Cent. Eur. J. Phys. • 12(5) • 2014 • 305-314
DOI: 10.2478/s11534-014-0454-y

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Central European Journal of **Physics**

The vortex flow caused by sound in a bubbly liquid

Research Article

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Received 21 September 2013; accepted 16 March 2014

Abstract: Generation of vorticity in the field of intense sound in a bubbly liquid in the free half-space is considered. The reasons for generation of vorticity are nonlinearity, diffraction, and dispersion. Acoustic streaming differs from that in a Newtonian fluid. Under some conditions, the vortex flow changes its direction. Conclusions concern streaming induced by a harmonic or an impulse Gaussian beam.

PACS (2008): 43.25Yw

Keywords: acoustic streaming • bubbly liquid • nonlinear acoustics • dispersion, diffraction

1. Introduction

Acoustical dispersion in unbounded acoustical media is usually a weak effect in contrast to strong dispersion of light in most optical media. Inhomogeneous media may reveal noticeable dispersion under some special conditions [1–3]. A liquid which involves gaseous bubbles possesses compressibility much lower than the gas phase, and the acoustical properties of a bubbly liquid differ considerably from that in a pure liquid [2, 4]. The sound speed becomes essentially reduced, nonlinearity increases by orders of magnitude, and features of sound propagation depend considerably on its frequency. That makes studies of nonlinear effects important not only relative to sound itself, but in connection with nonlinear phenomena induced in the sound field. Analysis of propagation of finite-amplitude sound in bubbly liquids is quite complicated. It relates to a number of theoretical models which describe inclusion of set of bubbles into the bulk of liquid differently [4–8]. A proper description of nonlinear sound propagation is the second starting point in all nonlinear effects caused by sound [2, 3]. The equation which governs sound beam is analogous to the famous Khokhlov-Zabolotskaya-Kuznetsov equation, but it includes a dispersive term instead of or supplementing standard attenuation [2, 9]. Analytical methods for solving the fully nonlinear form of the Khokhlov-Zabolotskaya equation (that is, the lossless form of the KZK equation) have been proposed only recently. One method incorporates analytical techniques used in nonlinear geometrical acoustics [10]. An approximate axial solution is derived for the preshock region of a beam radiated by a monofrequency source. The second method is more general in that it applies to pulses and takes into account shock formation [11]. A Gaussian profile of acoustic pressure at the transducer is assumed in both approaches. As for the full KZK equation, general analytical methods to solve it are still absent, all the more

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so for solutions of the equations which govern sound in a bubbly viscous liquid.

Among other nonlinear phenomena (such as acoustic heating and scattering of sound), understanding of acoustic streaming in bubbly liquids is important [12, 13]. Studies of nonlinear fluid dynamics must start from equations describing fluid dynamic of the mixture as a whole. These equations schedule all motions which may exist in a bubbly liquid and should be consequently decomposed in order to yield equations governing every mode. Modes of infinitely small amplitude do not interact. In three dimensions, both non-wave modes, i.e., vorticity and entropy modes (these names come from the linear theory of flows of Newtonian uniform unbounded fluids [14]) are nonlinearly generated in the field of sound. As far as the authors know, acoustic streaming in a bubbly liquid, that is, vortex modes which are caused by intense sound, is still an unexplored domain. The main difficulty is to describe as precisely as possible propagation of the sound beam itself and the interaction of sound and non-wave modes. We use an analytical approach in order to subdivide governing weakly nonlinear equations for every mode from the system of conservation equations, and numerical methods to solve them approximately.

As a result of the proper decomposition of equations, nonlinear terms become distributed between equations correctly. They include terms of all modes and may be considered as "driving forces" of specific modes. The procedure was applied by one of the authors in problems of acoustic heating and streaming in Newtonian and some non-Newtonian fluids [15–17].

2. Equations governing perturbations in bubbly liquid

We consider three-dimensional motions of a mixture which consists of compressible liquid and identical spherical bubbles of an ideal gas. All bubbles are of the same radii at equilibrium, and there is no heat and mass transfer between liquid and gas. To simplify the analysis, we assume that motions of the bubbles do not influence each other (i.e., they are well separated), and that they pulsate in their lowest, radially symmetric mode. The characteristic scale of perturbation in the mixture is much larger than a bubble radius, so that the mixture as a whole may be treated as a homogeneous continuum. The pressure of the mixture equals the pressure of the liquid [4, 18]. Quantities relating to gas, liquid, or to the mixture are marked by index g, l, and mix, respectively. Unperturbed quantities are marked by an additional zero, and disturbed ones are primed. The density of the mixture is given by

$$\rho_{mix} = \frac{\rho_g \rho_l}{\beta \rho_l + (1-\beta)\rho_g}, \qquad (1)$$

where β is the constant mass concentration of gas in the mixture. The initial volume concentration of gas in the mixture, α_0, equals

$$\alpha_0 = \beta \frac{\rho_{mix0}}{\rho_{g0}}. \qquad (2)$$

The acoustics of incompressible liquids (when $c_l \to \infty$) including bubbles was originally studied by van Wijngaarden [4]. In particular, involving liquid compressibility corrects the nonlinear sound parameter [2, 19]. The following equations in differential form declare conservation of momentum, energy and mass:

$$\begin{gathered}
\frac{\partial \vec{v}}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} + \frac{1}{\rho_{mix}}\vec{\nabla}p = 0, \\
\frac{\partial p}{\partial t} + c_l^2 \frac{\partial \rho_l}{\partial t} - \frac{c_l^2(\gamma_l - 1)}{\rho_{l0}}\rho_l \frac{\partial \rho_l}{\partial t} = 0, \\
\frac{\partial \rho_{mix}}{\partial t} + \vec{\nabla}(\vec{v}\rho_{mix}) = 0,
\end{gathered} \qquad (3)$$

where $\vec{v}$, p denote velocity and pressure in the mixture. The second equation in (3) is actually a result of linear combination of the continuity and energy equations for a pure liquid, with $\gamma_l = \frac{C_{p,l}}{C_{V,l}}\frac{\rho_{l0}}{p_{l0}}\left(\frac{\partial p_l}{\partial \rho_l}\right)_{T=const}$, where C_p and C_V denote the heat capacities at constant pressure and density. For water at normal conditions, it equals approximately 7. Some other equations complement the system (3). The first reflects a constant mass of gas inside a spherical bubble whose density is constantly distributed over a volume (R denotes the bubble's radius),

$$R^3 \rho_g = R_0^3 \rho_{g0}, \qquad (4)$$

and the second one describes the adiabatic behavior of gas in it,

$$p_g \rho_g^{-\gamma_g} = p_{g0} \rho_{g0}^{-\gamma_g}. \qquad (5)$$

Eq. (5) imposes, given spatially homogeneous distribution of density and pressure in a bubble, no energy exchange between bubbles and the surrounding liquid, $\gamma_g = \frac{C_{p,g}}{C_{V,g}}$. Pulsation of each bubble is described by the Rayleigh-Plesset equation [20]:

$$R\frac{\partial^2 R}{\partial t^2} + \frac{3}{2}\left(\frac{\partial R}{\partial t}\right)^2 - \frac{1}{c_l}\left(R^2\frac{\partial^3 R}{\partial t^3} + 6R\frac{\partial R}{\partial t}\frac{\partial^2 R}{\partial t^2} + 2\left(\frac{\partial R}{\partial t}\right)^3\right) = \frac{p'_g - p'_l}{\rho_l}. \qquad (6)$$

Surface tension is not taken into account by Eq. (6), but it accounts for compressibility of a liquid [5, 18]. Eqs. (4), (5), (6) permit rearranging the second equation from the system (3) in terms of quantities describing the mixture: p, ρ_{mix}, $\vec{v}$. Eqs. (3) in the dimensionless quantities ($\vec{x}$ denotes the vector of cartesian coordinates)

$$\vec{v}^d = \frac{\vec{v}'}{c_{mix}}, \ p^d = \frac{p'}{c_{mix}^2\rho_{mix0}}, \ \rho^d = \frac{\rho'_{mix}}{\rho_{mix0}},$$
$$\vec{x}^d = \frac{\vec{x}}{\lambda}, \ t^d = \frac{t c_{mix}}{\lambda}, \tag{7}$$

where excess quantities are denoted by primes, and c_{mix} is the velocity of sound of infinitely small magnitude in a bubbly liquid [4]:

$$\frac{1}{c_{mix}^2} = \frac{(1-\alpha_0)^2}{c_l^2} + \frac{\alpha_0(1-\alpha_0)\rho_{l0}}{\gamma_g p_{g0}}, \tag{8}$$

take the form [19]

$$\frac{\partial\vec{v}}{\partial t} + \vec{\nabla}p = -\left((\vec{v}\cdot\vec{\nabla})\vec{v} - \rho\vec{\nabla}p\right),$$
$$\frac{\partial p}{\partial t} + \vec{\nabla}\cdot\vec{v} - \frac{\alpha_0(1-\alpha_0)R_0^2\rho_{l0}^2c_{mix}^4}{3(\gamma_g p_{g0})^2}\frac{\partial^3 p}{\partial t^3} = (1-\alpha_0)c_{mix}^2\left(-\frac{\gamma_l+1}{c_l^2}\rho\vec{\nabla}\cdot\vec{v} - c_{mix}^2\frac{\alpha_0(1-\alpha_0)\rho_{l0}^2(\gamma_g+1)}{(\gamma_g p_{g0})^2}p\vec{\nabla}\cdot\vec{v}\right) - (\vec{v}\cdot\vec{\nabla})\rho + \rho\vec{\nabla}\cdot\vec{v}, \tag{9}$$
$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0.$$

Starting from Eqs. (9), upper indices by dimensionless quantities will be omitted. The largest, quadratic terms reside in the non-linear right-hand parts of all equations.

3. Decomposition of sound, the entropy and the vorticity mode in the flow of infinitely small magnitude

The linear analogue of the system (9) takes the form

$$\frac{\partial\Psi}{\partial t} + L\Psi = 0, \tag{10}$$

where Ψ is a vector of perturbations, $\Psi = \begin{pmatrix} v_x & v_y & v_z & p & \rho\end{pmatrix}^T$, and

$$L = \begin{pmatrix} 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial z} & 0 \\ \frac{\partial}{\partial x}(1+D\Delta) & \frac{\partial}{\partial y}(1+D\Delta) & \frac{\partial}{\partial z}(1+D\Delta) & 0 & 0 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 & 0 \end{pmatrix},$$
$$D = \frac{\alpha_0(1-\alpha_0)R_0^2\rho_{l0}^2c_{mix}^2}{3(\gamma_g p_{g0})^2\lambda^2} \tag{11}$$

are a linear matrix operator including spatial derivatives and a small parameter responsible for dispersion, D, respectively. In leading order, $\frac{\partial^3 p}{\partial t^3}$ equals $-\Delta(\vec{\nabla}\cdot\vec{v})$. That follows from the first and second equations in the system (9). Studies of the motions of infinitely-small amplitudes usually begin by representing of all perturbations as a sum of plane waves:

$$f(\vec{x},t) = \int\widetilde{f}(\vec{k},t)\exp(-i\vec{k}\cdot\vec{x})d\vec{k} = \int\widetilde{f}(\vec{k})\exp(i\omega t - i\vec{k}\cdot\vec{x})d\vec{k}, \tag{12}$$

($\widetilde{f}(\vec{k},t)$ denotes the Fourier transform of $f(\vec{x},t)$, $\widetilde{f}(\vec{k},t) = \frac{1}{(2\pi)^3}\int f(\vec{x},t)e^{i\vec{k}\cdot\vec{x}}d\vec{x}$). In all evaluations below only terms proportional to D^0 and D^1, are retained. There are five roots of the dispersion equation, the first two being acoustic (marked by indices 1 and 2, respectively), the third dispersion relation describing a stationary (or "entropy") mode, and the last two zero roots describing the stationary vortex motion,

$$\omega_1 = -i\sqrt{\widetilde{\Delta}}\left(1+\frac{D}{2}\widetilde{\Delta}\right), \ \omega_2 = i\sqrt{\widetilde{\Delta}}\left(1+\frac{D}{2}\widetilde{\Delta}\right),$$
$$\omega_3 = 0, \ \omega_4 = 0, \ \omega_5 = 0, \tag{13}$$

where

$$\widetilde{\Delta} = -k_x^2 - k_y^2 - k_z^2, \quad \sqrt{\widetilde{\Delta}} = i\sqrt{k_x^2+k_y^2+k_z^2}.$$

They determine relations of perturbations specific for every mode ($\widetilde{\Psi}$ denotes a vector of Fourier-transforms of perturbations):

$$\widetilde{\Psi}_1 = \begin{pmatrix} \frac{ik_x}{\sqrt{\widetilde{\Delta}}}\left(1+\frac{D}{2}\widetilde{\Delta}\right) \\ \frac{ik_y}{\sqrt{\widetilde{\Delta}}}\left(1+\frac{D}{2}\widetilde{\Delta}\right) \\ \frac{ik_z}{\sqrt{\widetilde{\Delta}}}\left(1+\frac{D}{2}\widetilde{\Delta}\right) \\ 1+D\widetilde{\Delta} \\ 1 \end{pmatrix} \widetilde{\rho}_1,$$

$$\widetilde{\Psi}_2 = \begin{pmatrix} -\frac{ik_x}{\sqrt{\widetilde{\Delta}}}\left(1+\frac{D}{2}\widetilde{\Delta}\right) \\ -\frac{ik_y}{\sqrt{\widetilde{\Delta}}}\left(1+\frac{D}{2}\widetilde{\Delta}\right) \\ -\frac{ik_z}{\sqrt{\widetilde{\Delta}}}\left(1+\frac{D}{2}\widetilde{\Delta}\right) \\ 1+D\widetilde{\Delta} \\ 1 \end{pmatrix} \widetilde{\rho}_2,$$

$$\widetilde{\Psi}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \widetilde{\rho}_3, \qquad (14)$$

where $\widetilde{\rho}_n$ $(n = 1, 2, 3)$ are Fourier transforms of perturbations in density belonging to the corresponding specific mode. Both branches of the vortex mode may be determined in the following way:

$$\vec{\nabla} \cdot \vec{v}_{4,5} = 0, \; p_{4,5} = 0, \; \rho_{4,5} = 0, \qquad (15)$$

where $\vec{v}_4$, $\vec{v}_5$ denote two arbitrary independent branches of the vortex flow: $\vec{v}_4 + \vec{v}_5 = \vec{v}_{vort}$. Eqs. (13),(14) may be expanded in series with respect to powers of D. That significantly simplifies evaluations. Based on the links specific for every mode, the projecting matrix operators may be determined. Projectors are matrix operators decomposing every mode from the total vector of perturbations,

$$\widetilde{P}_i \Psi = \Psi_i \quad (i = 1 \ldots 5). \qquad (16)$$

Every projector is a matrix of spatial operators consisting of five rows and five columns. In the Fourier space, they take the leading-order forms

$$\widetilde{P}_{1,2} = \frac{1}{2} \begin{pmatrix} -\frac{k_x^2}{\widetilde{\Delta}} & -\frac{k_x k_y}{\widetilde{\Delta}} & -\frac{k_x k_z}{\widetilde{\Delta}} & \pm\frac{ik_x}{\sqrt{\widetilde{\Delta}}}\left(1+\frac{D}{2}\widetilde{\Delta}\right) & 0 \\ -\frac{k_x k_y}{\widetilde{\Delta}} & -\frac{k_y^2}{\widetilde{\Delta}} & -\frac{k_y k_z}{\widetilde{\Delta}} & \pm\frac{ik_y}{\sqrt{\widetilde{\Delta}}}\left(1+\frac{D}{2}\widetilde{\Delta}\right) & 0 \\ -\frac{k_x k_z}{\widetilde{\Delta}} & -\frac{k_y k_z}{\widetilde{\Delta}} & -\frac{k_z^2}{\widetilde{\Delta}} & \pm\frac{ik_z}{\sqrt{\widetilde{\Delta}}}\left(1+\frac{D}{2}\widetilde{\Delta}\right) & 0 \\ \pm\frac{ik_x}{\sqrt{\widetilde{\Delta}}}\left(1+\frac{D}{2}\widetilde{\Delta}\right) & \pm\frac{ik_y}{\sqrt{\widetilde{\Delta}}}\left(1+\frac{D}{2}\widetilde{\Delta}\right) & \pm\frac{ik_z}{\sqrt{\widetilde{\Delta}}}\left(1+\frac{D}{2}\widetilde{\Delta}\right) & 1 & 0 \\ \pm\frac{ik_x}{\sqrt{\widetilde{\Delta}}}\left(1-\frac{D}{2}\widetilde{\Delta}\right) & \pm\frac{ik_y}{\sqrt{\widetilde{\Delta}}}\left(1-\frac{D}{2}\widetilde{\Delta}\right) & \pm\frac{ik_z}{\sqrt{\widetilde{\Delta}}}\left(1-\frac{D}{2}\widetilde{\Delta}\right) & 1-D\widetilde{\Delta} & 0 \end{pmatrix}, \qquad (17)$$

$$P_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1+D\widetilde{\Delta} & 1 \end{pmatrix},$$

$$\widetilde{P}_{vort} = \widetilde{P}_4 + \widetilde{P}_5 = \begin{pmatrix} -\frac{k_y^2+k_z^2}{\widetilde{\Delta}} & \frac{k_x k_y}{\widetilde{\Delta}} & \frac{k_x k_z}{\widetilde{\Delta}} & 0 & 0 \\ \frac{k_x k_y}{\widetilde{\Delta}} & -\frac{k_x^2+k_z^2}{\widetilde{\Delta}} & \frac{k_y k_z}{\widetilde{\Delta}} & 0 & 0 \\ \frac{k_x k_z}{\widetilde{\Delta}} & \frac{k_y k_z}{\widetilde{\Delta}} & -\frac{k_x^2+k_y^2}{\widetilde{\Delta}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Projectors analogous to (17) in the planar flow of a bubbly liquid were firstly derived in [19]. In the leading order, projecting operators form a full set of orthogonal operators, with properties

$$\sum_{i=1}^{5} P_i = \mathbf{I},\ P_i \cdot P_j = \mathbf{0},\ i \neq j,\ P_i^2 = P_i, \qquad (18)$$

where $\mathbf{I}$ and $\mathbf{0}$ denote unit and zero matrix operators.

4. Equations governing sound and the vorticity mode in a weakly nonlinear flow

The modes of a flow of infinitely small magnitude do not interact far from boundaries. The dynamic equations which govern any mode may be readily decomposed applying projectors on the linearized system (10). The projecting operators (17) point the way for successful decomposition of equations governing every mode also in the nonlinear flow. Projection results in dynamic equations with nonlinear terms responsible for the modes' interaction. These terms may be considered as "driving forces" for individual modes. Eqs. (9) accounting for nonlinear terms takes the form

$$\frac{\partial \Psi}{\partial t} + L\Psi = \Psi_{nl}. \qquad (19)$$

Application of P_{vort} to the system (19) cancels all acoustic and entropy terms in the linear part of the left-side vector, but yields nonlinear sources in the right-hand vector. We will consider among nonlinear terms only acoustic ones. These correspond to intense sound as compared to the non-wave modes. The vorticity projector in fact applies on three components of overall velocity. Its part, applying to the velocity vector, $P_{vort,\vec{v}}$, includes nine operators:

$$P_{vort,\vec{v}} = \frac{1}{\Delta}\begin{pmatrix} \frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} & -\frac{\partial^2}{\partial x\partial y} & -\frac{\partial^2}{\partial x\partial z} \\ -\frac{\partial^2}{\partial x\partial y} & \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2} & -\frac{\partial^2}{\partial y\partial z} \\ -\frac{\partial^2}{\partial x\partial z} & -\frac{\partial^2}{\partial y\partial z} & \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \end{pmatrix} \qquad (20)$$

Application of $P_{vort,\vec{v}}$ to the first three equations from the system (9) (they represent the momentum equation), results in the dynamic equation governing velocity of the vorticity mode:

$$\frac{\partial \vec{v}_{vort}}{\partial t} = P_{vort,\vec{v}}\begin{pmatrix} -(\vec{v}\cdot\vec{\nabla})v_x + \rho\frac{\partial p}{\partial x} \\ -(\vec{v}\cdot\vec{\nabla})v_y + \rho\frac{\partial p}{\partial y} \\ -(\vec{v}\cdot\vec{\nabla})v_z + \rho\frac{\partial p}{\partial z} \end{pmatrix}_a . \qquad (21)$$

The right-hand side of Eq. (21) includes, in general, terms belonging to both acoustic modes. By use of the first two eigenvectors from Eqs. (14), we express the acoustic pressure and components of the velocity in terms of excess density for each pair of acoustic modes, and readily rearrange Eq. (21) into the following equation:

$$\frac{\partial \vec{v}_{vort}}{\partial t} = P_{vort,\vec{v}}\left\{\frac{1}{2}\vec{\nabla}\left(-\sum_{n=1}^{2}\vec{v}_n\cdot\sum_{n=1}^{2}\vec{v}_n + \left(\sum_{n=1}^{2}\rho_n\right)^2\right) + D\sum_{n=1}^{2}\rho_n\vec{\nabla}\Delta\sum_{n=1}^{2}\rho_n\right\} = DP_{vort,\vec{v}}\left(\sum_{n=1}^{2}\rho_n\vec{\nabla}\Delta\sum_{n=1}^{2}\rho_n\right). \quad (22)$$

This yields the dynamic equation for the vorticity mode in the field of intense sound in two equivalent forms,

$$\frac{\partial \vec{\Omega}}{\partial t} = D\left(\vec{\nabla}\rho_a\right)\times\left(\vec{\nabla}\Delta\rho_a\right),$$
$$\frac{\partial \vec{v}_{vort}}{\partial t} = DP_{vort,\vec{v}}\left(\rho_a\vec{\nabla}\Delta\rho_a\right). \qquad (23)$$

where $\vec{\Omega}$ is the vorticity of the flow, $\vec{\Omega} = \vec{\nabla}\times\vec{v}_{vort}$, and $\rho_a = \rho_1 + \rho_2$. Application of the last row of P_1 on Eqs. (9), if only nonlinear terms belonging to the first mode are kept in the nonlinear part to leading-order, results in the equation governing an excess density of the first branch of sound,

$$\frac{\partial \rho_1}{\partial t} - \sqrt{\Delta}\left(1+\frac{D}{2}\Delta\right)\rho_1 + \left(\frac{2\varepsilon-1}{2}\rho_1(\vec{\nabla}\cdot\vec{v}_1) + \frac{1}{2}\vec{v}_1\cdot(\vec{\nabla}\rho_1)\right) = 0, \qquad (24)$$

where ε denotes the parameter of nonlinearity,

$$\varepsilon = \left(\frac{(1-\alpha_0)c_{mix}^2(\gamma_l+1)}{2c_l^2} + \frac{c_{mix}^4\alpha_0(1-\alpha_0)^2\rho_{l0}^2(\gamma_g+1)}{2(\gamma_g p_{g0})^2} \right). \quad (25)$$

and the square root of the Laplacian ($\sqrt{\Delta}f(\vec{r},t)$) means an integral operator which corresponds to the Fourier transform of $i\sqrt{k_x^2+k_y^2+k_z^2}\tilde{f}(\vec{k},t)$. The parameter of nonlinearity, given by Eq. (25), coincides with that evaluated in Ref. [2]. In the study [2], the expression obtained for incompressible liquid is completed by the terms following from the nonlinearity in equations different from the pressure-density relation for the mixture. In contrast, Eq. (25) is an immediate result of considering of the total system of conservation equations describing compressible liquid including bubbles.

5. Vortex flow induced in the field of a Gaussian beam

Until this point, no restriction concerning the type of flow geometry was done. Let y designate the nominal axis of the sound beam pointing in the propagation direction, and let x, z be the coordinates perpendicular to that axis. We will assume that all acoustic perturbations vary much faster in the direction of the OY axis than in the direction perpendicular to this axis: $k_y^2 \gg k_x^2+k_z^2$. This allows expansion of the relations for sound perturbations in a series of powers of the small parameter $\mu=(k_x^2+k_z^2)/k_y^2$. Using this assumption, we can rewrite the equation (24) in the form:

$$\frac{\partial\rho_1}{\partial t} + \frac{\partial\rho_1}{\partial y} + \frac{\Delta_\perp}{2}\int \rho_1 dy + \frac{D}{2}\frac{\partial^3\rho_1}{\partial y^3} + \varepsilon\rho_1\frac{\partial\rho_1}{\partial y} = 0, \quad (26)$$

where $\Delta_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$. This equation is in fact one of the forms of the KZ (the inviscid limit of KZK) equation including dispersion. The dynamic equation for the vorticity mode in the field of intense sound propagating in the positive direction of the OY axis takes the leading-order form:

$$\frac{\partial\vec{\Omega}}{\partial t} = D\left(\vec{\nabla}\rho_1\right) \times \left(\vec{\nabla}\frac{\partial^2\rho_1}{\partial y^2}\right). \quad (27)$$

Eq. (27) reveals that both nonlinearity and dispersion are necessary conditions for acoustic streaming in a bubbly liquid. The following assumptions will be made regarding the source: it is defined at the plane $y=0$ and is positioned symmetrically with respect to the y-axis. The system (Eqs. (26), (27)) may be readily rearranged in a cylindrically symmetric geometry with $r=\sqrt{\mu}\sqrt{x^2+z^2}$:

$$\frac{\partial\rho_1}{\partial t} + \frac{\partial\rho_1}{\partial y} + \frac{\mu}{2r}\frac{\partial}{\partial r}\left(r\int\frac{\partial\rho_1}{\partial r}dy\right) + \frac{D}{2}\frac{\partial^3\rho_1}{\partial y^3} + \varepsilon\rho_1\frac{\partial\rho_1}{\partial y} = 0, \quad (28)$$

$$\frac{\partial\vec{\Omega}}{\partial t} = \sqrt{\mu}D\begin{pmatrix}\cos\phi\\ 0\\ -\sin\phi\end{pmatrix}F(\rho_1), \quad (29)$$

where

$$F(\rho_1) = -\frac{\partial\rho_1}{\partial r}\frac{\partial^3\rho_1}{\partial y^3} + \frac{\partial\rho_1}{\partial y}\frac{\partial^3\rho_1}{\partial r\partial y^2} \quad (30)$$

and $\phi = \arccos\left(\frac{z}{\sqrt{x^2+z^2}}\right)$. The main difficulty in solving for $\vec{\Omega}$ is establishment of function ρ_1, which satisfies Eq. (28). As far as the authors know, there are no analytical solutions of Eq. (28). We rearranged Eq. (28) into the following equivalent form:

$$\frac{\partial\rho_1}{\partial t} + \frac{\partial\rho_1}{\partial y} + \frac{\sqrt{\mu}}{2r}\frac{\partial}{\partial r}(rg) + \frac{D}{2}\frac{\partial^3\rho_1}{\partial y^3} + \varepsilon\rho_1\frac{\partial\rho_1}{\partial y} = 0, \quad (31)$$

$$\frac{\partial g}{\partial y} = \sqrt{\mu}\frac{\partial\rho_1}{\partial r}, \quad (32)$$

where

$$g = -\sqrt{\mu}\int_y^\infty \frac{\partial\rho_1}{\partial r}dy. \quad (33)$$

and solved it numerically. The equation describing the radial-component of $\vec{\Omega}$ takes the form:

$$\frac{\partial\Omega_r}{\partial t} = \sqrt{\mu}DF(\rho_1), \quad (34)$$

where ρ_1 is a solution of Eqs. (31), (32). The y-component of $\vec{\Omega}$ is independent on time. Solutions of Eq. (34) with ρ_1 described by Eqs. (31), (32) have been calculated numerically for the following initial and boundary conditions:

$$\Omega_y(y,r,t=0) = \Omega_r(y,r,t=0) = 0, \quad \Omega_r(y=0,r,t) = 0, \quad (35)$$

$$\rho_1(y,r,t=0) = 0, \quad \rho_1(y=0,r,t) = M\sin(t)e^{-r^2-at^2}, \quad \frac{\partial\rho_1}{\partial r}(y,r=0,t) = 0. \quad (36)$$

In accordance with the system (29), the radial-component of the vorticity is proportional to M^2, D and $\sqrt{\mu}$. Its y-component equals zero for the chosen initial conditions. It can be concluded from numerical simulations that the vorticity achieves a maximum at some distance from a transducer and the axis of a beam. Vorticity decreases far from a transducer, which is due to decrease in

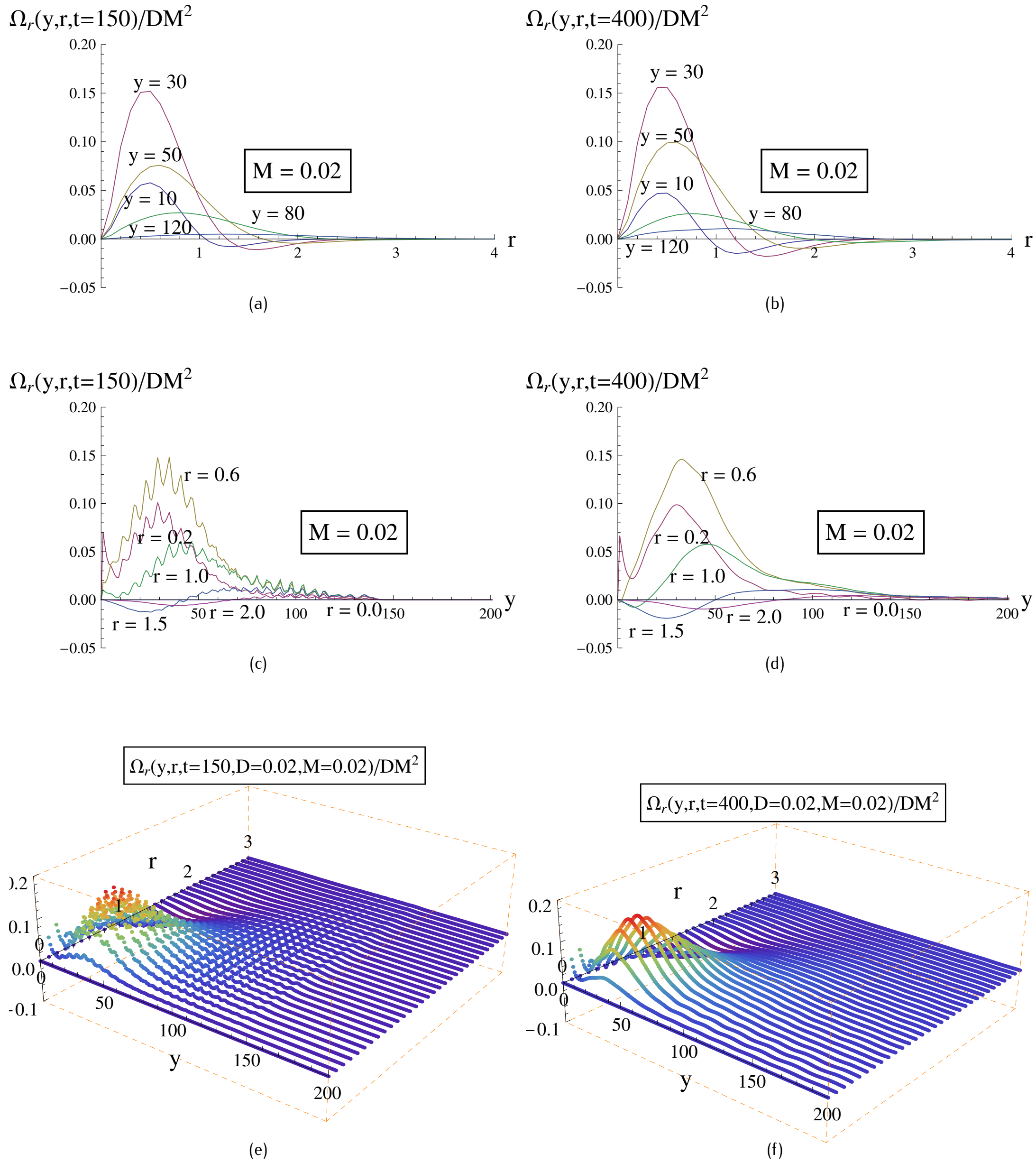

Figure 1. Transverse component of vorticity $\Omega_r(y, r, t)$. $a = 2.5 \cdot 10^{-5}$ (impulse), $M = 0.02$, $D = 0.02$, $\sqrt{\mu} = 0.1$, $\varepsilon = 1.2$, $t = 150$ or $t = 400$.

the magnitude of the acoustic density at large distances. For M less than D, vorticity constantly decreases with increased distance from the sound source, starting from the distance where it achieves maximum. Thus, we can conclude that the parameter D is responsible for the stabilization of vorticity, in the sense that it suppresses oscillations of vorticity. In this stabilization, D behaves like attenuation in nonlinear effects caused by sound in Newtonian fluids. The larger the ratio of M and D is, the larger vorticity oscillations are. For values of M larger than D, domains appear where vorticity changes to the opposite sign. This happens, among other locations, close to the source of sound for $D = 0.02$ and M between 0.01 and 0.02. The last parameter that affects the formation of vorticity is the diffraction parameter μ. The distance from a transducer where vorticity is maximal increases somewhat with decreased μ. In turn, maxima increase with enlargement of μ. Oscillations of the vorticity generated by pulses

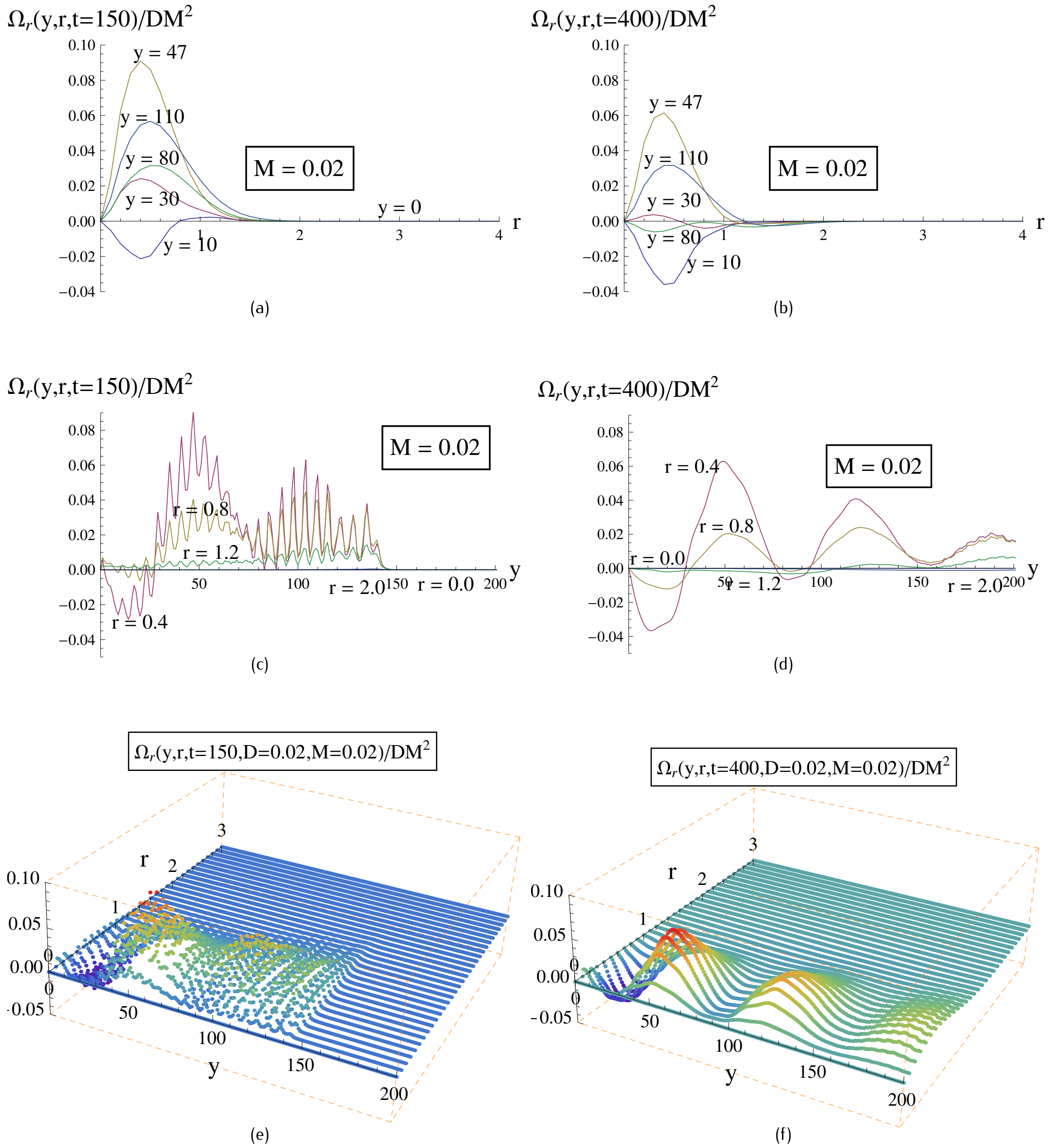

Figure 2. Transverse component of vorticity $\Omega_r(y, r, t)$. $a = 2.5 \cdot 10^{-5}$ (impulse), $M = 0.02$, $D = 0.02$, $\sqrt{\mu} = 0.05$, $\varepsilon = 1.2$, $t = 150$ or $t = 400$.

are much smaller than those caused by harmonic acoustic waves at a transducer. For periodic acoustic waves, vorticity increases in time, while for aperiodic case its tends to some limit level. Figures 1, 2 show vorticity generated by impulses for values of diffraction parameter $\sqrt{\mu}$ equal 0.1 and 0.05, respectively (a equals 0.000025). Figures 3, 4 show vorticity generated by periodic sound for different ratios of the dispersion parameter and the Mach number. Solutions of Eqs. (31), (32) have been obtained by means of a numerical scheme which uses an implicit Runge-Kutta method, and the second-order central difference scheme for the spatial derivatives, which for zero diffraction parameter μ is analogous to a scheme for solution of the Korteweg-de-Vries equation developed in [22]. The differences between two consecutive space points of 0.1 was chosen both for the dimensionless longitudinal variable and the dimensionless transversal variable in all evaluations. That corresponds to the following dimensional spatial steps: for the longitudinal variable 10^{-4} m and for the transverse variable 10^{-3} m (for $\sqrt{\mu} = 0.1$) and $2 \cdot 10^{-3}$ m

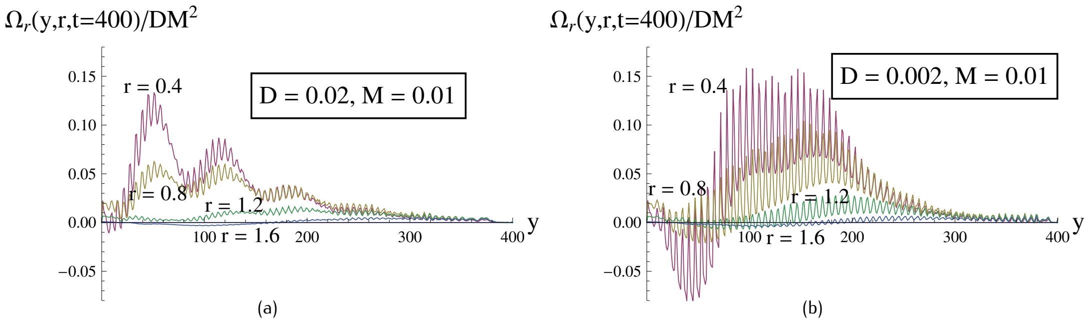

Figure 3. Values of $\Omega_r(y,r,t)$, $a = 0$ (the case of harmonic sound at a transducer), plots for $D = 0.02$ and $D = 0.002$, $M = 0.01$, $\sqrt{\mu} = 0.05$, $\varepsilon = 1.2$, $t = 400$.

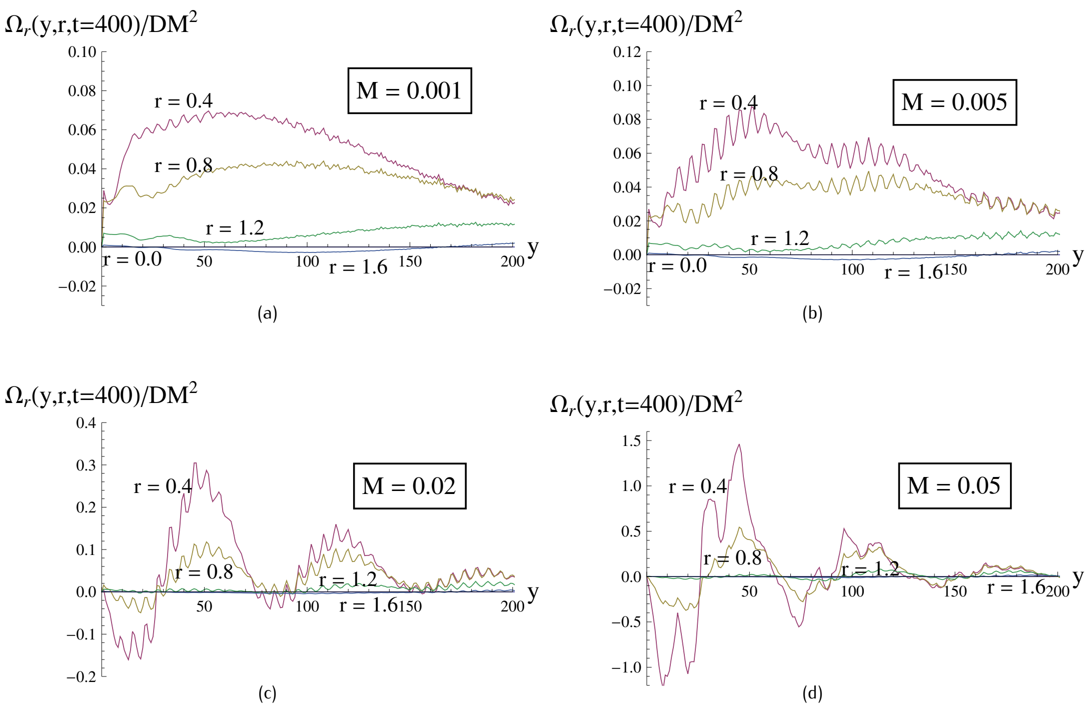

Figure 4. Transversal component of vorticity, $\Omega_r(y,r,t)$. $a = 0$ (the case of harmonic sound at a transducer), plots for different M, $D = 0.02$, $\sqrt{\mu} = 0.05$, $\varepsilon = 1.2$ and $t = 400$.

(for $\sqrt{\mu} = 0.05$). The dimensionless time step equals 0.01 in all calculations. It corresponds to a dimensional time step of about 10^{-7} s.

6. Conclusions

In this study, we consider acoustic streaming in a liquid including gaseous bubbles. The main results of this study are equations (23), which describe nonlinear generation of the vortex mode in the field of intense sound. They apply to periodic and aperiodic sound, they are instantaneous, and their derivation does not require averaging over the sound period. An inviscid liquid is considered, so that nonlinear generation of the vortex mode is caused exclusively by dispersion. Dispersion of a bubbly fluid originates mostly from the difference in compressibility of the liquid and gaseous phases. In some sense, it behaves like a Newtonian attenuation: it is a necessary (along with nonlinearity) condition for generation of the vortex motion in the field of sound. Dispersion also stabilizes the vortex velocity. This study accounts for liquid compressibility. Neither heat transfer between bubbles and surrounding liquid, nor non-uniformity of pressure and temperature inside a bubble, nor vaporization in the case of bubbles including vapor, were considered. Without accounting for nonlinearity, the dynamic equation for the vorticity mode in an inviscid fluid describes a stationary velocity which is independent on the sound field. In a viscous liquid, it takes the form of a diffusion equation, but it still develops independently from the other modes. Accounting for nonlinearity makes this mode enhance in the field of intense sound. We do not consider convective nonlinearity in the left-hand side of the dynamic equation for the vorticity, Eq. (27), which is described by the term $(\vec{v}\cdot\vec{\nabla})\vec{\Omega}$. It is well established that accounting for this term prevents enlargement of vorticity with time [21].

Numerical results show impact of non-linearity, dispersion, and diffraction of an acoustic beam on the vortex motion. The results concern a free half-space with a circular transducer which is situated at the boundary. Inclusion of boundaries may essentially change conclusions and influences on the very definition of modes, which, as a rule, refers to a discrete set of wavenumbers dependent on the geometry of a volume and boundary conditions.

Analysis of the results has revealed the impact of the ratio of the Mach number and dispersion parameter on the vorticity mode. If the Mach number of a flow M is larger than the dispersion D, the streamlines may change direction in some domains. This is shown in Figures 3, 4. The examples considered concern zero initial conditions. After the transducer is turned on, sound begins propagate generating vorticity. In the initial phase (up to dimensionless t less than 40), generation of vorticity is similar for a set of different dispersion parameters and Mach numbers. With longer time, differences become more evident. The vorticity achieves a maximum close to dimensionless $y = 50$. If the ratio of M and D is larger than unity, the maximum value is achieved for smaller y. More intense sound yields larger fluctuations of vorticity. This is especially evident for smaller distances from the transducer (for y less then 50).

Another remarkable conclusion is connected with diffraction. In accordance with Eq. (29), the vortex velocity is proportional to the square root of μ (it might be not so evident from Eqs. (23)). That is confirmed by the numerical evaluations (Figs. 1, 2). For smaller values of the diffraction parameter, the vorticity achieves its maximum at a somewhat larger distance from a transducer. The dependence of acoustic streaming on diffraction may be useful for configuring the vortex flow. A similar behavior of vorticity is observed for both periodic sound and for pulses. The difference between these two cases is that, for the pulses, the value of vorticity is limited in time, and for the periodic case, the vorticity constantly increases. Accounting for purely nonlinear attenuation (in an inviscid liquid, a shock wave forms sooner or later) or viscosity would prevent this unlimited growth in the magnitude of the vorticity.

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