has a fixed point. To do it we shall show that  $A$  is a contraction map.

Let us choose constants  $m, r$  such that  $1 < m < \frac{1}{1-q}$  and  $r = \frac{m}{m-1}$ . Now, we use the norm  $\|\cdot\|_*$  with a positive  $\lambda$  such that:

$$\lambda^{\frac{1}{r}} > \frac{K}{\Gamma(q)} T^{q-1+\frac{1}{m}} \left( \frac{\Gamma^2(m(q-1) + 1)}{\Gamma(2[m(q-1) + 1])} \right)^{\frac{1}{m}} \left( \frac{1}{r} \right)^{\frac{1}{r}} \equiv \rho. \tag{4}$$

Note that

$$B(t) = \int_t^T (s - t)^{m(q-1)} (T - s)^{m(q-1)} ds = \frac{\Gamma^2(m(q-1) + 1)}{\Gamma(2[m(q-1) + 1])} (T - t)^{2m(q-1)+1},$$

$$D(t) = \int_t^T e^{-\lambda rs} ds < \frac{1}{\lambda r} e^{-\lambda rt}.$$

Then using the Hölder inequality for integrals

$$\int_t^T |a(s)||b(s)| ds \leq \left( \int_t^T |a(s)|^m ds \right)^{\frac{1}{m}} \left( \int_t^T |b(s)|^r ds \right)^{\frac{1}{r}},$$

and assumption  $H_1$ , for  $x, y \in C_{1-q}(J, \mathbb{R})$  we obtain

$$\begin{aligned} \|Ax - Ay\|_* &\leq \frac{1}{\Gamma(q)} \max_{t \in J} (T - t)^{1-q} e^{\lambda(t-T)} \int_t^T (s - t)^{q-1} |(Fx)(s) - (Fy)(s)| ds \\ &\leq \frac{K}{\Gamma(q)} \|x - y\|_* \max_{t \in J} (T - t)^{1-q} e^{\lambda t} \int_t^T (s - t)^{q-1} (T - s)^{q-1} e^{-\lambda s} ds \\ &\leq \frac{K}{\Gamma(q)} \|x - y\|_* \max_{t \in J} (T - t)^{1-q} e^{\lambda t} [B(t)]^{\frac{1}{m}} [D(t)]^{\frac{1}{r}} \\ &\leq \frac{K}{\Gamma(q)} \left( \frac{\Gamma^2(m(q-1) + 1)}{\Gamma(2[m(q-1) + 1])} \right)^{\frac{1}{m}} \frac{1}{(\lambda r)^{\frac{1}{r}}} \|x - y\|_* \max_{t \in J} (T - t)^{q-1+\frac{1}{m}} \leq \frac{\rho}{\lambda^{\frac{1}{r}}} \|x - y\|_*. \end{aligned}$$

This and condition (4) prove that problem (2) has a unique solution, by the Banach fixed point theorem. This ends the proof. ■

**Remark 1.** If  $q = 1$ , then  $(D_1^1 x)(t) = -x'(t)$ , and operator  $A$  has the form

$$(Ax)(t) = k + \int_t^T (Fx)(s)ds.$$

In this case,  $\|x\|_* = \max_{t \in J} e^{\lambda(t-T)} |x(t)|$ , so choosing  $\lambda > K$ , we can show that operator  $A$  has a unique fixed point.

### 3. Linear fractional differential equations

Let us consider the linear fractional differential problem

$$(D_T^q x)(t) = \lambda x(t) + \sigma(t), \quad t \in J_0, \quad \bar{x}(T) = k, \quad (5)$$

where  $\lambda$  is a real number and  $\sigma \in C_{1-q}(J, T)$ .

**Theorem 2.** Let  $q \in (0, 1]$ ,  $\lambda \in \mathbb{R}$ ,  $\sigma \in C_{1-q}(J, \mathbb{R})$ . Then problem (5) has a unique solution given by formula

$$x(t) = k\Gamma(q)(T-t)^{q-1}E_{q,q}(\lambda(T-t)^q) + \int_t^T (s-t)^{q-1}E_{q,q}(\lambda(s-t)^q)\sigma(s)ds, \quad (6)$$

where  $E_{q,q}(\zeta) = \sum_{r=0}^{\infty} \frac{\zeta^r}{\Gamma(q(r+1))}$  is the Mittag-Leffler function.

**Proof.** Indeed, problem (5) is equivalent in the space  $C_{1-q}(J, \mathbb{R})$  to the following fractional integral equation

$$x(t) = x_0(t) + \frac{\lambda}{\Gamma(q)} \int_t^T (s-t)^{q-1}x(s)ds + \frac{1}{\Gamma(q)} \int_t^T (s-t)^{q-1}\sigma(s)ds, \quad t \in J_0, \quad (7)$$

or

$$x(t) = x_0(t) + \lambda(I_T^q x)(t) + (I_T^q \sigma)(t),$$

where

$$x_0(t) = \frac{\bar{k}}{\Gamma(q)}(T-t)^{q-1}, \quad \bar{k} = k\Gamma(q).$$

We apply the method of successive approximations to find the solution of problem (7), so for  $n = 0, 1, \dots$ , we have

$$x_{n+1}(t) = x_0(t) + \lambda(I_T^q x_n)(t) + (I_T^q \sigma)(t).$$

Hence,

$$\begin{aligned} x_1(t) &= x_0(t) + \lambda(I_T^q x_0)(t) + (I_T^q \sigma)(t) \\ &= x_0(t) + \frac{\lambda \bar{k}}{\Gamma^2(q)} \int_t^T (s-t)^{q-1}(T-s)^{q-1}ds + \frac{1}{\Gamma(q)} \int_t^T (s-t)^{q-1}\sigma(s)ds \\ &= \frac{\bar{k}}{\Gamma(q)}(T-t)^{1-q} + \frac{\lambda \bar{k}}{\Gamma(2q)}(T-t)^{2q-1} + \frac{1}{\Gamma(q)} \int_t^T (s-t)^{q-1}\sigma(s)ds, \end{aligned}$$

using the formula

$$\int_t^T (s-t)^{rq-1}(T-s)^{mq-1}ds = \frac{\Gamma(rq)\Gamma(mq)}{\Gamma((r+m)q)}(T-t)^{(r+m)q-1}, \quad r, m \in \mathbb{N} = \{1, 2, \dots\}.$$

Using this  $x_1$ , we find the next approximation  $x_2$  as

$$\begin{aligned} x_2(t) &= x_0(t) + \lambda(I_T^q x_1)(t) + (I_T^q \sigma)(t) \\ &= x_0(t) + \lambda(I_T^q [x_0 + \lambda I_T^q x_0 + I_T^q \sigma])(t) + (I_T^q \sigma)(t) \\ &= x_0(t) + \lambda(I_T^q x_0)(t) + \lambda^2(I_T^{2q} x_0)(t) + \lambda(I_T^{2q} \sigma)(t) + (I_T^q \sigma)(t) \\ &= x_0(t) + \frac{\lambda \bar{k}}{\Gamma(2q)}(T-t)^{2q-1} + \frac{\lambda^2 \bar{k}}{\Gamma(3q)}(T-t)^{3q-1} + \frac{1}{\Gamma(q)} \int_t^T (s-t)^{q-1}\sigma(s)ds \\ &\quad + \frac{\lambda}{\Gamma(2q)} \int_t^T (s-t)^{2q-1}\sigma(s)ds \\ &= \bar{k}(T-t)^{q-1} \left[ \frac{1}{\Gamma(q)} + \frac{\lambda}{\Gamma(2q)}(T-t)^q + \frac{\lambda^2}{\Gamma(3q)}(T-t)^{2q} \right] \\ &\quad + \int_t^T (s-t)^{q-1} \left[ \frac{1}{\Gamma(q)} + \frac{\lambda}{\Gamma(2q)}(s-t)^q \right] \sigma(s)ds, \end{aligned}$$

by the property

$$(I_T^\alpha I_T^\beta f)(t) = (I_T^{\alpha+\beta} f)(t), \quad \alpha, \beta > 0.$$

Thus, in general, we get by induction  $x_n$  as follows

$$x_n(t) = \bar{k}(T-t)^{q-1} \sum_{r=0}^n \frac{\lambda^r}{\Gamma(q(r+1))} (T-t)^{rq} + \int_t^T (s-t)^{q-1} \left( \sum_{r=0}^{n-1} \frac{\lambda^r}{\Gamma(q(r+1))} (s-t)^{rq} \right) \sigma(s) ds, \quad n = 0, 1, \dots,$$

where  $\sum_0^{-1} = 0$ . Taking the limit as  $n \rightarrow \infty$ , we obtain the solution  $x$  in terms of Mittag-Leffler's function given by formula (6). ■

**Remark 2.** Let  $q = 1$ . Then problem (5) takes the form

$$-x'(t) = \lambda x(t) + \sigma(t), \quad t \in J_0, \quad x(T) = k.$$

Since  $E_{1,1}(t) = e^t$ , then, in view of (6), the solution of this problem is given by

$$\begin{aligned} x(t) &= ke^{\lambda(T-t)} + \int_t^T e^{\lambda(s-t)} \sigma(s) ds \\ &= e^{\lambda(T-t)} \left[ k + \int_t^T e^{-\lambda(T-s)} \sigma(s) ds \right], \quad t \in J. \end{aligned}$$

**Example 1.** For  $q \in (0, 1]$ , let us consider the following problem

$$\begin{cases} (D_T^q x)(t) = x(t) + (T-t)^{2-q} \left[ \frac{\Gamma(3)}{\Gamma(3-q)} - (T-t)^q \right], & t \in J_0 = [0, T), \\ \bar{x}(T) = 0. \end{cases} \quad (8)$$

Comparing this problem with (5) we see that

$$\lambda = 1, \quad \sigma(t) = (T-t)^{2-q} \left[ \frac{\Gamma(3)}{\Gamma(3-q)} - (T-t)^q \right], \quad k = 0.$$

In view of Theorem 2, problem (8) has a unique solution given by

$$x(t) = \int_t^T (s-t)^{q-1} E_{q,q}((s-t)^q) \sigma(s) ds,$$

so

$$\begin{aligned} x(t) &= \int_t^T (s-t)^{q-1} \sum_{n=0}^{\infty} \frac{(s-t)^{qn}}{\Gamma((n+1)q)} (T-s)^{2-q} \left[ \frac{\Gamma(3)}{\Gamma(3-q)} - (T-s)^q \right] ds \\ &= \frac{\Gamma(3)}{\Gamma(3-q)} \sum_{n=0}^{\infty} \frac{1}{\Gamma((n+1)q)} \int_t^T (s-t)^{q(n+1)-1} (T-s)^{2-q} ds \\ &\quad - \sum_{n=0}^{\infty} \frac{1}{\Gamma((n+1)q)} \int_t^T (s-t)^{q(n+1)-1} (T-s)^2 ds \\ &= \frac{\Gamma(3)}{\Gamma(3-q)} \sum_{n=1}^{\infty} \frac{1}{\Gamma((n+1)q)} (T-t)^{qn+2} \frac{\Gamma((n+1)q)\Gamma(3-q)}{\Gamma(qn+3)} \\ &\quad - \sum_{n=0}^{\infty} \frac{1}{\Gamma((n+1)q)} (T-t)^{q(n+1)+2} \frac{\Gamma((n+1)q)\Gamma(3)}{\Gamma(q(n+1)+3)} \\ &= \Gamma(3) E_{q,3}((T-t)^q) (T-t)^2 - (T-t)^2 \left[ E_{q,3}((T-t)^q) - \frac{1}{\Gamma(3)} \right] \Gamma(3) \\ &= (T-t)^2, \end{aligned}$$

where  $E_{q,r}$  is the Mittag-Leffler function defined by

$$E_{q,r}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(qn+r)}.$$

It proves that  $x(t) = (T-t)^2$  is the unique solution of problem (8).

#### 4. Extremal solutions to problem (1)

Let us introduce the following definition.

We say that  $u \in C_{1-q}(J, \mathbb{R})$  is a lower solution of problem (1) if

$$(D_T^q u)(t) \leq (Fu)(t), \quad t \in J, \quad g(u(0), u(T)) \geq 0,$$

and it is an upper solution of (1) if the above inequalities are reversed.

Now, we give a result for fractional differential inequalities.

**Lemma 1.** Let  $q \in (0, 1]$ ,  $M \in \mathbb{R}$ . Assume that  $p \in C_{1-q}(J, \mathbb{R})$  satisfies problem

$$\begin{cases} (D_T^q p)(t) \leq -Mp(t), & t \in J_0, \\ \bar{p}(T) \leq 0. \end{cases}$$

Then  $p(t) \leq 0$ ,  $t \in J$ .

**Proof.** Let  $(D_T^q p)(t) = -Mp(t) + \sigma(t)$ ,  $t \in J_0$ ,  $\bar{p}(T) = c$ , where  $\sigma(t) \leq 0$ ,  $c \leq 0$ . Then, in view of formula (6), we have  $p(t) \leq 0$ ,  $t \in J$ . This ends the proof. ■

The next result concerns the case when problem (1) has extremal solutions.

**Theorem 3.** Assume that  $q \in (0, 1]$ ,  $f \in C(J \times \mathbb{R}, \mathbb{R})$ ,  $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ . Let  $y_0, z_0 \in C_{1-q}(J, \mathbb{R})$  be lower and upper solutions of (1), respectively and  $y_0(t) \leq z_0(t)$ ,  $t \in J$ . In addition, we assume that

$H_3$ : there exists a constant  $M$  such that

$$f(t, u_1) - f(t, v_1) \leq M[v_1 - u_1]$$

if  $y_0(t) \leq u_1 \leq v_1 \leq z_0(t)$ ,

$H_4$ : there exists a constant  $\nu > 0$  such that

$$g(v, u) - g(v_0, u_0) \leq \nu(u_0 - u)$$

if  $\bar{y}_0(T) \leq u \leq u_0 \leq \bar{z}_0(T)$ ,  $\bar{y}_0(0) \leq v \leq v_0 \leq \bar{z}_0(0)$ .

Then problem (1) has extremal solutions in the sector

$$[y_0, z_0]_* = \{w \in C_{1-q}(J, \mathbb{R}) : y_0(t) \leq w(t) \leq z_0(t), t \in J\}.$$

**Proof.** For  $n = 0, 1, \dots$ , let us define

$$(D_T^q y_{n+1})(t) = (Fy_n)(t) - M[y_{n+1}(t) - y_n(t)], \quad t \in J_0,$$

$$0 = g(\bar{y}_n(0), \bar{y}_n(T)) - \nu[\bar{y}_{n+1}(T) - \bar{y}_n(T)],$$

$$(D_T^q z_{n+1})(t) = (Fz_n)(t) - M[z_{n+1}(t) - z_n(t)], \quad t \in J_0,$$

$$0 = g(\bar{z}_n(0), \bar{z}_n(T)) - \nu[\bar{z}_{n+1}(T) - \bar{z}_n(T)].$$

In view of Theorem 1, functions  $y_1, z_1$  are well defined. First, we show that

$$z_0(t) \leq z_1(t) \leq y_1(t) \leq y_0(t), \quad t \in J. \tag{9}$$

Put  $p = y_0 - y_1$ . This and the assumption that  $y_0$  is a lower solution of problem (1) yield

$$(D_T^q p)(t) \leq (Fy_0)(t) - (Fy_1)(t) + M[y_1(t) - y_0(t)] = -Mp(t),$$

$$0 = g(\bar{y}_0(0), \bar{y}_0(T)) - \nu[\bar{y}_1(T) - \bar{y}_0(T)] \geq \nu\bar{p}(T).$$

Hence,  $y_0(t) \leq y_1(t)$ ,  $t \in J$ , by Lemma 1. By a similar way, we can show that  $z_1(t) \leq z_0(t)$ ,  $t \in J$ . Now, we put  $p = y_1 - z_1$ .

Hence, in view of assumptions  $H_4, H_3$  we have

$$0 = g(\bar{y}_0(0), \bar{y}_0(T)) - g(\bar{z}_0(0), \bar{z}_0(T)) - \nu[\bar{y}_1(T) - \bar{y}_0(T) - \bar{z}_1(T) + \bar{z}_0(T)] \leq -\nu\bar{p}(T),$$

$$(D_T^q p)(t) = (Fy_0)(t) - (Fz_0)(t) - M[y_1(t) - y_0(t) - z_1(t) + z_0(t)] \leq -Mp(t).$$

This and Lemma 1 prove that  $y_1(t) \leq z_1(t)$ ,  $t \in J$ , so relation (9) holds.

In the next step, we show that  $y_1, z_1$  are lower and upper solutions of problem (1), respectively. Note that

$$\begin{aligned} (D_T^q y_1)(t) &= (Fy_0)(t) - (Fy_1)(t) + (Fy_1)(t) - M[y_1(t) - y_0(t)] \\ &\leq M[y_1(t) - y_0(t)] - M[y_1(t) - y_0(t)] + (Fy_1)(t) = (Fy_1)(t), \end{aligned}$$

$$\begin{aligned} 0 &= g(\bar{y}_0(0), \bar{y}_0(T)) - g(\bar{y}_1(0), \bar{y}_1(T)) + g(\bar{y}_1(0), \bar{y}_1(T)) - \nu[\bar{y}_1(T) - \bar{y}_0(T)] \\ &\leq \nu[\bar{y}_1(T) - \bar{y}_0(T)] + g(\bar{y}_1(0), \bar{y}_1(T)) - \nu[\bar{y}_1(T) - \bar{y}_0(T)] = g(\bar{y}_1(0), \bar{y}_1(T)) \end{aligned}$$

by assumptions  $H_3, H_4$ . This proves that  $y_1$  is a lower solution of problem (1). Similarly, we can show that  $z_1$  is an upper solution of (1).

Using the mathematical induction, we can show that

$$y_0(t) \leq y_1(t) \leq \dots \leq y_n(t) \leq y_{n+1}(t) \leq z_{n+1}(t) \leq z_n(t) \leq \dots \leq z_1(t) \leq z_0(t)$$

for  $t \in J$  and  $n = 1, 2, \dots$ . Employing standard arguments we see that the sequences  $\{y_n, z_n\}$  converge to their limit functions  $y, z$ , respectively. Indeed,  $y$  and  $z$  are solutions of problem (1) and  $y_0(t) \leq y(t) \leq z(t) \leq z_0(t)$  on  $J$ .

To show that  $y, z$  are the minimum and maximum solutions of (1) we have to prove that if  $u \in [y_0, z_0]_*$  is any solution of (1), then  $y(t) \leq u(t) \leq z(t)$  on  $J$ . To do this, we assume that  $y_m(t) \leq u(t) \leq z_m(t)$ ,  $t \in J$  for some  $m$ . Let  $p = y_{m+1} - u$ ,  $P = u - z_{m+1}$ . Then, in view of assumptions  $H_3, H_4$ , we can prove that:

$$\begin{aligned} (D_7^q p)(t) &= (Fy_m)(t) - (Fu)(t) - M[y_{m+1}(t) - y_m(t)] \leq -Mp(t), \\ (D_7^q P)(t) &= (Fu)(t) - (Fz_m)(t) + M[z_{m+1}(t) - z_m(t)] \leq -MP(t), \\ 0 &= g(\bar{y}_m(0), \bar{y}_m(T)) - v[\bar{y}_{m+1}(T) - \bar{y}_m(T)] - g(\bar{u}(0), \bar{u}(T)) \leq -vp(T), \\ 0 &= g(\bar{u}(0), \bar{u}(T)) - g(\bar{z}_m(0), \bar{z}_m(T)) + v[\bar{z}_{m+1}(T) - \bar{z}_m(T)] \leq -vP(T). \end{aligned}$$

This and Lemma 1 show  $y_{m+1}(t) \leq u(t) \leq z_{m+1}(t)$ ,  $t \in J$ , so by induction,  $y_n(t) \leq u(t) \leq z_n(t)$  on  $J$  for all  $n$ . Taking the limit as  $n \rightarrow \infty$ , we conclude  $y(t) \leq u(t) \leq z(t)$ ,  $t \in J$ . ■

**Example 2.** Consider the following problem

$$\begin{cases} (D_1^q x)(t) = \frac{(1-t)^{-q}}{\Gamma(1-q)} + A[1-t-x(t)]^3 \equiv (Fx)(t), & t \in J_0 = [0, 1], \\ 0 = \bar{x}(1)[1 - \bar{x}(0)] \equiv g(\bar{x}(0), \bar{x}(1)), \end{cases} \quad (10)$$

where  $q \in (0, 1)$ ,  $A > 0$ ,  $t \in J = [0, 1]$ .

Let  $y_0(t) = 0$ ,  $z_0(t) = 2 - t$ ,  $t \in J$ . Then  $\bar{y}_0(0) = \bar{y}_0(1) = 0$ ,  $\bar{z}_0(0) = 2$ ,  $\bar{z}_0(1) = 0$ . Indeed,  $y_0$  is a lower solution of (10). Moreover,

$$(Fz_0)(t) = \frac{(1-t)^{-q}}{\Gamma(1-q)} - A \leq \frac{(1-t)^{-q}}{\Gamma(1-q)} + \frac{(1-t)^{1-q}}{\Gamma(2-q)} = (D_1^q z_0)(t), \quad g(\bar{z}_0(0), \bar{z}_0(1)) = 0.$$

Note that  $M = 3A$ ,  $v = 1$ . Obviously,  $z_0$  is an upper solution of problem (10).

By Theorem 3, problem (10) has extremal solutions in the region  $[y_0, z_0]_*$ .

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