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ASYMPTOTIC PROPERTIES OF QUADRATIC STOCHASTIC OPERATORS ACTING ON THE L^1 SPACE

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ABSTRACT. Quadratic stochastic operators can exhibit a wide variety of asymptotic behaviours and these have been introduced and studied recently in the ℓ^1 space. It turns out that in principle most of the results can be carried over to the L^1 space. However, due to topological properties of this space one has to restrict in some situations to kernel quadratic stochastic operators. In this article we study the uniform and strong asymptotic stability of quadratic stochastic operators acting on the L^1 space in terms of convergence of the associated (linear) nonhomogeneous Markov chains.

1. INTRODUCTION

The study of chains of Markov operators has become a subject of interest due to their applications in many different areas of science and technology. However, a description of many phenomena requires the use of nonlinear methods. Population and disease dynamics, physics, evolutionary biology and economic and social systems are some examples of fields where stochastic nonlinear dynamics are encountered (see e.g. the book of [20] for a detailed overview of results and models related with nonlinear Markov processes). This work here is devoted to the study of quadratic stochastic operators which are bilinear by nature (cf. Definition 2.2). They were first introduced by [8] to describe the evolution of a distribution of classes of individuals (i.e. groups of individuals possessing a particular trait) in a population. Since then the theory has grown in different directions with biology often motivating the development [7, 12, 15, 16, 17, 13]. A detailed review of mathematical results and open problems is presented by [18].

A fundamental issue is the study of limit properties of quadratic stochastic operators (see e.g. [14, 17, 13] for very recent studies on non–ergodicity of quadratic stochastic operators). However, because of the inherent nonlinearity it appears rather difficult. On the other hand, there exists a relation between quadratic stochastic operators and (linear) Markov operators [11]. This correspondence allows one to study a linear model instead of a nonlinear one. The aim of this paper is to define different types of asymptotic behaviour (mixing) of a quadratic stochastic operator considered on the L^1 space and express them in terms of convergence of a nonhomogeneous chain of (linear) Markov operators. Hence, while studying the limit properties of a quadratic stochastic operator one can apply the theory of nonhomogeneous Markov chains.

The asymptotic stability and the uniform asymptotic stability of quadratic stochastic operators considered on the ℓ^1 space of all absolutely summable real sequences was described by [6]. In this work we move to the L^1 space and generalize the results of [6]. Most can be carried over, but is some cases we have to restrict ourselves to a very particular type

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of quadratic stochastic operators, which we call kernel (cf. Definition 3.2). The concepts and proofs in this paper often follow those of [6], but for the convenience of the reader and clarity of the work we present all the arguments.

2. Basic definitions and properties

Let $(X, \mathscr{A}, \mu)$ be a separable σ-finite measure space. Throughout the paper we consider the (separable) Banach lattice of real and $\mathscr{A}$-measurable functions f such that $|f|$ is μ-integrable, equipped with the norm $\|f\|_1 := \int_X |f| d\mu$ and we denote it by L^1. By $\mathscr{D} := \mathscr{D}(X, \mathscr{A}, \mu)$ we denote the convex set of all *densities* on X, i.e.

$$\mathscr{D} = \left\{ f \in L^1 : f \geq 0, \|f\|_1 = 1 \right\}.$$

We say that a linear operator $P\colon L^1 \to L^1$ is *Markov* (or *stochastic*) if

$$Pf \geq 0 \quad \text{and} \quad \|Pf\|_1 = \|f\|_1$$

for all $f \geq 0$, $f \in L^1$. It follows that $|||P||| := \sup_{\|f\|_1 = 1} \|Pf\|_1 = 1$ and $P(\mathscr{D}) \subset \mathscr{D}$. The sequence of such operators denoted by $\mathbf{P} = (P^{[n,n+1]})_{n \geq 0}$ is called a *(discrete time) nonhomogeneous chain of stochastic operators* or shorter, a *nonhomogeneous Markov chain*. For $m, n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $n - m \geq 1$, and any $f \in L^1$ we set

$$P^{[m,n]} f = P^{[n-1,n]}(P^{[n-2,n-1]}(\cdots(P^{[m,m+1]} f)\cdots))$$

and $P^{[n,n]} = \mathbf{I}$. If for all $n \geq 0$ one has $P^{[n,n+1]} = P$, then we say that $\mathbf{P} = (P)_{n \geq 0}$ is *homogeneous*. The set of all chains of Markov operators $\mathbf{P} = (P^{[n,n+1]})_{n \geq 0}$ will be denoted by $\mathfrak{S}$.

The geometric structure of the set $\mathfrak{S}$ (endowed with suitable natural metric topology) has been recently intensively studied by [24, 25]. Let us recall

Definition 2.1. *A discrete time nonhomogeneous chain of stochastic operators* $\mathbf{P} \in \mathfrak{S}$ *is called*

(1) *uniformly asymptotically stable if there exists a unique* $f_* \in \mathscr{D}$ *such that for any* $m \geq 0$

$$\lim_{n\to\infty} \sup_{f \in \mathscr{D}} \left\| P^{[m,n]} f - f_* \right\|_1 = 0,$$

(2) *almost uniformly asymptotically stable if for any* $m \geq 0$

$$\lim_{n\to\infty} \sup_{f,g \in \mathscr{D}} \left\| P^{[m,n]} f - P^{[m,n]} g \right\|_1 = 0,$$

(3) *strong asymptotically stable if there exists a unique* $f_* \in \mathscr{D}$ *such that for all* $m \geq 0$ *and* $f \in \mathscr{D}$

$$\lim_{n\to\infty} \left\| P^{[m,n]} f - f_* \right\|_1 = 0,$$

(4) *strong almost asymptotically stable if for all* $m \geq 0$ *and* $f, g \in \mathscr{D}$

$$\lim_{n\to\infty} \left\| P^{[m,n]} f - P^{[m,n]} g \right\|_1 = 0\ .$$

The asymptotic behavior of nonhomogeneous chains of Markov operators has an extensive literature, e.g. [19] presents a detailed classification of different types of limit behaviour of nonhomogeneous Markov chains. The papers of [21, 22, 23] are examples of very recent studies in this direction.

We proceed with the concept of a quadratic stochastic operator acting on the L^1 space.

Definition 2.2. *A bilinear operator $\mathbf{Q}\colon L^1 \times L^1 \to L^1$ is called a quadratic stochastic operator if*

$$\mathbf{Q}(f,g) \geq 0\,, \quad \mathbf{Q}(f,g) = \mathbf{Q}(g,f) \quad \text{and} \quad \|\mathbf{Q}(f,g)\|_1 = \|f\|_1\|g\|_1$$

for all $f,g \geq 0$, $f,g \in L^1$.

It follows that $\mathbf{Q}$ is bounded as $\sup_{\|f\|_1=1,\|g\|_1=1}\|\mathbf{Q}(f,g)\|_1 = 1$. Moreover if $\widetilde{f} \geq f \geq 0$ and $\widetilde{g} \geq g \geq 0$ then $\mathbf{Q}(\widetilde{f},\widetilde{g}) \geq \mathbf{Q}(f,g)$. Clearly, $\mathbf{Q}(\mathscr{D}\times\mathscr{D}) \subseteq \mathscr{D}$. The family of all quadratic stochastic operators will be denoted by $\mathfrak{Q}$.

In this paper we pay special attention to a nonlinear mapping $\mathscr{D} \ni f \mapsto \mathbb{Q}(f) := \mathbf{Q}(f,f) \in \mathscr{D}$. This restriction to the “diagonal” is very relevant from a biological application point of view. The iterates $\mathbb{Q}^n(f)$, where $n = 0,1,2,\ldots$, can describe the evolution of a distribution of some trait of an inbreeding or hermaphroditic population. Clearly, $\mathbb{Q}(\mathscr{D}) \subseteq \mathscr{D}$.

Remark 2.1. *Let us note that any homogeneous chain of stochastic operators $\mathbf{P} = (P)_{n\geq 0} \in \mathfrak{S}$ may be represented by such a nonlinear mapping $\mathbb{Q}\colon \mathscr{D} \to \mathscr{D}$. In fact, let $P\colon L^1 \to L^1$ be a (linear) Markov operator. Consider $\mathbf{Q} \in \mathfrak{Q}$ defined by*

$$\mathbf{Q}(f,g) = \tfrac{1}{2}\left(\left(\int_X g\,d\mu\right)Pf + \left(\int_X f\,d\mu\right)Pg\right)$$

for any $f,g \in L^1$. Then for $f \in \mathscr{D}$ we have $\mathbb{Q}(f) = Pf$. In particular $\mathbb{Q}^n$ and P^n restricted to $\mathscr{D}$ are identical. It follows that any homogeneous Markov dynamics on $\mathscr{D}$ may be viewed as a quadratic mapping $\mathbb{Q}\colon \mathscr{D} \to \mathscr{D}$.

It follows from the triangle inequality and the fact that $\mathbf{Q}$ is bilinear (by definition) that for any $f,g,\widetilde{f},\widetilde{g} \in L^1$

$$\begin{aligned}\left\|\mathbf{Q}(f,g)-\mathbf{Q}(\widetilde{f},\widetilde{g})\right\|_1 &\leq \left\|\mathbf{Q}(f,g)-\mathbf{Q}(\widetilde{f},g)\right\|_1 + \left\|\mathbf{Q}(\widetilde{f},g)-\mathbf{Q}(\widetilde{f},\widetilde{g})\right\|_1\\ &\leq \left\|f-\widetilde{f}\right\|_1\|g\|_1 + \|g-\widetilde{g}\|_1\left\|\widetilde{f}\right\|_1 .\end{aligned}$$

If all the vectors $f,g,\widetilde{f},\widetilde{g}$ are from the unit ball

$$\left\|\mathbf{Q}(f,g)-\mathbf{Q}(\widetilde{f},\widetilde{g})\right\|_1 \leq \left\|f-\widetilde{f}\right\|_1 + \|g-\widetilde{g}\|_1$$

and hence

$$\|\mathbb{Q}(f)-\mathbb{Q}(g)\|_1 = \|\mathbf{Q}(f,f)-\mathbf{Q}(g,g)\|_1 \leq 2\,\|f-g\|_1 .$$

Thus $\mathbf{Q}$ is continuous on $L^1 \times L^1$ and uniformly continuous if applied to vectors from the unit ball in L^1. In particular, $\mathbb{Q}$ is uniformly continuous on the unit ball in L^1.

3. Mutual correspondence between L^1 and ℓ^1 spaces and its consequences

The fact that the properties of quadratic stochastic operators acting on ℓ^1 carry over to L^1 is that these spaces are quite closely connected.

Let us recall that a measurable countable partition $\xi := \{B_k\}$ of X is called consistent with σ–finite measure μ if $0 < \mu(B_k) < \infty$ for all k. The existence of such partitions is evident due to σ–finiteness of the measure μ. Given a consistent measurable countable partition $\xi := \{B_k\}$ and any $f_1, f_2 \in L^1$ we write

$$f_1 \sim f_2 \Leftrightarrow \forall_i \int_{B_i} f_1 \mathrm{d}\mu = \int_{B_i} f_2 \mathrm{d}\mu \ .$$

We can see that we have defined an equivalence relationship on the L^1 space. Each equivalence class (taking $f \in L^1$ as its representative) can be then associated with an element $p_f \in \ell^1$ in a natural way, namely take

$$\ell^1 \ni p_f = \left(\int_{B_1} f \mathrm{d}\mu, \int_{B_2} f \mathrm{d}\mu, \dots \right).$$

Notice that the coordinates of the vector p_f are actually the conditional expectations $\mathbb{E}\left[\cdot|B_i\right]$ for the density f and measure μ.

The presented connection between L^1 and ℓ^1 spaces gives that every counterexample for quadratic stochastic operators acting on ℓ^1 carries over to the L^1 case. We will also see in the further proofs that via this relation we can utilize arguments from [6].

Let us recall the definition of a quadratic stochastic operator on ℓ^1 [6]:

Definition 3.1. *A quadratic stochastic operator is defined as a cubic array of nonnegative real numbers* $\mathbf{Q}_{seq} = [q_{ij,k}]_{i,j,k\geq 1}$ *if it satisfies*

(D1) $0 \leq q_{ij,k} = q_{ji,k} \leq 1$ *for all* $i, j, k \geq 1$,
(D2) $\sum_{k=1} q_{ij,k} = 1$ *for any pair* (i, j).

Such a cubic matrix $\mathbf{Q}_{seq}$ may be viewed as a bilinear mapping $\mathbf{Q}_{seq} : \ell^1 \times \ell^1 \to \ell^1$ if we set $\mathbf{Q}_{seq}((x_1,x_2,\dots),(y_1,y_2,\dots))_k = \sum_{i,j=1} x_i y_j q_{ij,k}$ for any $k \geq 1$.

The following definition is motivated by the mutual correspondence between L^1 and ℓ^1 spaces discussed above.

Definition 3.2. *A quadratic stochastic operator* $\mathbf{Q}\colon L^1 \times L^1 \to L^1$ *is called a* kernel quadratic stochastic operator *if there exists an* $\mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A}$*–measurable, nonnegative function* $q\colon X \times X \times X \to \mathbb{R}_+$ *such that*

$$\mathbf{Q}(f,g)(z) = \int_X \int_X f(x)g(y)q(x,y,z)d\mu(x)d\mu(y)$$

and $q(x,y,z) = q(y,x,z)$ *for any* $x,y,z \in X$, *and* $\int_X q(x,y,z)d\mu(z) = 1$ *for every* $(x,y) \in X \times X$.

One can immediately see a close relation of kernel quadratic stochastic operators to quadratic stochastic operators on ℓ^1, namely the latter have an obvious kernel form.

4. Ergodic structure

We will consider different types of asymptotic behaviours of quadratic stochastic operators.

Definition 4.1. *A quadratic stochastic operator* $\mathbf{Q} \in \mathfrak{Q}$ *is called:*

(1) *norm mixing (uniformly asymptotically stable) if there exists a density* $f \in \mathscr{D}$ *such that*

$$\lim_{n\to\infty} \sup_{g\in\mathscr{D}} \|\mathbb{Q}^n(g) - f\|_1 = 0 \ ,$$

(2) *strong mixing (asymptotically stable) if there exists a density $f \in \mathscr{D}$ such that for all $g \in \mathscr{D}$ we have*

$$\lim_{n\to\infty} \|\mathbb{Q}^n(g) - f\|_1 = 0\ ,$$

(3) *strong almost mixing if for all $g,h \in \mathscr{D}$ we have*

$$\lim_{n\to\infty} \|\mathbb{Q}^n(g) - \mathbb{Q}^n(h)\|_1 = 0\ .$$

The sets of all norm mixing, strong mixing, strong almost mixing quadratic stochastic operators are denoted respectively by $\mathfrak{Q}_{nm}$, $\mathfrak{Q}_{sm}$, $\mathfrak{Q}_{sam}$. Taking into account the mutual correspondence between L^1 and ℓ^1 and using the results from [6] we can easily see that $\mathfrak{Q}_{nm} \subsetneq \mathfrak{Q}_{sm} \subsetneq \mathfrak{Q}_{sam}$.

The key issue for our further studies is the existence of the relation between quadratic stochastic operators and (linear) Markov operators. The idea of studying the so–called associated Markov chains comes from [11], which was successfully employed by others [6].

Definition 4.2. *Let $\mathbf{Q} \in \mathfrak{Q}$. For arbitrarily fixed initial density function $g \in \mathscr{D}$ a nonhomogeneous Markov chain associated with $\mathbf{Q}$ and $g \in \mathscr{D}$ is defined as a sequence $\mathbf{P}_g = (P_g^{[n,n+1]})_{n\geq 0}$ of Markov operators $P_g^{[n,n+1]} : L^1 \to L^1$ of the form*

$$P_g^{[n,n+1]}(h) := \mathbf{Q}(\mathbb{Q}^n(g), h)\ .$$

Let us notice that if the initial density f is $\mathbb{Q}$–invariant (i.e. $\mathbb{Q}(f) = f$), then the associated Markov chain $\mathbf{P}_f$ is homogeneous as for any $h \in L^1$ the expression $\mathbf{Q}(\mathbb{Q}^n(f), h) = \mathbf{Q}(f,h)$ does not depend on n. Then we write $P_f^{[n,n+1]} =: P_f$ and $P_f^{[0,n]} =: P_f^n$.

We proceed with studying the ergodic structure of the set $\mathfrak{Q}$. Recall that the support of $g \in L^1$ is defined to be the set $\operatorname{supp}(g) := \{x \in X : g(x) \neq 0\}$.

Definition 4.3. *Let $\mathbf{Q} \in \mathfrak{Q}$. We say that the subset $D \subseteq X$ is $\mathbf{Q}$–invariant (or $\mathbf{Q}$–absorbing) if*

$$\int_{X\setminus D} \mathbf{Q}(g,h)\mathrm{d}\mu = 0$$

for all $g,h \in \mathscr{D}(D) := \{g \in \mathscr{D} : \operatorname{supp}(g) \subseteq D\}$, i.e. $\mathbf{Q}(g,h)$ is supported on D.

We will now show that if f is a $\mathbb{Q}$–invariant density then its support $\operatorname{supp}(f) := \{x \in X : f(x) \neq 0\} := D_f$ is a $\mathbf{Q}$–invariant set. It is sufficient to show that for every $\varepsilon > 0$ and all $A^\varepsilon, B^\varepsilon \subseteq \{x : f(x) \geq \varepsilon\}$ one has $\mathbf{Q}(\mathbf{1}_{A^\varepsilon}, \mathbf{1}_{B^\varepsilon}) = 0$ on $D_f^C := X \setminus D_f$. Indeed, the set $\{\sum_{j=1}^J a_j \mathbf{1}_{A_j^\varepsilon} : J \in \mathbb{N}, \varepsilon > 0\}$ is L^1–norm dense in $L^1(D_f)$ and $\mathbf{Q}(\cdot,\cdot)$ is continuous. Clearly

$$\left\|\mathbf{Q}(\mathbf{1}_{A^\varepsilon}, \mathbf{1}_{B^\varepsilon})\mathbf{1}_{D_f^C}\right\|_1 \leq \left\|\mathbf{Q}(\tfrac{1}{\varepsilon}f, \tfrac{1}{\varepsilon}f)\mathbf{1}_{D_f^C}\right\|_1 = \tfrac{1}{\varepsilon^2}\left\|\mathbf{Q}(f,f)\mathbf{1}_{D_f^C}\right\|_1 = \tfrac{1}{\varepsilon^2}\left\|f\mathbf{1}_{D_f^C}\right\|_1 = 0.$$

Let f be a $\mathbb{Q}$–invariant density. Using the fact that D_f is $\mathbf{Q}$–invariant we will show that the Markov operator P_f (associated with $\mathbf{Q}$) restricted to D_f *overlaps supports*, i.e. for any pair of $g,h \in \mathscr{D}(D_f)$ there exists a natural number n such that $P_f^n(g) \wedge P_f^n(h) \neq 0$ (where $\wedge$ stands for the ordinary minimum in L^1). Let $g,h \in \mathscr{D}(D_f)$ and suppose that $P_f(g) \wedge P_f(h) = 0$. For some $B, G \in \mathscr{A}$ of positive measure we have $g\mathbf{1}_B \geq \varepsilon f \mathbf{1}_B$, $h\mathbf{1}_G \geq \varepsilon f \mathbf{1}_G$ for some $\varepsilon > 0$. Now,

$$P_f(g) = \mathbf{Q}(f,g) \geq \mathbf{Q}(f,\varepsilon f\mathbf{1}_B) = \varepsilon\mathbf{Q}(f,f\mathbf{1}_B),$$
$$P_f(h) = \mathbf{Q}(f,h) \geq \mathbf{Q}(f,\varepsilon f\mathbf{1}_G) = \varepsilon\mathbf{Q}(f,f\mathbf{1}_G)$$

and by our assumption

$$\mathbf{Q}(f,f\mathbf{1}_B) \wedge \mathbf{Q}(f,f\mathbf{1}_G) = 0$$

giving

$$\mathbf{Q}(f\mathbf{1}_G,f\mathbf{1}_B) \leq \mathbf{Q}(f,f\mathbf{1}_B),$$
$$\mathbf{Q}(f\mathbf{1}_G,f\mathbf{1}_B) \leq \mathbf{Q}(f,f\mathbf{1}_G).$$

Hence $\mathbf{Q}(f\mathbf{1}_G,f\mathbf{1}_B) = 0$. But $\|\mathbf{Q}(f\mathbf{1}_G,f\mathbf{1}_B)\|_1 = \|f\mathbf{1}_G\|_1 \cdot \|f\mathbf{1}_B\|_1 > 0$. This is a contradiction and hence $\|P_f(g) \wedge P_f(h)\|_1 > 0$ if $g,h \in \mathscr{D}(D_f)$.

[5] show that if a stochastic operator overlaps supports, has a strictly positive invariant density and is also at least partially kernel then it is asymptotically stable. Hence we obtain the following lemma.

Lemma 4.1. *Let $\mathbf{Q}$ be a kernel quadratic stochastic operator. If f is a $\mathbb{Q}$–invariant density, then $L^1(\mathrm{supp}\,(f))$ is a P_f–invariant set, i.e. $P_f(L^1(\mathrm{supp}\,(f))) \subseteq L^1(\mathrm{supp}\,(f))$, for the associated Markov operator P_f and*

$$\lim_{n\to\infty} \left\| P_f^n(h) - \left(\int_X h\mathrm{d}\mu\right) f \right\|_1 = 0$$

for all $h \in L^1(\mathrm{supp}\,(f))$ (i.e. P_f is asymptotically stable on $L^1(\mathrm{supp}\,(f))$).

The following theorem assumes that the quadratic stochastic operator is a kernel one. This is as we use Lemma 4.1. If one can show that another set of quadratic stochastic operators will be asymptotically stable and also have the property that trajectory sets of the associated nonhomogeneous Markov operator will be norm relatively compact then Thm. 4.1 will also hold for them. These are in fact a non–trivial problems and the latter was discussed by [5] and [26, 27]. Notice that [27] provides an example of a Markov operator that overlaps supports, possesses an invariant density but is unstable.

Theorem 4.1. *Let $\mathbf{Q}$ be a kernel quadratic stochastic operator and let $g \in \mathscr{D}$. If*

$$\lim_{n\to\infty} \|\mathbb{Q}^n(g) - f\|_1 = 0$$

(then f is a $\mathbb{Q}$–invariant density), then

$$\lim_{n\to\infty} \left\| P_g^{[0,n]}(h) - f \right\|_1 = 0,$$

for any $h \in L^1$ satisfying $\mathrm{supp}\,(h) \subseteq \mathrm{supp}\,(g)$.

Proof. Let $g \in \mathscr{D}$ be such that $P_g^{[0,n]}(g) = \mathbb{Q}^n(g) \to f$. If h satisfies $0 \leq h \leq g$ (h need not belong to $\mathscr{D}$) then by monotonicity of quadratic stochastic operators we get $0 \leq P_g^{[0,n]}(h) \leq P_g^{[0,n]}(g)$. Indeed,

$$0 \leq P_g^{[0,1]}(h) = \mathbf{Q}(g,h) \leq \mathbf{Q}(g,g) = \mathbb{Q}(g)$$

and hence

$$0 \leq P_g^{[0,2]}(h) = P_g^{[1,2]}(P_g^{[0,1]}(h)) = \mathbf{Q}(\mathbb{Q}(g),P_g^{[0,1]}(h)) \leq \mathbf{Q}(\mathbb{Q}(g),\mathbb{Q}(g)) = \mathbb{Q}^2(g)\ .$$

Thus by iterating, for any natural n

$$0 \leq P_g^{[0,n]}(h) = \mathbf{Q}(\mathbb{Q}^{n-1}(g), P_g^{[0,n-1]}(h)) \leq \mathbf{Q}(\mathbb{Q}^{n-1}(g), \mathbb{Q}^{n-1}(g)) = \mathbb{Q}^n(g).$$

We conclude that $A_g := \{P_g^{[0,n]}(h) : 0 \leq h \leq g, n \geq 1\}$ is relatively weakly compact as ordered intervals in the Banach lattice L^1 are weakly compact [L^1 has order continuous norm; see e.g. Theorem 4.9 in 1]. Hence for any fixed h and any sequence $P_g^{[0,k_m]}(h)$ there exists a subsequence $n_i = k_{m_i}$ such that $P_g^{[0,n_i]}(h)$ converges weakly (obviously $P_g^{[0,n_i+1]}(h)$ converges weakly too). Let us denote $w := \lim_{i\to\infty} P_g^{[0,n_i]}(h)$ (weakly). We will now show that choosing a subsequence we may guarantee that $P_g^{[0,n_i]}(h)$ converges in norm. We can see immediately that

$$\begin{aligned}
&\left\| P_g^{[0,n_i+1]}(h) - \mathbf{Q}(f, P_g^{[0,n_i]}(h) \wedge f) \right\|_1 \\
&= \left\| \mathbf{Q}(\mathbb{Q}^{n_i}(g), P_g^{[0,n_i]}(h)) - \mathbf{Q}(f, P_g^{[0,n_i]}(h) \wedge f) \right\|_1 \\
&\leq \|\mathbb{Q}^{n_i}(g) - f\|_1 \left\| P_g^{[0,n_i]}(h) \right\|_1 + \left\| P_g^{[0,n_i]}(h) - P_g^{[0,n_i]}(h) \wedge f \right\|_1 \|f\|_1 \to 0
\end{aligned}$$

as

$$\left\| P_g^{[0,n_i]}(h) \wedge f - P_g^{[0,n_i]}(h) \right\|_1 \to 0. \tag{1}$$

This means that we can choose a pointwise convergent subsequence (for μ almost all $z \in X$). With a little abuse of notation we still denote this subsequence as n_j. Now for μ almost all $z \in X$

$$P_g^{[0,n_j+1]}(h)(z) - \left(\mathbf{Q}(f, P_g^{[0,n_j]}(h) \wedge f) \right)(z) \to 0. \tag{2}$$

We consider the second term of Eq. (2), assuming that $z \in X$ is temporarily fixed,

$$\int_X \frac{\left(P_g^{[0,n_j]}(h) \wedge f \right)(x)}{f(x)} \int_X f(x) f(y) q(x,y,z) \mathrm{d}\mu(y) \mathrm{d}\mu(x)$$

and see that $\left(P_g^{[0,n_j]}(h) \wedge f \right)(x)/f(x) \in L^\infty$ as both the numerator and denominator are positive, and $\int_X f(x) f(y) q(x,y,z) \mathrm{d}\mu(y) \in L^1$. If $A \subset X$ such that $\mu(A) < \infty$ then by Eq. (1) and the weak convergence

$$\int_X \left(P_g^{[0,n_j]}(h) \wedge f \right) \frac{\mathbf{1}_A}{f} \mathrm{d}\mu$$

converges. This gives us that for any $r \in L^1$,

$$\int_X \frac{P_g^{[0,n_j]}(h) \wedge f}{f} r \mathrm{d}\mu$$

converges. Hence as for almost all $z \in X$ the left hand side of Eq. (2) converges to 0 and the second term converges too, we obtain that $P_g^{[0,n_j+1]}(h)$ converges pointwise as required. Then this combined with the weak convergence stated above will give norm convergence

[Theorem 4.21.5, Corollary 4.21.6 in 10]. Summarizing: from any sequence $P_g^{[0,k_m]}(h)$ we managed to select a subsequence $P_g^{[0,n_j+1]}(h)$ converging in the L^1–norm. In particular $\{P_g^{[0,m]}(h) : m = 1,2,\ldots\}$ is norm relatively compact.

Let us introduce the L^1–norm ω–limit set

$$\omega_g(h) = \{w = \lim_{\iota\to\infty} P_g^{[0,n_\iota]}(h) \text{ for those subsequences for whom such a limit exists}\}.$$

It follows from the above that

$$\omega_g(h) = \{w = \lim_{i\to\infty} P_g^{[0,n_i]}(h) \text{ weakly}\}.$$

Clearly $\omega_g(h)$ is weakly closed as well. If $\|P_g^{[0,n_j]}(h) - w\|_1 \to 0$ then

$$\begin{aligned}\left\|P_g^{[0,n_j+1]}(h) - P_f(w)\right\|_1 &= \left\|\mathbf{Q}(\mathbb{Q}^{n_j}(g), P_g^{[0,n_j]}(h)) - \mathbf{Q}(f,w)\right\|_1 \\ &\le \|\mathbb{Q}^{n_j}(g) - f\|_1 + \left\|P_g^{[0,n_j]}(h) - w\right\|_1 \xrightarrow{j\to\infty} 0.\end{aligned}$$

It follows that $P_f(w) \in \omega_g(h)$ whenever $w \in \omega_g(h)$. Similarly considering $\|P_g^{[0,n_j-1]}(h) - \widetilde{w}\|_1 \to 0$ we get $\|P_g^{[0,n_j]}(h) - P_f(\widetilde{w})\|_1 \to \|w - P_f(\widetilde{w})\|_1 = 0$. Hence $P_f(\omega_g(h)) = \omega_g(h)$.

Now let us temporarily summarize some properties of the set $\omega_g(h)$:

- $\omega_g \subseteq \{w : 0 \le w \le f\}$ as $0 \le g \le h$ implies $0 \le \underbrace{P_g^{[0,n_\iota]}(h)}_{\to w} \le \mathbb{Q}^{n_\iota}(g) \to f$ and hence $\mathrm{supp}\,(w) \subseteq \mathrm{supp}\,(f)$,
- $\omega_g(h)$ is norm compact,
- every element of $w \in \omega_g(h)$ is of the form $w = P_f(v_w)$,
- $\omega_g(h)$ is P_f invariant (i.e. $P_f^m(\omega_g(h)) = \omega_g(h)$ for all $m \ge 0$),
- if $w \in \omega_g(h)$ then $\|w\|_1 = \|h\|_1$.

Therefore for every $m \in \mathbb{N}$ and $w \in \omega_g(h)$ let $w_m \in \omega_g(h)$ be such that $P_f^m(w_m) = w$ and by compactness taking an appropriate subsequence w_{m_ι} we have $\lim_{\iota\to\infty} w_{m_\iota} = w_* \in \omega_g(h)$. We obtained

$$\left\|P_f^{m_\iota}(w_*) - w\right\|_1 = \left\|P_f^{m_\iota}(w_*) - P_f^{m_\iota}(w_{m_\iota})\right\|_1 \le \|w_* - w_{m_\iota}\|_1 \to 0.$$

As $w_* \le f$ we have $\mathrm{supp}\,(w_*) \subseteq \mathrm{supp}\,(f)$ therefore by Lemma 4.1

$$\lim_{m\to\infty} P_f^m(w_*) = \|w_*\|_1 f = \|h\|_1 f = w.$$

Hence every element $w \in \omega_g(h)$ is of the form $w = \|h\|_1 f$. Thus $\omega_g(h) = \{\|h\|_1 f\}$ and therefore the norm limit $\lim_{n\to\infty} P_g^{[0,n]}(h) = \|h\|_1 f$ exists.

As the operators $P_g^{[0,n]}$ are linear, h comes from an ordered interval, $[0,g]$ spans the whole space $L^1(\mathrm{supp}\,(g))$, we obtained that $\lim_{n\to\infty} P_g^{[0,n]}(h) = \|h\|_1 f$ for $h \in L^1(\mathrm{supp}\,(g))$. If in addition $h \in \mathscr{D}$ and $\mathrm{supp}\,(h) \subseteq \mathrm{supp}\,(g)$ we obtain $\lim_{n\to\infty} P_g^{[0,n]}(h) = f$. □

Corollary 4.1. *Let $\mathbf{Q}$ be a kernel quadratic stochastic operator and let $g \in \mathscr{D}$. The iterates $\mathbb{Q}^n(g)$ converge in norm to a $\mathbb{Q}$–invariant density f if and only if*

$$\lim_{n\to\infty}\left\|P_g^{[0,n]}(u)-f\right\|_1=0$$

for all $u\in\mathscr{D}$ *such that* $\operatorname{supp}(u)\subseteq\operatorname{supp}(g)$.

Proof. Only the sufficient condition needs to be proved. Take $u=g$ and notice that $P_g^{[0,n]}(g)=\mathbb{Q}^n(g)$. □

Example 4.1. *Let* $\{B_k\}$ *be a consistent measurable countable partition of* X. *Define* $\mathbf{Q}\in\mathfrak{Q}$ *for any* $g,h\in L^1$ *by*

$$\begin{aligned}
\mathbf{Q}(g,h) \quad = \quad & \tfrac{1}{\mu(B_1)}\mathbf{1}_{B_1}\int_{B_1}g\mathrm{d}\mu\int_{B_1}h\mathrm{d}\mu\\
&+\tfrac{1}{\mu(B_2)}\mathbf{1}_{B_2}\left(\int_{B_1}g\mathrm{d}\mu\int_{B_2}h\mathrm{d}\mu+\int_{B_2}g\mathrm{d}\mu\int_{B_1}h\mathrm{d}\mu+\int_{B_2}g\mathrm{d}\mu\int_{B_2}h\mathrm{d}\mu\right)\\
&+\tfrac{1}{\mu(B_4)}\mathbf{1}_{B_4}\left(\int_{B_1}g\mathrm{d}\mu\int_{B_3}h\mathrm{d}\mu+\int_{B_3}g\mathrm{d}\mu\int_{B_1}h\mathrm{d}\mu\right)\\
&+\sum_{k=5}^{\infty}\left(\sum_{j=1}^{k-1}\int_{B_j}g\mathrm{d}\mu\int_{B_{k-j}}h\mathrm{d}\mu\right)\tfrac{1}{\mu(B_k)}\mathbf{1}_{B_k}.
\end{aligned}$$

We can see that $f=\frac{1}{\mu(B_1)}\mathbf{1}_{B_1}(\cdot)$ *is* $\mathbb{Q}$*–invariant and* $\frac{1}{\mu(B_2)}\mathbf{1}_{B_2}(\cdot)$ *is* P_f*–invariant. Hence* $P_f^{[0,n]}(u)$ *does not converge to* f *whenever* $\int_{\bigcup_{i=2}^{\infty}B_i}u\mathrm{d}\mu>0$. *Moreover if* $\int_{\bigcup_{i=3}^{\infty}B_i}u\mathrm{d}\mu>0$ *then the sequence* $P_f^{[0,n]}(u)$ *does not converge weakly.*

5. Asymptotic stability properties

In this section we give the equivalent conditions of asymptotic stability and uniform asymptotic stability of quadratic stochastic operators expressed in terms of convergence of the associated nonhomogeneous (linear) Markov chain. The first result follows from Theorem 4.1 and hence we formulate it for kernel quadratic stochastic operators.

Theorem 5.1. *Let* $\mathbf{Q}$ *be a kernel quadratic stochastic operator. The following conditions are equivalent:*

(1) *There exists* $f\in\mathscr{D}$ *such that for every* $g\in\mathscr{D}$ *we have*

$$\lim_{n\to\infty}\|\mathbb{Q}^n(g)-f\|_1=0$$

(i.e. $\mathbf{Q}$ *is asymptotically stable).*

(2) *There exists* $f\in\mathscr{D}$ *such that for all* $g,h\in\mathscr{D}$ *such that* $\operatorname{supp}(h)\subseteq\operatorname{supp}(g)$ *we have*

$$\lim_{n\to\infty}\left\|P_g^{[0,n]}(h)-f\right\|_1=0.$$

(3) *There exists* $f\in\mathscr{D}$ *such that for all* $m\geq 0$ *and* $g,h\in\mathscr{D}$ *satisfying* $\operatorname{supp}(h)\subseteq\operatorname{supp}(\mathbb{Q}^m(g))$ *we have*

$$\lim_{n\to\infty}\left\|P_g^{[m,n]}(h)-f\right\|_1=0.$$

Proof. The implication $(1)\Rightarrow(2)$ follows from Theorem 4.1 (hence we need to assume that $\mathbf{Q}$ is kernel). To see that $(2)\Rightarrow(3)$ holds true, notice that for any associated nonhomogeneous Markov chain $\mathbf{P}_g$ we have $P_g^{[m,n]}(h)=P_{\mathbb{Q}^m(g)}^{[0,n-m]}(h)$, where $h\in\mathscr{D}$ and $0\leq m<n$.

Using (2) we get $\lim_{n\to\infty}\|P_g^{[m,n]}(h)-f\|_1=\lim_{n\to\infty}\|P_{\mathbb{Q}^m(g)}^{[0,n-m]}(h)-f\|_1=0$. The implication (3) $\Rightarrow$ (1) is trivial as it is enough to take $m=0$ and $g=h$. $\square$

Similarly as it was observed by [6], we can suspect that the assumptions $\operatorname{supp}(h)\subseteq\operatorname{supp}(g)$ in (2) and $\operatorname{supp}(h)\subseteq\operatorname{supp}(\mathbb{Q}^n(g))$ in (3) might be possible to drop. We construct the below example.

Example 5.1. *As before, let $\{B_k\}$ be a consistent measurable countable partition of X. We define the operator $\mathbf{Q}$ on the space $\{g\in L^1:\operatorname{supp}(g)\subseteq B_1\cup B_2\}$ as follows: for any g,h*

$$\begin{aligned}\mathbf{Q}(g,h)\quad:=\quad&\tfrac{1}{\mu(B_1)}\mathbf{1}_{B_1}\left(\left(\int_{B_1}g\mathrm{d}\mu\right)\left(\int_{B_1}h\mathrm{d}\mu\right)+\tfrac{1}{2}\left(\int_{B_1}g\mathrm{d}\mu\right)\left(\int_{B_2}h\mathrm{d}\mu\right)\right.\\&\left.+\tfrac{1}{2}\left(\int_{B_2}g\mathrm{d}\mu\right)\left(\int_{B_1}h\mathrm{d}\mu\right)+\left(\int_{B_2}g\mathrm{d}\mu\right)\left(\int_{B_2}h\mathrm{d}\mu\right)\right)\\&+\tfrac{1}{\mu(B_2)}\mathbf{1}_{B_2}\left(\tfrac{1}{2}\left(\int_{B_1}g\mathrm{d}\mu\right)\left(\int_{B_2}h\mathrm{d}\mu\right)+\tfrac{1}{2}\left(\int_{B_2}g\mathrm{d}\mu\right)\left(\int_{B_1}h\mathrm{d}\mu\right)\right).\end{aligned}$$

If $g\in\mathscr{D}$, $\operatorname{supp}(g)\subseteq B_1\cup B_2$ then for all $n\geq 1$

$$\begin{aligned}\mathbb{Q}^{n+1}(g)_{|B_2}\quad&=\quad\tfrac{\mathbf{1}_{B_2}}{\mu(B_2)}\int_{B_1}\mathbb{Q}^n(g)\mathrm{d}\mu\int_{B_2}\mathbb{Q}^n(g)\mathrm{d}\mu\\&=\quad\tfrac{\mathbf{1}_{B_2}}{\mu(B_2)}\int_{B_2}\mathbb{Q}^n(g)\mathrm{d}\mu\left(1-\int_{B_2}\mathbb{Q}^n(g)\mathrm{d}\mu\right)\mathrm{d}\mu.\end{aligned}$$

Denote $b_n:=\int_{B_2}\mathbb{Q}^n(g)\mathrm{d}\mu$. Clearly $0\leq b_n\leq 1$ and $b_{n+1}=b_n(1-b_n)$. Hence obviously $b_n\to 0$ entailing $\int_{B_1}\mathbb{Q}^n(g)\mathrm{d}\mu\to 1$. We conclude that $\mathbb{Q}^n(g)\to\frac{1}{\mu(B_1)}\mathbf{1}_{B_1}$. Moreover, $\frac{1}{\mu(B_1)}\mathbf{1}_{B_1}$ is $\mathbb{Q}$–invariant and hence we proved that $\mathbf{Q}$ is asymptotically stable. Clearly, the (linear) Markov operator associated with $\mathbf{Q}$ and the density $\frac{1}{\mu(B_1)}\mathbf{1}_{B_1}$ is of the form

$$P_{\frac{1}{\mu(B_1)}\mathbf{1}_{B_1}}(h)=\frac{\mathbf{1}_{B_1}}{\mu(B_1)}\left[\int_{B_1}h\mathrm{d}\mu+\frac{1}{2}\int_{B_2}h\mathrm{d}\mu\right]+\frac{\mathbf{1}_{B_2}}{\mu(B_2)}\frac{1}{2}\int_{B_2}h\mathrm{d}\mu,$$

where $h\in\mathscr{D}$, $\operatorname{supp}(h)\subseteq B_1\cup B_2$. By induction we get

$$P^{[0,n]}_{\frac{1}{\mu(B_1)}\mathbf{1}_{B_1}}(h)=\frac{\mathbf{1}_{B_1}}{\mu(B_1)}\left[\int_{B_1}h\mathrm{d}\mu+\left(1-\left(\frac{1}{2}\right)^n\right)\int_{B_2}h\mathrm{d}\mu\right]+\frac{\mathbf{1}_{B_2}}{\mu(B_2)}\left(\frac{1}{2}\right)^n\int_{B_2}h\mathrm{d}\mu$$

and hence

$$P^{[0,n]}_{\frac{1}{\mu(B_1)}\mathbf{1}_{B_1}}(h)\to\frac{1}{\mu(B_1)}\mathbf{1}_{B_1}.$$

The inclusion $\operatorname{supp}(h)\subseteq\operatorname{supp}(g)$ appear to be redundant.

Theorem 5.2. *Let $\mathbf{Q}$ be a kernel quadratic stochastic operator such that for all $g,h\in\mathscr{D}$ the trajectories $\{P_g^{[0,n]}(h):n=1,2,\ldots\}$ are norm relatively compact. The following conditions are equivalent:*

(1) *There exists $f\in\mathscr{D}$ such that for every $g\in\mathscr{D}$ we have*

$$\lim_{n\to\infty}\|\mathbb{Q}^n(g)-f\|_1=0.$$

(2) *There exists $f\in\mathscr{D}$ such that for all $g,h\in\mathscr{D}$ we have*

$$\lim_{n\to\infty}\left\|P_g^{[0,n]}(h)-f\right\|_1=0.$$

(3) *There exists $f \in \mathscr{D}$ such that for all $m \geq 0$ and $h, g \in \mathscr{D}$ we have*

$$\lim_{n\to\infty} \left\| P_g^{[m,n]}(h) - f \right\|_1 = 0.$$

Proof. The implication (2) $\Rightarrow$ (3) follows from the equality $P_g^{[m,n]}(h) = P_{\mathbb{Q}^m(g)}^{[0,n-m]}(h)$ for arbitrary $g, h \in \mathscr{D}$ and $0 \leq m < n$. The proof of (3) $\Rightarrow$ (1) is trivial as it is enough to substitute $m = 0$ and $h = g$ in (3). We now show that (1) $\Rightarrow$ (2).

Fix $g, h \in \mathscr{D}$ and denote

$$\omega_g(h) := \overline{\{w : \exists_{n_j \nearrow \infty}\, P_g^{[0,n_j]}(h) \to w\}}^{\|\cdot\|_1}$$

the closed ω–limit set. By the norm relative compactness assumption $\omega_g(h)$ is nonempty and compact in $\|\cdot\|_1$. By the same argument as before $P_f(\omega_g(h)) = \omega_g(h)$.

Now, for fixed $g \in \mathscr{D}$ and arbitrary $h \in \mathscr{D}$ let us introduce,

$$L(h) := \limsup_{n\to\infty} \int_{D_f} P_g^{[0,n]}(h)\mathrm{d}\mu = \sup_{w\in\omega_g(h)} \int_{D_f} w\mathrm{d}\mu,$$

where $D_f = \operatorname{supp}(f)$. By the norm relative compactness of $\omega_g(h)$ there exists $w_* \in \omega_g(h)$ such that $L(h) = \int_{D_f} w_*\mathrm{d}\mu$. We recall that D_f is a $\mathbf{Q}$–invariant set (as $\mathbb{Q}(f) = f$) giving for kernel quadratic stochastic operators $\mathbf{Q}$ that for almost all $(x,y) \in D_f \times D_f$

$$\int_{D_f} q(x,y,z)\mathrm{d}\mu(z) = 1.$$

Denote by $P_f^* \colon (L^1)^* \to (L^1)^*$ the adjoint operator of P_f. For any $B \subseteq D_f$

$$\begin{aligned}
\textstyle\int_B P_f^*(\mathbf{1}_{D_f})(y)\mathrm{d}\mu(y) &= \textstyle\int_B \left[\int_X \int_X f(x)q(x,y,z)\mathrm{d}\mu(x)\mathbf{1}_{D_f}(z)\mathrm{d}\mu(z)\right]\mathrm{d}\mu(y) \\
&= \textstyle\int_B \left[\int_{D_f} f(x)\int_X q(x,y,z)\mathbf{1}_{D_f}(z)\mathrm{d}\mu(z)\mathrm{d}\mu(x)\right]\mathrm{d}\mu(y) \\
&= \textstyle\int_B \mathrm{d}\mu(y) = \mu(B).
\end{aligned}$$

Therefore $P_f^*(\mathbf{1}_{D_f}) = 1$ on D_f. Obviously $P_f^*(\mathbf{1}_{D_f}) \geq 0$. Hence $P_f^*(\mathbf{1}_{D_f}) \geq \mathbf{1}_{D_f}$. Using the standard notation for dual operations,

$$\int_{D_f} P_f(w_*)\mathrm{d}\mu = \langle P_f(w_*), \mathbf{1}_{D_f}\rangle = \langle w_*, P_f^*(\mathbf{1}_{D_f})\rangle \geq \langle w_*, \mathbf{1}_{D_f}\rangle = \int_X w_*\mathbf{1}_{D_f}\mathrm{d}\mu = L(h).$$

Since $P_f(w_*) \in \omega_g(h)$, then $L(h) \geq \int_{D_f} P_f(w_*)\mathrm{d}\mu$ and so $L(h) = \int_{D_f} P_f(w_*)\mathrm{d}\mu$. By the induction method we obtain that $L(h) = \int_{D_f} P_f^n(w_*)\mathrm{d}\mu$ for $n = 0, 1, \ldots$.

If $L(h) = 1$ then $\operatorname{supp}(w_*) \subseteq D_f$. Therefore by Lemma 4.1 we have,

$$\lim_{n\to\infty} \left\| P_f^n(w_*) - f \right\|_1 = 0.$$

In particular $f \in \omega_g(h)$ as $P_f^n(\omega_g(h)) = \omega_g(h)$. Let $n_j \nearrow \infty$ be a sequence such that $\|P_g^{[0,n_j]}(h) - f\|_1 \to 0$. For any $w \in \omega_g(h)$ there exists $k_j \nearrow \infty$ such that $\lim_{j\to\infty} \|P_g^{[0,n_j+k_j]}(h) - w\|_1 = 0$ (by definition of $\omega_g(h)$). Therefore

$$\begin{aligned}
\|w-f\|_1 &= \lim_{j\to\infty}\left\|P_g^{[0,n_j+k_j]}(h)-\mathbb{Q}^{n_j+k_j}(g)\right\|_1\\
&= \lim_{j\to\infty}\left\|P_{\mathbb{Q}^{n_j}(g)}^{[0,k_j]}\left(P_g^{[0,n_j]}(h)\right)-P_{\mathbb{Q}^{n_j}(g)}^{[0,k_j]}(\mathbb{Q}^{n_j}(g))\right\|_1\\
&= \lim_{j\to\infty}\left\|P_{\mathbb{Q}^{n_j}(g)}^{[0,k_j]}\left(P_g^{[0,n_j]}(h)-\mathbb{Q}^{n_j}(g)\right)\right\|_1\\
&\le \lim_{j\to\infty}\left\|P_g^{[0,n_j]}(h)-\mathbb{Q}^{n_j}(g)\right\|_1\\
&= \lim_{j\to\infty}\left\|P_g^{[0,n_j]}(h)-f\right\|_1=0.
\end{aligned}$$

Hence $L(h)=1$ implies $\omega_g(h)=\{f\}$ and $\lim_{n\to\infty}P_g^{[0,n]}(h)=f$.

Let us assume now that $0<L(h)<1$. Then we represent w_* in the form

$$w_*=L(h)\mathbf{1}_{D_f}\frac{w_*}{L(h)}+(1-L(h))u_*,$$

where $u_*:=\mathbf{1}_{D_f^C}\frac{w_*}{1-L(h)}$. Notice that for every natural j the function $P_f^j(u_*)$ is concentrated on D_f^C, otherwise

$$\limsup_{n\to\infty}\int_{D_f}P_g^{[0,n+j]}(h)\mathrm{d}\mu\ge L(h)+(1-L(h))\int_{D_f}P_f^j(u_*)\mathrm{d}\mu>L(h).$$

Recalling that we assumed relative compactness of trajectories we apply the Eberlein mean ergodic theorem [see Theorem 5.1 in 9] and obtain that the Cesàro means converge

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}P_f^n(u_*)=:a$$

and $a\in L^1$ is P_f–invariant. Denoting $D_a:=\operatorname{supp}(a)$ we additionally get $D_a\subseteq D_f^C$.

For a fixed $u\in L^1$ let us denote $A(u):=\int_{D_f}u\mathrm{d}\mu$ and $B(u):=\int_{D_a}u\mathrm{d}\mu$. We will show that $B(\mathbb{Q}(u))\ge 2A(u)B(u)$. Notice that

$$a(z)=P_f(a)(z)=\mathbf{Q}(f,a)(z)=\int_X\int_X f(x)a(y)q(x,y,z)\mathrm{d}\mu(x)\mathrm{d}\mu(y)$$

implies $\operatorname{supp}(q(x,y,\cdot))\subseteq\operatorname{supp}(a)$ for almost all $(x,y)\in\operatorname{supp}(f)\times\operatorname{supp}(a)$. It follows that $\int_{D_a^C}\mathbf{Q}(h|_{D_f},v|_{D_a})\mathrm{d}\mu=0$ for any pair of densities h,v. In particular for any density u we have

$$\int_X\mathbf{Q}(u_{|D_f},u_{|D_a})\mathrm{d}\mu=\|u_{|D_f}\|_1\|u_{|D_a}\|_1=A(u)B(u)=\int_{D_a}\mathbf{Q}(u_{|D_f},u_{|D_a})\mathrm{d}\mu.$$

Thus using the symmetry of $\mathbf{Q}(\cdot,\cdot)$

$$\begin{aligned}
B(\mathbb{Q}(u)) &= \textstyle\int_{D_a}\mathbb{Q}(u)\mathrm{d}\mu=\int_{D_a}\mathbb{Q}(u_{|D_a}+u_{|D_a^C})\mathrm{d}\mu\\
&= \textstyle\int_{D_a}\mathbb{Q}(u_{|D_a})\mathrm{d}\mu+\int_{D_a}\mathbb{Q}(u_{|D_a^C})\mathrm{d}\mu+2\int_{D_a}\mathbf{Q}(u_{|D_a},u_{|D_a^C})\mathrm{d}\mu\\
&\ge \textstyle 2\int_{D_a}\mathbf{Q}(u_{|D_a},u_{|D_f}+u_{|D_a^C\setminus D_f})\mathrm{d}\mu\\
&\ge \textstyle 2\int_{D_a}\mathbf{Q}(u_{|D_a},u_{|D_f})\mathrm{d}\mu=2\|u_{|D_f}\|_1\|u_{|D_a}\|_1\\
&= 2A(u)B(u).
\end{aligned}$$

Now consider the P_f–invariant density $s:=L(h)f+(1-L(h))a$ (recall that we are in the $0<L(h)<1$ regime). Since we assume that $\mathbf{Q}$ is asymptotically stable, we clearly have that

$\lim_{n\to\infty} \mathbb{Q}^n(s) = f$ implying $\lim_{n\to\infty} A(\mathbb{Q}^n(s)) = 1$ and so there exists n_0 such that $A(\mathbb{Q}^n(s)) \geq \frac{3}{4}$ for $n \geq n_0$. Obviously $B(\mathbb{Q}^n(s)) > 0$ as $L(h) < 1$. Therefore

$$\begin{aligned}
\limsup_{k\to\infty} B(\mathbb{Q}^{n_0+k}(s)) &= \limsup_{k\to\infty} B(\mathbb{Q}(\mathbb{Q}^{n_0+k-1}(s))) \\
&\geq \limsup_{k\to\infty} 2A(\mathbb{Q}^{n_0+k-1}(s))B(\mathbb{Q}^{n_0+k-1}(s)) \\
&\geq \limsup_{k\to\infty} \tfrac{3}{2}B(\mathbb{Q}^{n_0+k-1}(s)) \geq \cdots \geq \limsup_{k\to\infty} \left(\tfrac{3}{2}\right)^k B(\mathbb{Q}^{n_0}(s)) = \infty,
\end{aligned}$$

a contradiction.

Assume $L(h) = 0$. Then $\int_{D_f} P_f^n(w)\mathrm{d}\mu = 0$ for every $w \in \omega_g(h)$ and any $n = 0, 1, 2, \ldots$. As before, using the Eberlein mean ergodic theorem the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} P_f^n(w) =: a$$

exists, it is P_f–invariant and a is concentrated outside D_f (i.e. $D_a \subseteq D_f^C$). Taking P_f–invariant density $s := \frac{1}{2}f + \frac{1}{2}a$ similarly to the previous case we obtain $B(\mathbb{Q}^n(s)) \to \infty$, a contradiction.

□

In the next theorem we describe the uniform asymptotic stability of quadratic stochastic operators. We give equivalent conditions for norm mixing of nonlinear transformation $\mathbf{Q}$ expressed in terms of convergence of associated with $\mathbf{Q}$ nonhomogeneous linear Markov chain. We first show an auxiliary lemma.

Lemma 5.1. *Let $P : L^1 \to L^1$ be a stochastic operator with invariant density f. If there exist $\rho > 0$ and $k_0 \in \mathbb{N}$ such that*

$$\left\| P^{k_0}(w) \wedge f \right\|_1 \geq \rho \tag{3}$$

for all $w \in \mathscr{D}$ then there exists $d_0 \in \mathbb{N}$ such that P^{nd_0} is convergent in the operator norm to a finite dimensional projection.

Proof. We will show that Eq. (3) entails that the operator P is uniformly η–smoothing [4], i.e. there exist $0 < \eta < 1$, $F \subseteq X$ with $\mu(F) < \infty$, and $0 < \delta$ such that for some natural k_0

$$\sup_{w\in\mathscr{D}} \int_{E\cup F^C} P^{k_0}(w)\mathrm{d}\mu \leq \eta$$

for all $E \subseteq X$ satisfying $\mu(E) < \delta$.

Let $0 < \varepsilon < \rho/3$ and take κ such that $\|f\mathbf{1}_{\{1/\kappa<f\leq\kappa\}} - f\|_1 \leq \varepsilon$. Define $F = \{x \in X : 1/\kappa < f(x) \leq \kappa\}$. Obviously $\mu(F) < \infty$. Let $0 < \delta < \rho/(3\kappa)$ and $E \subseteq F$ such that $\mu(E) < \delta$. Then $\int_E f\mathrm{d}\mu < \kappa \cdot \rho/(3\kappa) = \rho/3$. Hence

$$\begin{aligned}
\textstyle\int_{E\cup F^C} P^{k_0}(w)\mathrm{d}\mu &= \textstyle\int_{E\cup F^C} P^{k_0}(w) \wedge f\mathrm{d}\mu + \int_{E\cup F^C} P^{k_0}(w) - P^{k_0}(w) \wedge f\mathrm{d}\mu \\
&\leq \textstyle\int_{F^C} f\mathrm{d}\mu + \int\limits_E f\mathrm{d}\mu + \int_{E\cup F^C} P^{k_0}(w) - P^{k_0}(w) \wedge f\mathrm{d}\mu \\
&\leq \varepsilon + \delta\kappa + 1 - \rho \leq \rho/3 + \rho/3 + 1 - \rho \leq 1 - \rho/3.
\end{aligned}$$

We therefore have that P is uniformly smoothing with $\eta = 1 - \rho/3$ and hence it is quasi–compact [4], i.e. $|||P^n - K||| < 1$ for some compact operator K and natural n. It follows from [4] [but see also 2, 3] that there is a $d_0 \in \mathbb{N}$ such that P^{nd_0} is norm convergent as $n \to \infty$. In particular there are pairwise disjoint densities $f_1, \ldots, f_d \in \mathscr{D}$ and functionals $\Lambda_1, \ldots, \Lambda_d \in L^\infty$ such that

$$\sup_{w \in \mathscr{D}} \left\| P^{nd_0}(w) - \sum_{j=1}^{d} \Lambda_j(w) f_j \right\|_1 \xrightarrow{n \to \infty} 0.$$

□

Theorem 5.3. *Let* $\mathbf{Q}$ *be a quadratic stochastic operator. The following conditions are equivalent:*

(1) *There exists* $f \in \mathscr{D}$ *such that*

$$\lim_{n \to \infty} \sup_{g \in \mathscr{D}} \|\mathbb{Q}^n(g) - f\|_1 = 0.$$

(2) *There exists* $f \in \mathscr{D}$ *such that*

$$\lim_{n \to \infty} \sup_{g,h \in \mathscr{D}} \left\| P_h^{[0,n]}(g) - f \right\|_1 = 0.$$

(3) *There exists* $f \in \mathscr{D}$ *such that for every* $m \geq 0$ *we have*

$$\lim_{n \to \infty} \sup_{g,h \in \mathscr{D}} \left\| P_h^{[m,n]}(g) - f \right\|_1 = 0,$$

i.e. independently of the seed $g \in \mathscr{D}$*, all nonhomogeneous Markov chains* $\mathbf{P}_g = (P_g^{[n,n+1]})_{n \geq 0}$ *are norm mixing with a common limit distribution* f *and the rate of convergence is uniform for* g*.*

Proof. Similarly as in the case of Theorem 5.2 it suffices to show the implication $(1) \Rightarrow (2)$. The proof is ad absurdum; assume that (1) holds true and suppose that (2) is false, i.e. there exists $\varepsilon > 0$ such that for sufficiently large n there are $g, h \in \mathscr{D}$ satisfying $\|P_g^{[0,n]}(h) - f\|_1 \geq \varepsilon$. We will proceed in three steps.

Step 1 Let $m \in \mathbb{N}$ be fixed. By assumption (1) and uniform continuity of $\mathbb{Q}$ (on the unit ball in L^1) there exists $L \in \mathbb{N}$ such that for any $g \in \mathscr{D}$ and $l \geq L$ we have $\|\mathbb{Q}^l(g) - f\|_1 \leq \delta$ and $\delta > 0$ is chosen such that $|||P_{g_1}^{[0,m]} - P_{g_2}^{[0,m]}||| < \varepsilon/2$ if $\|g_1 - g_2\|_1 \leq \delta$. Let $n = l + m$ and $g, h \in \mathscr{D}$ be such that $\|P_g^{[0,l+m]}(h) - f\|_1 \geq \varepsilon$. Then

$$\begin{aligned}
\varepsilon &\leq \left\| P_g^{[0,l+m]}(h) - f \right\|_1 \\
&= \left\| P_{\mathbb{Q}^l(g)}^{[0,m]}(P_g^{[0,l]}(h)) - f \right\|_1 \\
&\leq \left\| P_{\mathbb{Q}^l(g)}^{[0,m]}(P_g^{[0,l]}(h)) - P_f^m(P_g^{[0,l]}(h)) \right\|_1 + \left\| P_f^m(P_g^{[0,l]}(h)) - f \right\|_1 \\
&< \tfrac{\varepsilon}{2} + \left\| P_f^m(P_g^{[0,l]}(h)) - f \right\|_1
\end{aligned}$$

giving

$$\frac{\varepsilon}{2} < \left\| P_f^m(P_g^{[0,l]}(h)) - f \right\|_1.$$

Hence $\|P_f^m(r) - f\|_1 > \frac{\varepsilon}{2}$, where $r = P_g^{[0,l]}(h) \in \mathscr{D}$ and $m = n - l$ can be arbitrarily large. Thus the iterates P_f^m would not converge uniformly (in the operator norm).

Step 2 Suppose that there exist $\rho > 0$ and $k_0 \in \mathbb{N}$ such that for all $w \in \mathscr{D}$ we have $\|P_f^{k_0}(w) \wedge f\|_1 \geq \rho$. Therefore by Lemma 5.1 we obtain that P_f is quasi–compact. Furthermore it overlaps supports (see the remark prior to Lemma 4.1) and hence asymptotic periodicity turns to convergence, i.e. $d = 1$ [2, 3]. In particular,

$$\sup_{w\in\mathscr{D}} \left\|P_f^n(w) - f\right\|_1 \xrightarrow{n\to\infty} 0$$

and by Step 1 we obtain a contradiction. Hence we showed that if the iterates P_f^j do not converge in the operator norm to $\mathbf{1} \otimes f$, then $\inf_{w\in\mathscr{D}} \|P_f^k(w) \wedge f\|_1 = 0$ for all k, or equivalently, $\sup_{w\in\mathscr{D}} \|P_f^k(w) - f\|_1 = 2$.

Step 3 Let $\mathscr{J} \in \mathbb{N}$ be such that $\|\mathbb{Q}^j(u) - f\|_1 < \varepsilon/4$ for all $u \in \mathscr{D}$ and $j \geq \mathscr{J}$, where $\varepsilon > 0$. First we show that for any $\beta > 0$ and any fixed $k_* \in \mathbb{N}$ there exists $w \in \mathscr{D}$ such that $\|f - w\|_1 \geq 2 - \beta$ (or equivalently $\|f \wedge w\|_1 < \beta/2$) and $\|P_f^l(w) - w\|_1 < \beta$ for all $0 \leq l \leq k_*$. For this fix $\beta > 0$ and $k_* \in \mathbb{N}$ and consider natural $k \geq \max\{k_*, 2\beta\}$. Applying Step 2 we find a $v \in \mathscr{D}$ such that $\|P_f^{k^2}(v) - f\|_1 > 2 - \beta/k^2$. Since P_f is stochastic, the sequence $l \mapsto \|P_f^l(v) - f\|_1$ is non–increasing and hence $\|P_f^l(v) - f\|_1 > 2 - \beta/k^2$ for any $0 \leq l \leq k^2$, or equivalently $\|k^2 P_f^l(v) \wedge k^2 f\|_1 \leq \beta/2$ for $l = 0, 1, \ldots, k^2$. Let

$$w = \left(v + P_f(w) + \ldots + P_f^{k^2-1}(v)\right)/k^2.$$

Clearly $w \in \mathscr{D}$. We have

$$\begin{aligned} \|f \wedge w\|_1 &= \tfrac{1}{k^2} \left\|k^2 f \wedge \left(v + P_f(v) + \cdots + P_f^{k^2}(v)\right)\right\|_1 \\ &\leq \tfrac{1}{k^2} \textstyle\sum_{l=0}^{k^2-1} \left\|k^2 f \wedge k^2 P_f^l(v)\right\|_1 < \tfrac{\beta}{2}, \end{aligned}$$

or equivalently $\|f - w\|_1 > 2 - \beta$, and for all $0 \leq l \leq k$

$$\begin{aligned} \left\|P_f^l(w) - w\right\|_1 &= \tfrac{1}{k^2} \left\|\textstyle\sum_{j=0}^{l-1} \left(P_f^{k^2+l-1-j}(v) - P_f^j(v)\right)\right\|_1 \\ &\leq \tfrac{1}{k^2} \textstyle\sum_{j=0}^{l-1} \left\|P_f^{k^2+l-1-j}(v) - P_f^j(v)\right\|_1 \leq \tfrac{1}{k^2} \cdot 2l \leq \tfrac{1}{k^2} \cdot 2k < 2 \cdot \tfrac{\beta}{2} = \beta. \end{aligned}$$

We call the density w constructed above a (k_*, β)–approximate fixed point of $P_f = \mathbf{Q}(f, \cdot)$ as $P_f(w) \approx w$ and $w \approx\perp f$.

Let $\eta \in (0, \varepsilon]$ (in the further part of the proof we take $\eta = \varepsilon < 1/2$ and without loss of generality assume $\beta < \varepsilon/4$) and let $g, u \geq 0$. Denote

$$C(\eta, u, g) := \begin{cases} \sup\{\tau \in (0,1] : \|\underline{u} \wedge \frac{g}{\tau}\|_1 \geq (1-\eta)\|u\|_1\}, \\ 0, \qquad \text{if the set of such } \tau \text{ is empty.} \end{cases}$$

Notice that

$$C(\eta, u, g) = \sup\{\tau \in [0,1] : \exists_{r\in[0,u]} \ \|r\|_1 \geq (1-\eta)\|u\|_1 \text{ and } \tau r \leq g\}.$$

If $C(\eta, u, g) > 0$ then for any positive $\tau < C(\eta, u, g)$ there exists $0 \leq r \leq u \wedge \frac{g}{\tau}$ such that $\|r\|_1 = (1-\eta)\|u\|_1$. Clearly $\tau r \leq u$ and $\tau r \leq g$, hence $\tau r \leq u \wedge g$. Thus $\tau(1-\eta)\|u\|_1 = \tau\|r\|_1 \leq \|u \wedge g\|_1$ implying

$$\tau \leq \frac{\|u \wedge g\|_1}{(1-\eta)\,\|u\|_1}$$

and resulting

$$C(\eta,u,g) \le \frac{\|u\wedge g\|_1}{(1-\eta)\|u\|_1}.$$

We apply the above estimation for w, where $w \in \mathscr{D}$ is an arbitrary (k_*,β)–approximate fixed point of P_f, and for $g \in \mathbb{Q}^j(u)$, $j \ge \mathscr{J}$ and where $u \in \mathscr{D}$. We have

$$\begin{array}{rcl} C(\eta,w,\mathbb{Q}^j(u)) & \le & \frac{\|w\wedge\mathbb{Q}^j(u)\|_1}{(1-\eta)\|w\|_1} = \frac{1-\frac{1}{2}\|w-\mathbb{Q}^j(u)\|_1}{1-\eta} \\ & \le & \frac{1-\frac{1}{2}\left(\|w-f\|_1-\|f-\mathbb{Q}^j(u)\|_1\right)}{1-\eta} \\ & < & \frac{1-\frac{1}{2}(2-\beta)+\frac{1}{2}\cdot\frac{\varepsilon}{4}}{1-\eta} = \frac{\frac{1}{2}\beta+\frac{1}{2}\cdot\frac{\varepsilon}{4}}{1-\eta} < \frac{\frac{\varepsilon}{4}}{1-\varepsilon} < \frac{\varepsilon}{2}. \end{array}$$

By norm continuity of lattice operations on L^1 we have

$$\lim_{\eta_n\to\eta, w_n\to w, g_n\to g} C(\eta_n,w_n,g_n) = C(\eta,w,g)$$

where the convergence of sequences $(w_n)_{n\ge 0}$, $(g_n)_{n\ge 0}$ holds in L^1 norm.

We estimate $C(\eta,w,\mathbb{Q}^j(h))$ from below where $w\in\mathscr{D}$ is a $(2^{\mathscr{J}},\beta)$–approximate fixed point of P_f and $h=(w+f)/2$. We have

$$\mathbb{Q}(h) = \frac{1}{4}f + \frac{1}{2}\mathbf{Q}(f,w) + \frac{1}{4}\mathbb{Q}(w).$$

Passing with $\beta\to 0^+$ we get $\liminf_{\beta\to 0^+} C(\eta,w,\mathbb{Q}(h)) \ge 1/2$. We take

$$\mathbb{Q}^2(h) = \frac{1}{4^2}f + 2\cdot\frac{1}{4}\cdot\frac{1}{2}\mathbf{Q}(f,\mathbf{Q}(f,w)) + \cdots = \frac{1}{2^4}f + \frac{1}{4}P_f^2(w) + \cdots$$

and we obtain $\liminf_{\beta\to 0^+} C(\eta,w.\mathbb{Q}^2(h)) \ge 1/4$. Iterating further until reaching $\mathscr{J}$

$$\mathbb{Q}^{\mathscr{J}}(h) = 2^{-2\mathscr{J}}f + 2^{\mathscr{J}-2\mathscr{J}}P_f^{\mathscr{J}}(w) + \text{ some nonnegative function.}$$

Using the continuity of $\mathbf{Q}(\cdot,\cdot)$ for a fixed $\mathscr{J}$ (choosing β sufficiently small) we can assume that a $(2^{\mathscr{J}},\beta)$–approximate fixed point $w\in\mathscr{D}$ of P_f satisfies

$$C(\eta,w,\mathbb{Q}^{\mathscr{J}}) > \frac{1}{2}\cdot 2^{\mathscr{J}-2\mathscr{J}} > 2^{\mathscr{J}-1-2\mathscr{J}}.$$

For $j\ge\mathscr{J}$ we can write $\mathbb{Q}^j(h) = v(j) + C(\eta,w,\mathbb{Q}^j(h))r(j)$, where $v(j)\ge 0$, $r(j)\le w$ and $\|r(j)\|_1 = 1-\eta = 1-\varepsilon$. We can find such a $r(j)$ from the ordered interval $[0,w]$ such that $\|r(j)\|_1 \ge (1-\eta)\|w\|_1$ (remembering $\eta=\varepsilon$) and $\tau r\le w$, $\tau \le C(\eta,w,\mathbb{Q}^j(h))$. We now get (for $j\ge\mathscr{J}$)

$$\|v(j)-f\|_1 \le \left\|v(j)-\mathbb{Q}^j(h)\right\|_1 + \left\|\mathbb{Q}^j(h)-f\right\|_1 \le C(\eta,w,\mathbb{Q}^j(h))\,\|r(j)\|_1 + \frac{\varepsilon}{4} < \frac{3}{4}\varepsilon.$$

If we apply the $\mathbb{Q}$ transformation once more we obtain,

$$\begin{array}{rcl} \mathbb{Q}^{j+1}(h) & = & \mathbb{Q}(v(j)+C(\eta,w,\mathbb{Q}^j(h))r(j)) \\ & = & \mathbb{Q}(v(j)) + 2C(\eta,w,\mathbb{Q}^j(h))\mathbf{Q}(v(j),r(j)) + C(\eta,w,\mathbb{Q}^j(h))^2\mathbb{Q}(r(j)). \end{array}$$

Since $\mathbf{Q}(v(j),r(j)) \le \mathbf{Q}(\mathbb{Q}^j(h),w)$ as $v(j)\le\mathbb{Q}^j(h)$, $r(j)\le w$, we arrive at

$$
\begin{aligned}
\|\mathbf{Q}(v(j),r(j))-w\|_1 &\le \|\mathbf{Q}(v(j),r(j))-\mathbf{Q}(f,w)\|_1+\|\mathbf{Q}(f,w)-w\|_1\\
&\le \|v(j)-f\|_1\,\|r(j)\|_1+\|r(j)-w\|_1\,\|f\|_1+\beta\\
&\le \|v(j)-f\|_1+\|r(j)-w\|_1+\beta\le \tfrac{\varepsilon}{2}+\tfrac{\varepsilon}{4}+\varepsilon+\tfrac{\varepsilon}{4}\\
&= 2\varepsilon
\end{aligned}
$$

and in particular

$$\|\mathbf{Q}(v(j),r(j))\wedge w\|_1 \ge 1-\varepsilon = 1-\eta .$$

Hence

$$C(\eta,w,\mathbb{Q}^{j+1}(h)) \ge 2C(\eta,w,\mathbb{Q}^{j}(h)).$$

Iterating the above estimation $k=2^{\mathscr{J}}-\mathscr{J}$ times we obtain

$$\frac{\varepsilon}{2} \ge C(\eta,w,\mathbb{Q}^{\mathscr{J}+k}(h)) \ge 2^k C(\eta,w,\mathbb{Q}^{\mathscr{J}}(h)) \ge 2^k 2^{\mathscr{J}-2^{\mathscr{J}}-1} = \frac{1}{2} \ge \varepsilon$$

a contradiction, proving our theorem. □

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