

# MODULE STRUCTURE IN CONLEY THEORY WITH SOME APPLICATIONS

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**Abstract.** A multiplicative structure in the cohomological version of Conley index is described following a joint paper by the author with K. Gęba and W. Uss. In the case of equivariant flows we apply a normalization procedure known from equivariant degree theory and we propose a new continuation invariant. The theory is applied then to obtain a mountain pass type theorem. Another illustrative application is a result on multiple bifurcations for some elliptic PDE.

**1. Introduction.** In this paper, we consider a module structure of cohomology Conley index of local flows determined by equivariant smooth vector fields in  $\mathbb{R}^n$ . By using this structure, in [7] a continuation invariant called *a relative cup-length* has been described. Motivated also by [3], we present here a normalization technique known from equivariant degree theory. This allows to define a version of the relative cup-length as an element of the Euler ring of a group  $G$  (comp. [6]). Let us observe that the module structure can be used also in the infinite-dimensional version of the Conley index (see [11]), since it is preserved after suspension. Some application to PDE is also briefly presented in the last section.

The paper is organized as follows. Section 2 contains some standard notation for compact Lie group actions. In Section 3 we recall necessary notions from (equivariant) Conley index theory and in Section 4 a normalization process is described. In Section 5 we describe an abstract notion of a relative cup-length of an index pair with respect to an isolating domain. Some simple applications are given in the last two sections. A mountain pass type theorem is proved in Section 6. Elliptic PDE with Dirichlet boundary condition and with  $\mathbb{Z}_2$ -symmetry is considered in Section 7. We prove a multiple bifurcation theorem.

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2010 *Mathematics Subject Classification*: Primary 37B30; Secondary 35J25, 37J20, 37J45.

*Key words and phrases*: Conley index, cup-length, equivariant flow, elliptic PDE.

The paper is in final form and no version of it will be published elsewhere.

**2. Preliminaries on group actions.** We start from some notation about group actions (see [6] for more details). Let  $G$  be a group. If  $H \subset G$  is a subgroup, we denote by  $G/H$  the set of left cosets  $gH$ . Two subgroups  $H$  and  $K$  of  $G$  are *conjugate* if there exists  $g \in G$  such that  $K = g^{-1}Hg$ . The conjugacy class of  $H$  is denoted by  $(H)$ . There is a natural partial order in the set  $\Phi(G)$  of conjugacy classes:

$$(K) \leq (H) \text{ if there exist } \bar{K} \in (K) \text{ and } \bar{H} \in (H) \text{ such that } \bar{K} \subset \bar{H}.$$

Throughout the whole paper we consider only compact Lie groups and their closed subgroups. Given a subgroup  $H \subset G$  let  $N(H)$  be the normalizer of  $H$ . The Weyl group of  $H$  is the quotient  $W(H) := N(H)/H$ . Let us define the set

$$\Phi_0(G) := \{(H) \in \Phi(G) : \dim W(H) = 0\}.$$

A  $G$ -set is a pair  $(X, \xi)$ , where  $X$  is a set and  $\xi : G \times X \rightarrow X$  is an action of  $G$  on  $X$ , i.e., a map such that:

- (i)  $\xi(g_1, \xi(g_2, x)) = \xi(g_1 g_2, x)$  for  $g_1, g_2 \in G$  and  $x \in X$ ,
- (ii)  $\xi(e, x) = x$  for  $x \in X$ , where  $e \in G$  is the group unit.

In the sequel we write  $gx$  instead of  $\xi(g, x)$ . For every subgroup  $H \subset G$  the set  $G/H$  is a  $G$ -set by the action  $g(\tilde{g}H) = g\tilde{g}H$ . If  $\xi$  is continuous, we call  $(X, \xi)$  a  $G$ -space. We say that a real (resp. complex) Banach space  $\mathbb{E}$  is a *real* (resp. *complex*) *Banach representation of  $G$*  if  $\mathbb{E}$  is  $G$ -space and, for each  $g \in G$ , the map  $\xi_{\mathbb{E}}(g, \cdot) : \mathbb{E} \ni x \mapsto gx$  is linear and bounded.

For  $x \in X$ , the subgroup  $G_x = \{g \in G : gx = x\}$  is called the *isotropy group* of  $X$  of the point  $x$ . The conjugacy class of an isotropy group is called an *isotropy type*. Denote by  $\text{Iso}(X)$  the set of all isotropy types in  $X$ . The set  $Gx = \{gx : g \in G\}$  is called an *orbit* through  $x$ .

For a given subgroup  $H \subset G$  we specify several subspaces of a given  $G$ -space  $X$ :  $X_H = \{x \in X : H = G_x\}$ ,  $X_{(H)} = \{x \in X : (H) = (G_x)\}$ ,  $X^H = \{x \in X : H \subset G_x\}$ ,  $X^{(H)} = \{x \in X : (H) \leq (G_x)\}$ .

Now we define the *Burnside ring* of  $G$  as follows (cf. [1] for details and examples):

As a group  $A(G)$  is a free abelian group generated by  $(H) \in \Phi_0(G)$ , i.e., an element  $a \in A(G)$  is a finite sum  $a = n_{H_1}(H_1) + \dots + n_{H_m}(H_m)$  with  $n_{H_i} \in \mathbb{Z}$  and  $(H_i) \in \Phi_0(G)$ .

The operation of multiplication in  $A(G)$  is a bit more sophisticated. Let  $(H), (K) \in \Phi_0(G)$ . Consider the diagonal action of  $G$  on  $G/H \times G/K$ . Then for any  $(L) \in \Phi_0(G)$ , the spaces  $G/H^L$  and  $G/K^L$  consist of finitely many  $W(L)$ -orbits. Therefore the space  $(G/H \times G/K)_{(L)}/G$  is finite. Let  $n_L(H, K)$  denote the number of elements of this space. Define

$$(H) \cdot (K) := \sum_{(L) \in \Phi_0(G)} n_L(H, K)(L).$$

A free abelian group  $U(G) = \mathbb{Z}(\Phi(G))$  can also be equipped with a natural multiplicative structure and it is called then an *Euler ring* of  $G$  in [6].

### 3. Local flows and flow generators.

Let  $X$  be a space.

DEFINITION 3.1. A *flow* on  $X$  is a map  $\varphi : X \times \mathbb{R} \rightarrow X$  such that

- $\varphi(x, 0) = x$  for all  $x \in X$ ;
- $\varphi(\varphi(x, t), s) = \varphi(x, s + t)$  for all  $x \in X, t, s \in \mathbb{R}$ .

A *local flow* is defined on an open subset

$$\text{dom } \varphi = \{(x, t) : t \in (a_x, b_x), a_x < 0 < b_x\} \subset X \times \mathbb{R},$$

with the above properties whenever  $\varphi$  is defined.

Let  $U$  be an open subset of  $\mathbb{R}^n$  and  $F : U \rightarrow \mathbb{R}^n$  a vector field which is at least locally Lipschitz (we consider here smooth vector fields for simplicity). Then  $F$  *generates* a local flow  $\eta$  on  $U$  by the rule that  $\eta(x, t)$  is the value of a unique solution to the Cauchy problem

$$\begin{cases} \dot{\eta}(x, \cdot) = F(\eta(x, \cdot)), \\ \eta(x, 0) = x \end{cases}$$

at the time  $t$ .

Throughout the whole paper we denote by  $V$  a finite-dimensional orthogonal representation of a compact Lie group  $G$ . Let  $U \subset V$  be an open  $G$ -invariant subset. A local vector field  $F : U \rightarrow V$  is  *$G$ -equivariant* if  $F(gx) = gF(x)$  for all  $g \in G, x \in U$ . It is easy to observe that the local flow  $\eta_F$  generated by an equivariant vector field is also equivariant, i.e.,  $g(\eta_F(x, t)) = \eta_F(gx, t)$  for each  $g \in G$ , whenever defined.

Given a local flow  $\eta$  on  $U$ , we define a *maximal invariant* part of  $U$ :

$$\text{Inv}_\eta(U) := \{x \in U : \eta(x, t) \in U \text{ for all } (x, t) \in \text{dom } \eta\}.$$

One easily observes that for a  $G$ -equivariant local flow  $\eta$  the above set is a  $G$ -invariant subset of  $U$ .

DEFINITION 3.2. An open set  $W \subseteq U$  is an *isolating domain* for a local flow  $\eta$ , if  $\text{Inv}_\eta(W) \subset W$  is a compact subset.

Observe that if  $W$  is an isolating domain then there exists an open relatively compact neighbourhood  $W'$  of  $\text{Inv}_\eta(W)$  such that  $\text{Inv}_\eta(W) = \text{Inv}_\eta(\overline{W}') \subset W' \subset \overline{W}' \subset W$ . The compact set  $\overline{W}'$  is usually called an *isolating neighbourhood* in Conley index theory (cf. [5]). A compact set  $S$  is an *isolated invariant set*, if  $S = \text{Inv}_\eta(W) \subset W$  for some isolating domain  $W$ .

DEFINITION 3.3. Let  $S$  be an isolated invariant set. A pair  $(X, A)$  of compact  $G$ -invariant sets is called a  *$G$ -index pair* for  $S$ , if

- $\text{int}(X \setminus A)$  is an isolating domain and  $S = \text{Inv}(X \setminus A) \subset \text{int}(X \setminus A)$ ;
- $A$  is positively invariant in  $X$ , i.e., for each  $x \in A$ : if  $\eta(x, [0, t]) \subset X$  for some  $t > 0$  then  $\eta(x, [0, t]) \subset A$ ;
- $A$  is an *exit set* from  $X$ : if  $x \in X$  and  $\eta(x, t) \notin X$  for some  $t > 0$ , then there exists  $s \in [0, t)$  such that  $\eta(x, s) \in A$ .

The following existence result is an easy consequence of the non-equivariant case (see [17], comp. [9], [10]).



**THEOREM 3.4.** *If  $U \subset V$  is a  $G$ -invariant isolating domain, then there exists a  $G$ -index pair for the compact isolated set  $S = \text{Inv}_\eta(U)$ .*

If  $(X, A)$  is a pair of compact  $G$ -spaces, then  $X/A$  denotes the pointed compact  $G$ -space obtained by identifying all points of  $A$  with the distinguished point. In case of empty set  $A$  we take  $X/A = X^+$ , where  $X^+$  denotes the pointed space with a separate base point added. Denote by  $[X/A]$  the  $G$ -equivariant homotopy class of the pointed space  $X/A$ . The following proposition is true.

**PROPOSITION 3.5.** *If  $(X_1, A_1), (X_2, A_2)$  are two  $G$ -index pairs in an isolating domain  $U$ , then  $[X_1/A_1] = [X_2/A_2]$ .*

The proof can be also carried from [17], Theorem 4.10, and it is enough to observe that all maps defined in [17] are  $G$ -equivariant if the local flow is  $G$ -equivariant.

**DEFINITION 3.6.** If  $U \subset V$  is a  $G$ -invariant isolating domain for a  $G$ -equivariant local flow  $\eta$ , then the  $G$ -equivariant Conley index is defined to be

$$\mathcal{CIG}(\eta, U) := [X/A],$$

where  $(X, A)$  is a  $G$ -index pair in  $U$ .

The independence of the index pair is assured by Proposition 3.5. In fact one can define the index for isolated invariant sets because of the following obvious localization property.

**PROPOSITION 3.7.** *Let  $U \subset V$  be an isolating domain and let  $U_1 \subset U$  be open and  $G$ -invariant with  $\text{Inv}_\eta(U) \subset U_1$ . Then  $U_1$  is an isolating domain and  $\mathcal{CIG}(\eta, U_1) = \mathcal{CIG}(\eta, U)$ .*

**DEFINITION 3.8.** A local equivariant flow generator is a pair  $(F, U)$ , where  $U \subset V$  is open and  $G$ -invariant subset of  $V$ , and  $F : U \rightarrow V$  is a  $G$ -equivariant vector field generating a local flow  $\eta$  on  $U$  with  $\text{Inv}_\eta(U) \subset U$  compact.

We consider here only local flows generated by vector fields. Motivated by [3] we introduce a convenient relation of *otopy* which plays a role of continuation. If  $V$  is a representation of  $G$  then  $V \times [0, 1]$  is a  $G$ -space with the action  $g(x, t) = (gx, t)$ . If  $\Omega$  is an open ( $G$ -invariant) subset of  $V \times [0, 1]$ , then for each  $\tau \in [0, 1]$  we put  $\Omega_\tau := \{x \in V : (x, \tau) \in \Omega\}$ . A map (family of vector fields)  $h : \Omega \rightarrow V$  generates a family of local flows  $\eta_\tau$ , i.e.,  $h_\tau = h|_{\Omega_\tau} : \Omega_\tau \rightarrow V$  generates  $\eta_\tau$ .

**DEFINITION 3.9.** An (equivariant) *otopy* is a pair  $(h, \Omega)$ , such that  $\Omega \subset V \times [0, 1]$  is open and  $G$ -invariant, and the equivariant map  $h : \Omega \rightarrow V$  generates a family of local flows  $\eta_\tau$  such that the invariant part

$$S := \bigcup_{\tau \in [0, 1]} \text{Inv}_{\eta_\tau}(\Omega_\tau) \subset V$$

is compact.

We admit that  $\Omega_\tau$  is empty for some  $\tau \in [0, 1]$ . Note that  $h_\tau$  are local flow generators. We say then also that  $h_1, h_0$  are *otopic local flow generators*. This defines an equivalence relation among local (equivariant) flow generators.



PROPOSITION 3.10. *Let  $(h, \Omega)$  be an otopy generating a family of equivariant flow generators of  $\{\eta_\tau\}_{\tau \in [0,1]}$ . Then  $CIG(\eta_0, \Omega_0) = CIG(\eta_1, \Omega_1)$ .*

*Proof.* This is an immediate consequence of Proposition 3.7 and the continuation property of the Conley index (see e.g. [10], Proposition 5.5). ■

Obviously all the properties are valid for non-equivariant Conley index, when a trivial action of  $G$  is considered not necessarily on subsets of  $\mathbb{R}^n$ , but on locally compact metric spaces [5].

**4. Normalization.** For an invariant subset  $X \subset V$  of a finite-dimensional representation of  $G$  we define

$$\text{Iso}(X) := \{(H) \in \Phi(G) : X_{(H)} \neq \emptyset\},$$

where  $X_{(H)} := \{x \in X : (G_x) = (H)\}$ . For every closed subgroup  $H \subset G$  the set  $M = V_{(H)}$  is a submanifold of  $V$  (in fact it is a linear subspace). Then

$$\nu(M) := \{(x, v) \in M \times V : x \in M, v \in N_x = (T_x M)^\perp\}$$

denotes a normal bundle over  $M$ . We have the map  $\mathcal{N} : \nu(M) \rightarrow V$ ,  $\mathcal{N}(x, v) := x + v$ .

We shall use the following version of the equivariant tubular neighbourhood theorem (see [3], Theorem 3.1, or [14], Theorem 4.8 for a proof).

THEOREM 4.1. *There exists an open  $G$ -invariant subset  $T$  (tubular neighbourhood) containing  $M$  and such that the map  $\mathcal{N}$  restricted to  $\mathcal{N}^{-1}(T)$  is a homeomorphism.*

DEFINITION 4.2. Let  $(H) \in \text{Iso}(V)$ . A local vector field  $(f, U)$  is  $H$ -normal if there is an open  $G$ -invariant subset  $U_0 \subset V_{(H)}$  and  $\varepsilon > 0$  such that

- $T = \{x + v : x \in U_0, v \in N_x, |v| < \varepsilon\} \subset U$ , where  $T$  is a tubular neighbourhood of  $V_{(H)}$ ;
- $f(x + v) = f(x) + v$  for  $x \in U_0, v \in N_x, |v| < \varepsilon$ .

LEMMA 4.3. *Let  $(f, U)$  be a local equivariant flow generator and let  $(H)$  be a maximal orbit type in  $\text{Iso}(U)$ . Then there exist two equivariant local flow generators  $(f_1, U_1), (f_2, U_2)$  such that*

- $U_1 \subset U, U_2 \subset U \setminus U_{(H)}, U_1 \cap U_2 = \emptyset$ ,
- $(f_1, U_1)$  is  $(H)$ -normal and  $f_1(x) = f(x)$  for all  $x \in (U_1)_{(H)}$ ,
- $(f, U)$  is otopic to the disjoint union  $(f_1, U_1)$  and  $(f_2, U_2)$ .

*Proof.* First we find an open bounded set  $U_0 \subset U_{(H)}$  such that  $\overline{U_0} \subset U_{(H)}$  and

$$\text{Inv}_\eta(U)_{(H)} = \text{Inv}_\eta(U) \cap U_{(H)} \subset U_0.$$

Given  $\rho > 0$ , we define two sets

$$\begin{aligned} X(\rho) &:= \{u + v \in V : u \in \overline{U_0}, v \in N_u, |v| \leq \rho\}, \\ Y(\rho) &:= \{u + v \in V : u \in \partial_{(H)} U_0, v \in N_u, |v| < \rho\}. \end{aligned}$$

Let  $T$  be a tubular neighbourhood of  $U_{(H)}$  in  $V$ . Then there exists  $\varepsilon > 0$  such that  $X(4\varepsilon) \subset T$  and  $\text{Inv}(U, \eta) \cap Y(4\varepsilon) = \emptyset$ .

Next we find a smooth function  $\alpha : [0, 4\varepsilon] \rightarrow [0, 1]$  such that  $\alpha(t) = 0$  for  $t \in [0, 2\varepsilon]$ ,  $\alpha(t) = 1$  for  $t \in [3\varepsilon, 4\varepsilon]$  and  $\alpha'(t) > 0$  for  $t \in (2\varepsilon, 3\varepsilon)$ . Define  $r : X(4\varepsilon) \rightarrow X(4\varepsilon)$  by the formula  $r(x) := u + \alpha(|v|)v$ , where  $x = u + v$ ,  $u \in \bar{U}_0$ ,  $v \in N_u$ .

Let  $\widehat{U} := U \setminus Y(4\varepsilon)$ , and  $\widehat{f} : \widehat{U} \rightarrow V$  be defined by

$$\widehat{f}(x) := \begin{cases} f(r(x)) & \text{for } x \in X(4\varepsilon) \setminus Y(4\varepsilon), \\ f(x) & \text{for } x \in \widehat{U} \setminus X(4\varepsilon). \end{cases}$$

Clearly  $(f, \widehat{U})$  and  $(\widehat{f}, \widehat{U})$  are otopic local flow generators. Observe that  $f|_{\widehat{U} \setminus \text{int } X(4\varepsilon)} = \widehat{f}|_{\widehat{U} \setminus \text{int } X(4\varepsilon)}$ .

Take another smooth function  $\theta : [0, 3\varepsilon] \rightarrow [0, 1]$  such that  $\theta(t) = 1$  for  $t \leq \varepsilon$  and  $\theta(t) = 0$  for all  $t \in [2\varepsilon, 3\varepsilon]$ . Define a vector field  $g : \widehat{U} \rightarrow V$  by

$$g(x) := \begin{cases} \theta(|v|)v & \text{for } x = u + v \in X(3\varepsilon) \setminus Y(3\varepsilon), \\ 0 & \text{for } x \in \widehat{U} \setminus X(3\varepsilon). \end{cases}$$

Consider a homotopy  $h : \widehat{U} \times [0, 1] \rightarrow V$  given by  $h(x, t) := \widehat{f}(x) + tg(x)$ . Observe that  $g|_{V(H)} \equiv 0$  and  $g|_{\widehat{U} \setminus X(2\varepsilon)} \equiv 0$ . Therefore  $h$  defines an otopy relation between  $(\widehat{f}, \widehat{U})$  and  $(\widehat{f} + g, \widehat{U})$ .

Now let

$$U_1 := \{x = u + v : u \in U_0, v \in N_u, |v| < \varepsilon\}, \quad U_2 := \widehat{U} \setminus [X(\varepsilon) \cup U(H)].$$

Define  $f_i$  to be a restriction of  $\widehat{f} + g$  to  $U_i$ ,  $i = 1, 2$ . Since  $\text{Inv}(\widehat{U}, \eta_{\widehat{f}+g})$  is a compact subset of  $U_1 \cup U_2$ , the generators  $(\widehat{f} + g, \widehat{U})$  and  $(f_1 \sqcup f_2, U_1 \sqcup U_2)$  are otopic. The proof is complete. ■

A similar procedure as in Lemma 4.3 can be applied to otopies and we obtain the following:

LEMMA 4.4. *Let  $(h, \Omega)$  be an otopy and  $(H)$  a maximal orbit type in  $\Omega$ . Then there exist two otopies  $(k, \widehat{\Omega}), (l, \widetilde{\Omega})$  such that*

- $\widehat{\Omega} \subset \Omega, \widetilde{\Omega} \subset \Omega \setminus \Omega(H), \widehat{\Omega} \cap \widetilde{\Omega} = \emptyset;$
- $(k_t, \widehat{\Omega})$  is  $(H)$ -normal for all  $t \in [0, 1];$
- $k(x, t) = h(x, t)$  for all  $(x, t) \in \widehat{\Omega}(H);$
- $(h, \Omega)_i$  is otopic to the disjoint union  $(k, \widehat{\Omega})_i \sqcup (l, \widetilde{\Omega})_i$  for  $i = 0, 1.$

Thus Lemma 4.4 gives the uniqueness of the decomposition in Lemma 4.3 up to an otopy, i.e., it is invariant under equivariant continuation.

Now let  $\text{Iso}(V) = \{(H_1), (H_2), \dots, (H_k)\}$ . Let the order be such that  $(H_i) < (H_j)$  implies  $i < j$ .

THEOREM 4.5. *Let  $(f, U)$  be an equivariant local flow generator. Then there exists a collection of local flow generators  $(f_i, U_i), i = 1, 2, \dots, k$ , such that  $f_i$  is  $(H_i)$ -normal for each  $i = 1, 2, \dots, k$  and  $(f, U)$  is otopic to the disjoint sum  $(f_1, U_1) \sqcup (f_2, U_2) \sqcup \dots \sqcup (f_k, U_k).$*



*Proof.* The proof is by induction. We start from  $(H_k)$ . Applying Lemma 4.3 we obtain that  $(f, U)$  is otopic to  $(f_1, U_1) \sqcup (f_2, U_2)$ , where  $U_2$  is disjoint from  $V_{(H_k)}$ . Thus  $(H_{k-1})$  is maximal in  $\text{Iso}(U_2)$  and thus we can use Lemma 4.3 again. After  $k$  steps we obtain the desired collection. ■

We call a collection obtained in Theorem 4.5 a *normal collection of local equivariant flow generators*. We have just proved the existence of a normal collection in the otopy class of any equivariant local flow generator. The uniqueness up to homotopy follows from the following:

**THEOREM 4.6.** *Assume we are given two normal collections  $\{(f_i^\alpha, U_i^\alpha)\}$  of local equivariant flow generators  $(\alpha = 1, 2, i = 1, 2, \dots, k)$ . Then there exists a normal collection of otopies  $\{(h_i, U_i)\}$  such that, for every  $i = 1, 2, \dots, k$ ,  $(h_i, U_i)$  is an otopy between  $(f_i, U_i^1)$  and  $(f_i, U_i^2)$ .*

*Proof.* An induction argument is the same as in the proof of Theorem 4.5. We only have to apply Lemma 4.4 instead of Lemma 4.3. ■

**COROLLARY 4.7.** *Let  $(f, U)$  be an equivariant local flow generator. Then its equivariant Conley index is homotopy equivalent to a  $G$ -CW complex which can be described as a union:*

$$CIG(\eta, U) = CIG(\eta, U_1) \vee CIG(\eta, U_2) \vee \dots \vee CIG(\eta, U_k)$$

where  $U_i$  are domains of a normal collection of local equivariant flow generators which is otopic to  $(f, U)$ .

**5. Relative cup-length.** Throughout this section we assume that  $A \subset X \subset Y$  are compact metric spaces and denote by  $H^*$  the Alexander–Spanier cohomology with the coefficients in a fixed abelian group  $G$ .

The cup product (see e.g. [20], Section 5.6)

$$\smile : H^k(X) \times H^l(X, A) \rightarrow H^{k+l}(X, A),$$

endows  $H^*(X, A)$  with a structure of an  $H^*(X)$ -module. If  $k : X \rightarrow Y$  denotes the inclusion map, then the formula

$$\beta \cdot \alpha := k^*(\beta) \smile \alpha$$

defines on  $H^*(X, A)$  a structure of an  $H^*(Y)$ -module. The following remark is a simple consequence of the naturality property of the cup product (see e.g. [12], Proposition 3.10).

**REMARK 5.1.** If  $B \subset A$  is compact, then

$$H^*(X, A) \rightarrow H^*(X, B) \rightarrow H^*(A, B)$$

is an exact sequence of  $H^*(Y)$ -modules, where the maps are induced by inclusions.

**DEFINITION 5.2.** Let  $\beta \in H^p(Y)$ ,  $p > 0$ ,  $\beta \neq 0$ , and  $A \subset X \subset Y$  be CW-complexes. The *relative cup-length* of  $\beta$  with respect to  $(X, A)$  is the number  $\chi(\beta; X, A) \in \mathbb{N}$  defined as follows:

- $\chi(\beta; X, A) = 0$  if  $H^*(X, A) = 0$ ;
- $\chi(\beta; X, A) = 1$  if  $H^*(X, A) \neq 0$  and  $\beta \cdot \alpha = 0$  for every  $\alpha \in H^*(X, A)$ ;
- $\chi(\beta; X, A) = k \geq 2$  if there exists  $\alpha_0 \in H^*(X, A)$  such that  $\beta^{k-1} \cdot \alpha_0 \neq 0$  and  $\beta^k \cdot \alpha = 0$  for every  $\alpha \in H^*(X, A)$ .



DEFINITION 5.3. The *relative cup-length* of the  $H^*(Y)$ -module  $H^*(X, A)$  is the number given by

$$\Upsilon(X, A; Y) := \max\{\chi(\beta; X, A) : 0 \neq \beta \in H^k(Y), k > 0\}.$$

If  $H^k(Y)$  are trivial for all  $k > 0$ , but  $H^*(X, A)$  is non-zero, we set  $\Upsilon(X, A; Y) = 1$ ; and if  $H^l(X, A)$  are trivial for all  $l \geq 0$ , then  $\Upsilon(X, A; Y) := 0$ .

LEMMA 5.4. If  $B \subset A \subset X \subset Y$ , then

$$\Upsilon(X, B; Y) \leq \Upsilon(X, A; Y) + \Upsilon(A, B; Y).$$

*Proof.* Let  $k_1 := \Upsilon(X, A; Y)$ ,  $k_2 := \Upsilon(A, B; Y)$  and

$$0 \neq \alpha \in H^p(X, B), p \geq 0, 0 \neq \beta \in H^q(Y), q > 0.$$

Let also

$$i : (X, B) \rightarrow (X, A), \quad j : (A, B) \rightarrow (X, B)$$

be inclusions.

Since  $k_2 = \Upsilon(A, B; Y)$  we have  $j^*(\beta^{k_2} \cdot \alpha) = 0$ .

By Remark 5.1 there exists  $\gamma \in H^*(X, A)$  such that  $\beta^{k_2} \cdot \alpha = i^*(\gamma)$ . Therefore

$$\beta^{k_1+k_2} \cdot \alpha = i^*(\beta^{k_1} \cdot \gamma).$$

But  $\beta^{k_1} \cdot \gamma = 0$  by definition of  $k_1$ , and thus  $\beta^{k_1+k_2} \cdot \alpha = 0$ . This means that

$$\Upsilon(X, B; Y) \leq k_1 + k_2,$$

which ends the proof. ■

LEMMA 5.5. If  $A \subset X \subset Y_1 \subset Y_2$ , then

$$\Upsilon(X, A; Y_2) \leq \Upsilon(X, A; Y_1)$$

*Proof.* Let us denote inclusions by

$$s : X \hookrightarrow Y, \quad k : A \hookrightarrow X, \quad t : A \hookrightarrow Y.$$

If  $\beta \in H^q(Y_2)$ ,  $q > 0$  and  $\alpha \in H^*(X, A)$ , then  $\beta \cdot \alpha = t^*(\beta) \smile \alpha = k^*(s^*(\beta)) \smile \alpha$ . Therefore  $\chi(\beta; X, A) = \chi(s^*(\beta); X, A)$  for all  $\beta \in H^q(Y_2)$ ,  $q > 0$ . But  $t = k \circ s$ , thus the condition  $t^*(\beta) \smile \alpha \neq 0$  implies  $s^*(\beta) \smile \alpha \neq 0$ , and our inequality follows. ■

Recall that the *cross product* is defined by the formula

$$a \times b := p_1^*(a) \smile p_2^*(b)$$

where  $p_1, p_2$  denote projections  $(X, A) \times (Y, B)$  onto  $(X, A)$  and  $(Y, B)$ . For algebraic properties of the maps  $\times : H^k(X) \times H^l(Y) \rightarrow H^{k+l}(X \times Y)$  and  $\times : H^k(X, A) \times H^l(Y, B) \rightarrow H^{k+l}(X \times Y, X \times B \cup A \times Y)$  we refer to [12].

Let  $\sigma$  be a generator of  $H^1(I, \partial I)$ , where  $I := [-1, 1]$ . The formula

$$\mathfrak{S}(a) := a \times \sigma$$

defines a mapping

$$\mathfrak{S} : H^k(X, A) \rightarrow H^{k+1}((X, A) \times (I, \partial I)) = H^{k+1}(X \times I, X \times \partial I \cup A \times I).$$

The following lemma holds (comp. [12], Theorem 3.21 for more general version).

LEMMA 5.6. If  $X \subset Y$  then  $\mathfrak{S}$  is an isomorphism of  $H^*(Y)$ -modules. More exactly

$$\mathfrak{S}(b \cdot a) = p^*(b) \cdot \mathfrak{S}(a),$$

where  $p$  denotes the projection  $Y \times I$  onto  $Y$ .





*Proof.* Let  $b \in H^*(Y)$ ,  $a \in H^*(X, A)$ . Consider the following projections:

$$p_1 : (X \times I, A \times I) \rightarrow (X, A), \quad p_2 : (X \times I, X \times \partial I) \rightarrow (I, \partial I), \quad \bar{p}_1 : X \times I \rightarrow X.$$

The following diagram is commutative ( $i_1(x, t) = (i(x), t)$ ):

$$\begin{array}{ccc} X \times I & \xrightarrow{i_1} & Y \times I \\ \downarrow \bar{p}_1 & & \downarrow p \\ X & \xrightarrow{i} & Y. \end{array}$$

Using this diagram together with the naturality and associativity properties of the cup product we obtain

$$\begin{aligned} \mathfrak{S}(b \cdot a) &= (b \cdot a) \times \sigma = p_1^*(i^*(b) \smile a) \smile p_2^*(\sigma) = \bar{p}_1^*(i^*(b)) \smile p_1^*(a) \smile p_2^*(\sigma) \\ &= \bar{p}_1^*(i^*(b)) \smile \mathfrak{S}(a) = i_1^*(p^*(b)) \smile \mathfrak{S}(a) = p^*(b) \cdot \mathfrak{S}(a), \end{aligned}$$

which ends the proof. ■

THEOREM 5.7.

$$\Upsilon((X, A) \times (I, \partial I); Y) = \Upsilon(X, A; Y).$$

*Proof.* Let us notice that formally  $X \times I \subset Y \times I$  and thus  $H^*(X \times I, X \times \partial I \cup A \times I)$  is an  $H^*(Y \times I)$ -module, but  $p^* : H^*(Y) \rightarrow H^*(Y \times I)$  is an isomorphism which gives the naturally isomorphic  $H^*(Y)$ -module structure:  $b \odot a := p^*(b) \cdot a$  for  $b \in H^*(Y)$  and  $a \in H^*(X \times I, X \times \partial I \cup A \times I)$ . By using this into account the desired equality follows directly from Lemma 5.6. ■

Now we apply the above notion to the Conley index, first in the nonequivariant case. It is useful to consider the *cohomology Conley index* defined by

$$CH^*(S) := H^*(N, L) = H^*(N/L),$$

where  $H^*$  denotes the Alexander–Spanier cohomology and  $(N, L)$  is an index pair for the isolated invariant set  $S$ . The last equality is understood that we identify  $H^*(N, L)$  and  $H^*(N/L)$  via the isomorphism induced by the quotient map.

It is convenient to extend the index to an index of isolating neighbourhoods: if  $N$  is an isolating neighbourhood for  $\eta$  then the *homotopy* (resp. *cohomology*) *Conley index* of  $N$  is defined to be

$$h(N) = h(N, \eta) := h(\text{Inv}(N, \eta)), \quad \text{resp.} \quad CH^*(N) = CH^*(N, \eta) := CH^*(\text{Inv}(N, \eta)).$$

Before giving the definition of the relative cup-length of Conley index we need some useful lemmas. If  $(N_0, N_1)$  is an index pair and  $t \geq 0$  then, following [18], we set

$$\begin{aligned} N_1^t &:= \{x \in N_1 : \eta(x, [-t, 0]) \subset N_1\}, \\ N_0^{-t} &:= \{x \in N_1 : \text{there is a point } x' \in N_0 \text{ and } t' \in [0, t] \\ &\quad \text{with } \eta(x', [-t', 0]) \subset N_1 \text{ and } \eta(x't) = x\}. \end{aligned}$$

For  $t \geq 0$  define a map of pointed spaces

$$g : (N_1/N_0^{-t}, *) \rightarrow (N_1^t/N_0 \cap N_1^t, *)$$

by

$$g([x]) := \begin{cases} [\eta(x, t)] & \text{if } \eta(x, [0, t]) \subset N_1 \setminus N_0, \\ * & \text{otherwise.} \end{cases}$$



It is known ([18], Lemma 23.14) that  $g$  is a homeomorphism. Therefore  $g$  induces an isomorphism

$$g^* : H^*(N_1^t, N_0 \cap N_1^t) \rightarrow H^*(N_1, N_0^{-t}).$$

LEMMA 5.8. *Assume that  $N$  is an isolating neighbourhood for  $\eta$  and  $(N_1, N_0)$  is an index pair for  $S \subset N$ . If  $N_1 \subset N$  then the inclusion  $i : (N_1, N_0 \cap N_1^t) \rightarrow (N_1, N_0^{-t})$  induces an isomorphism*

$$i^* = (g^*)^{-1} : H^*(N_1, N_0^{-t}) \rightarrow H^*(N_1, N_0 \cap N_1^t).$$

*Proof.* Consider the following diagram, where the vertical arrows denote the quotient maps:

$$\begin{array}{ccc} (N_1, N_0^{-t}) & \xleftarrow{i} & (N_1, N_0 \cap N_1^t) \\ \downarrow & & \downarrow \\ N_1/N_0^{-t} & \xrightarrow{g} & N_1/(N_0 \cap N_1^t). \end{array}$$

From the definition of  $g$  it is obvious that the diagram is homotopy commutative and the conclusion follows. ■

DEFINITION 5.9. Let  $N$  be an isolating neighbourhood for the flow  $\eta$ . We define the *relative cup-length* of  $\eta$  with respect to  $N$  to be

$$\Upsilon(\eta, N) := \Upsilon(N_1, N_0; N),$$

where  $(N_1, N_0)$  is an index pair for  $S$ .

The following lemma states that  $\Upsilon(\eta, N)$  is well defined.

LEMMA 5.10. *Let  $N$  be an isolating neighbourhood for  $\eta$  and let  $S \subset N$  be an isolated invariant set. If  $(N_1, N_0)$  and  $(\bar{N}_1, \bar{N}_0)$  are index pairs for  $S$  such that  $N_1, \bar{N}_1 \subset N$  then*

$$\Upsilon(\bar{N}_1, \bar{N}_0; N) = \Upsilon(N_1, N_0; N).$$

*Proof.* As in the proof of Lemma 23.17 in [18], we consider the following sequence of maps, where  $j, \hat{i}, \hat{i}_1$  are defined by inclusion maps of pairs of spaces and  $g, \hat{g}$  are as above. All of them are homotopy equivalences of pointed spaces, as is in details proved in [18].

$$\begin{aligned} N_1/N_0 &\xrightarrow{j} N_1/N_0^{-t} \xrightarrow{g} N_1^t/(N_0 \cap N_1^{-t}) \xrightarrow{\hat{i}_1} \bar{N}_1/\bar{N}_0^{-t} \\ &\xrightarrow{\hat{g}} N_1^t/(\bar{N}_0 \cap \bar{N}_1^t) \xrightarrow{\hat{i}} N_1^t/(\bar{N}_0 \cap \bar{N}_1^t). \end{aligned}$$

By Lemma 5.8 and definition of  $j$  it follows that the following sequence of isomorphisms

$$\begin{aligned} H^*(N_1, N_0) &\xleftarrow{\approx} H^*(N_1, N_0^{-t}) \xleftarrow{\approx} H^*(N_1^t, N_0 \cap N_1^{-t}) \xrightarrow{\approx} H^*(\bar{N}_1, \bar{N}_0^{-t}) \\ &\xrightarrow{\approx} H^*(N_1^t, \bar{N}_0 \cap \bar{N}_1^t) \xrightarrow{\approx} H^*(N_1^t, \bar{N}_0 \cap \bar{N}_1^t) \end{aligned}$$

all are induced by inclusions. Therefore they all are isomorphisms of  $H^*(N)$ -modules and the conclusion follows. ■

The continuation property holds for the relative cup-length.

LEMMA 5.11. *Consider a continuous family of flows  $\eta_\lambda : X \times \mathbb{R} \rightarrow X$ ,  $\lambda \in [0, 1]$ . Let  $N \subset X$  be an isolating neighbourhood for all flows  $\eta_\lambda$ . Then*

$$\Upsilon(\eta_0, N) = \Upsilon(\eta_1, N).$$

*Proof.* Similarly as in the proof of Lemma 5.10 we shall use parts of the proof of Theorem 23.31 in [18]. Given  $\mu \in [0, 1]$ , there exists a neighbourhood  $W$  of  $\mu$  in  $[0, 1]$  with the property that for all  $\lambda \in W$  we can find pairs  $(N_1, N_0) \subset (P_1^\lambda, P_0^\lambda) \subset (\bar{N}_1, \bar{N}_0)$  such that  $(N_1, N_0), (\bar{N}_1, \bar{N}_0)$  are index pairs for  $\eta_\mu$  in  $N$ , and  $(P_1^\lambda, P_0^\lambda)$  is an index pair for  $\eta_\lambda$  in  $N$  (see Lemma 23.28 in [18]). Then it is shown in the proof of Theorem 23.31 in [18] that the inclusion  $i : (N_1, N_0) \rightarrow (P_1^\lambda, P_0^\lambda)$  induces a homotopy equivalence of pointed spaces  $N_1/N_0$  and  $P_1^\lambda/P_0^\lambda$ . The same argument applies to show that  $i^* : H^*(P_1^\lambda, P_0^\lambda) \approx H^*(N_1, N_0)$  is an isomorphism of  $H^*(N)$ -modules. Therefore  $\Upsilon(\eta_\lambda, N) = \Upsilon(\eta_\mu, N)$ . Since  $[0, 1]$  is compact and connected, this completes the proof. ■

One easily sees that the relative cup length is also invariant under otopy (the proof is practically the same).

Now we turn back to the equivariant case. Let  $V$  be a finite-dimensional orthogonal representation of a compact Lie group  $G$  and let  $\text{Iso}(V) = \{(H_1), (H_2), \dots, (H_k)\}$ . Consider an equivariant flow generator which is already a collection of local flow generators  $(f_1, U_1) \sqcup (f_2, U_2) \sqcup \dots \sqcup (f_k, U_k)$  such that  $f_i$  is  $(H_i)$ -normal,  $i = 1, 2, \dots, k$ .

Consider one component  $(f_i, U_i)$  which is  $(H_i)$ -normal. Choose a representative  $H \in (H_i)$ . Then  $V^H$  is a linear subspace of  $V$  and it is a representation of the Weyl group  $WH$ . The set  $U_i^H = U_i \cap V^H$  is  $WH$ -invariant and  $f^h : U_i^H \rightarrow V^H$  is a  $WH$ -equivariant local flow generator (comp. [1]). Therefore we can consider the local flow  $\hat{\eta}$  defined on the quotient space  $N := U_i^H/WH \subset V^H/WH$ . Then the relative cup-length  $\Upsilon(\hat{\eta}, N)$  is well-defined. It is easy to observe that for any other representative  $\tilde{H} = gHg^{-1}$  in the same conjugacy class we obtain a conjugated local flow and therefore it gives the same number  $\Upsilon(\hat{\eta}, N)$ . Therefore we are ready to define a  $G$ -equivariant otopy invariant.

DEFINITION 5.12. Let us consider a  $G$ -equivariant local flow generator  $(f, U)$ , which generates a local flow  $\eta$ . We define

$$\Upsilon_G(\eta, U) := \sum_{(H_i) \in \Phi(G)} \Upsilon(\hat{\eta}_i, N) \cdot (H_i) \in U(G),$$

where  $\hat{\eta}_i, N$  are as above.

Because of Lemma 5.11 this notion is well defined. Now, by use of Theorem 4.5 for an arbitrary equivariant flow generator we find in the otopy class a collection of local flow generators which is normal and apply the above definition. Thus we obtain an invariant with immediate properties:

THEOREM 5.13.

- a) If two equivariant local flow generators  $(f_0, U_0), (f_1, U_1)$  are otopic then

$$\Upsilon_G(\eta_0, U_0) = \Upsilon_G(\eta_1, U_1),$$

where  $\eta_i$  are the local flows generated by  $f_i$ , respectively.

- b) If the  $H$ -component of  $\Upsilon_G(\eta, U)$  is non-zero, then  $\text{Inv}(\eta, U^H) \neq \emptyset$ . Moreover, if  $f = \nabla\varphi$  is a gradient, then this coefficient is a lower bound of critical  $WH$ -orbits of  $\varphi$ .

*Proof.* The first statement is a consequence of the continuation property of each coefficient  $\Upsilon(\widehat{\eta}_i, N)$ , see Lemma 5.11. As for the second one, we can apply Theorem 4.1 of [7] for the local flow  $\widehat{\eta}$ , defined by a  $H$ -normal component of the normal collection of local flow generators in the otopy class of  $(f, U)$ . ■

**6. Mountain Pass type theorems.** In this section we give simple applications of the module structure described above. We start with a classic result.

Let  $M$  be a smooth closed manifold and let  $f : M \rightarrow \mathbb{R}$  be a function of class  $C^1$ . Assume that  $f$  has only a finite number of critical points. Let  $c_1 < c_2 < \dots < c_p$  denote critical values of  $f$ . We choose numbers  $a_0, a_1, \dots, a_p$  such that

$$a_0 < c_1 < a_1 < c_2 < \dots < c_p < a_p$$

As usual, we consider sublevel sets  $f^a := \{x \in M : f(x) \leq a\}$ .

Denote by  $\varphi : M \times \mathbb{R} \rightarrow M$  the flow generated by a vector field  $-\nabla f : M \rightarrow TM$ . The following is well-known.

**THEOREM 6.1.** *For every  $i = 1, 2, \dots, p$  the sets  $f^{a_i}$  are isolating neighbourhoods, and  $(f^{a_i}, f^{a_{i-1}})$  are index pairs for  $\varphi$ .*

We can assume that  $f^{a_i}$  are CW-complexes ( $a_i$  are regular values). One observes that  $f^{a_p} = M$ , because  $c_p$  is a maximum of  $f$ . Let  $R$  be any ring of coefficients.

**LEMMA 6.2.** *Let  $\beta \in H^k(M; R)$ ,  $k > 0$ . Then for every  $i = 1, 2, \dots, p$  we have the inequality  $\chi(\beta; f^{a_i}, f^{a_{i-1}}) \leq 1$ .*

*Proof.* Since  $\text{Inv}(\overline{f^{a_i} \setminus f^{a_{i-1}}}) \subset f^{-1}(c_i)$  is finite, the set  $A = \{x_1, x_2, \dots, x_s\}$ , which corresponds to one critical level  $c_i$  and each of the singletons  $\{x_j\}$ , is an isolated invariant set. Moreover as an isolating neighbourhood we can choose a small disc  $D_j$  which is contained in  $f^{a_i} \setminus f^{a_{i-1}}$  and is disjoint with the other discs  $D_k$ . Thus the disjoint sum  $D = \bigcup D_j$  is an isolating neighbourhood for  $A$ . We find an index pair  $(Y, Z)$  for  $A$  in  $D$ . Since  $D_j$  are contractible, we have  $\chi(\beta; Y, Z) \leq 1$ . On the other hand, both pairs  $(Y, Z)$  and  $(f^{a_i}, f^{a_{i-1}})$  are index pairs for  $A$  in  $(f^{a_i} \setminus f^{a_{i-1}})$ , thus the inclusion gives an isomorphism of  $H^*(Y, Z)$  and  $H^*(f^{a_i}, f^{a_{i-1}})$  as  $H^*(M)$ -modules. ■

**PROPOSITION 6.3.** *For every  $i = 0, 1, 2, \dots, p$  and  $\beta \in H^k(M)$  we have  $\chi(\beta; f^{a_i}) \leq i$ .*

*Proof.* The set  $f^{a_0}$  is empty thus we start the induction. The inequality

$$\chi(\beta; f^{a_i}) \leq \chi(\beta; f^{a_i}, f^{a_{i-1}}) + \chi(\beta; f^{a_{i-1}})$$

can be proved identically to Lemma 5.6 (with  $B = \emptyset$ ). Then we apply Lemma 4.3 to complete the proof. ■

Therefore we have proved

**THEOREM 6.4.** *If  $M$  is a smooth closed manifold and a  $C^1$ -function  $f : M \rightarrow \mathbb{R}$  has a finite number of critical points on at most  $p$  levels, then  $\chi(\beta; M) \leq p$  for every  $\beta \in H^k(M)$ ,  $k > 0$ .*

**EXAMPLE 6.5.** Let  $M$  be an  $n$ -dimensional real projective space  $\mathbb{R}P^n$ . We have  $H^*(M; \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/\alpha^{n+1}$ , where  $\alpha \in H^1(M; \mathbb{Z}_2)$ . This means that  $\chi(\alpha; M) = n + 1$ . Therefore each smooth function  $f : \mathbb{R}P^n \rightarrow \mathbb{R}$  has at least  $n + 1$  critical points.



The following example is a straightforward consequence of the last one.

EXAMPLE 6.6. Let  $S^{n-1}$  be a unit sphere in  $\mathbb{R}^n$  and consider an even function  $f : S^{n-1} \rightarrow \mathbb{R}$  of class  $C^1$  with a finite number of critical points. Then  $f$  has at least  $n$  pairs of antipodal critical points with different values.

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be an even function of class  $C^1$  such that the vector field  $-\nabla f$  generates a flow  $\varphi$  on  $\mathbb{R}^n$ . Assume that the annulus  $\Omega := \{x \in \mathbb{R}^n : r \leq \|x\| \leq R\}$  is an isolating neighbourhood for  $\varphi$ . Moreover, assume

- (H1)  $f(x) \leq 0$  for  $x \in \partial\Omega$  and  $\partial\Omega$  is an exit set (e.g.  $-\nabla f$  is directed outward of  $\Omega$  at points in  $\partial\Omega$ ).
- (H2) There exists  $\rho \in (r, R)$  such that for  $x$  with  $\|x\| = \rho$  we have  $f(x) \geq \alpha > 0$ .

THEOREM 6.7. Under the above assumptions  $f$  has at least  $n$  pairs of critical points in  $\Omega$ .

Let us begin with some notation. First, since the antipodal action of the group  $\mathbb{Z}_2$  on  $\Omega$  is free, the quotient space  $M = \Omega/\mathbb{Z}_2$  is a compact manifold with boundary. Indeed,  $M$  is diffeomorphic to the product  $\mathbb{R}P^{n-1} \times [r, R]$ , and the boundary  $\partial M \approx \mathbb{R}P^{n-1} \times \{r, R\}$ . If we denote by  $\sigma$  the generator of the group  $H^1([r, R], \{r, R\}; \mathbb{Z}_2)$ , then by Lemma 5.6  $H^*(M, \partial M; \mathbb{Z}_2)$  is a  $H^*(M)$ -module with the generator  $\beta = 1 \times \sigma \in H^1(M, \partial M; \mathbb{Z}_2)$ . Nontrivial elements are of the form  $\beta \cdot \gamma^k \in H^{k+1}(M, \partial M; \mathbb{Z}_2)$ , where  $\gamma \in H^1(\mathbb{R}P^{n-1}; \mathbb{Z}_2)$ .

We denote with the same letter  $f$  the induced map  $f : M \rightarrow \mathbb{R}$ . As before, we consider the sublevel sets  $f^a = \{x \in M : f(x) \leq a\}$ . We have natural inclusions  $i_a : f^a \hookrightarrow M$ .

DEFINITION 6.8. Let  $\xi \in H^*(M, \partial M; \mathbb{Z}_2)$ . The depth of  $\xi$  is the number

$$\nu(\xi) = \inf\{a : i_a^*(\xi) \neq 0\}.$$

Notice that the depth of any element is always a critical level of  $f$ . Indeed, if  $a$  is a regular value of  $f$ , then for some  $\varepsilon > 0$  the interval  $[a - \varepsilon, a]$  consists of regular values. By Deformation Lemma,  $f^{a-\varepsilon}$  is a deformation retract of  $f^a$ . Thus, if  $i_a^*(\xi) \neq 0$  then also  $i_{a-\varepsilon}^*(\xi) \neq 0$ .

We start with

LEMMA 6.9.  $\nu(\beta) \geq \alpha$ , where  $\alpha$  is from (H2).

Proof. Let  $0 < a < \alpha$ . Consider the commutative diagram

$$\begin{array}{ccccc} H^0(M) & \xrightarrow{\text{mono}} & H^0(\partial M) & \xrightarrow{\delta_1} & H^1(M, \partial M) \\ \downarrow i_a^* & & \downarrow id^* & & \downarrow i_a^* \\ H^0(f^a) & \xrightarrow{k} & H^0(\partial M) & \xrightarrow{\delta} & H^1(f^a, \partial M) \end{array}$$

We have to prove that  $i_a^*(\beta) = 0$ . Since  $\delta_1$  is an epimorphism, there exists  $\bar{\beta}$  such that  $\beta = \delta_1(\bar{\beta})$ . Therefore  $i_a^*(\beta) = i_a^*(\delta_1(\bar{\beta})) = \delta(id^*(\bar{\beta}))$ .

On the other hand, by our assumptions  $\partial M \subset f^a$  has two connected components and the generators of  $H^0(\partial M)$  correspond to the generators of  $H^0(f^a)$  given by different



components of  $f^a$ , containing them. Therefore  $\text{id}^*(\bar{\beta}) \in \text{Im}(k)$  and thus  $\delta(\text{id}^*(\bar{\beta})) = 0$ . This completes the proof. ■

LEMMA 6.10. *Assume that  $f$  has only a finite number of critical points. Let  $c = \nu(\beta \cdot \gamma^k)$  be the only critical value in the interval  $[a_1, a_2]$ . Then  $\nu(\beta \cdot \gamma^{k+1}) > \nu(\beta \cdot \gamma^k)$ .*

*Proof.* We have  $i_{a_2}^*(\beta \cdot \gamma^k) \neq 0$  and  $i_{a_1}^*(\beta \cdot \gamma^k) = 0$ . We can repeat the argument from Lemma 6.2. The set of critical points  $A \subset f^{-1}(c)$  is finite and  $(f^{a_2}, f^{a_1})$  is an index pair of it. We have then  $\chi(\gamma; f^{a_2}, f^{a_1}) \leq 1$ . On the other hand, applying the proof of Lemma 5.4 with  $B = \partial M \subset A = f^{a_1} \subset X = f^{a_2} \subset Y = M$  we obtain the inequalities

$$k \leq \chi(\gamma; f^{a_2}, \partial M) \leq \chi(\gamma; f^{a_2}, f^{a_1}) + \chi(\gamma; f^{a_1}, \partial M) \leq 1 + (k - 1) = k.$$

Therefore  $i_{a_2}^*(\beta \cdot \gamma^{k+1}) = 0$ , which ends the proof. ■

*Proof of Theorem 6.7.* Now the proof of Theorem 6.7 is immediate. If the number of critical points is finite, then the above lemmas give us  $n$  different critical levels of  $f$ , which are greater than  $\alpha$ . ■

A more general abstract result of this type can be found in [7], Theorem 4.1.

**7. Elliptic BVP.** Consider the following family of Dirichlet boundary problems with a parameter  $\lambda \in \mathbb{R}$ :

$$\Delta + \lambda u = g(u) \quad \text{in } \Omega \tag{1}$$

$$u = 0 \quad \text{in } \partial\Omega \tag{2}$$

where

- $\Omega \subset \mathbb{R}^n$  is an open Lipschitzian domain;
- $g \in C^1(\mathbb{R}, \mathbb{R})$  defines a  $C^1$ -operator  $\mathcal{G} : L^2(\Omega) \rightarrow L^2(\Omega)$ ,  $\mathcal{G}(u)(x) := g(u(x))$  such that  $\mathcal{G}(u) = o(\|u\|)$  when  $u \rightarrow 0$ ;
- $g = \gamma'$ , where  $\gamma \in C^2(\mathbb{R}, \mathbb{R})$ .

It is clear that  $u \equiv 0$  is a solution of the above problem for every  $\lambda \in \mathbb{R}$ .

We are interested in the existence and multiplicity of bifurcation for this problem. To this aim we formulate an appropriate problem in the Hilbert space  $L^2(\Omega)$ . Let  $\langle \cdot, \cdot \rangle$  denote the inner product in  $L^2(\Omega)$ . Consider the Sobolev space  $H^1(\Omega) \subset L^2(\Omega)$  with the inner product

$$\langle u, v \rangle_1 = \sum_{i=1}^n \langle u'_{x_i}, v'_{x_i} \rangle + \langle u, v \rangle,$$

where derivatives are weak derivatives. By  $H_0^1(\Omega)$  we denote the closure of a subspace  $C_0^\infty(\Omega) \subset H^1(\Omega)$  in the norm  $\|\cdot\|_1$ .

A variational reformulation of the problem (1)(2) is the following integral equation

$$t(u, v) - \langle \lambda u, v \rangle + \langle \mathcal{G}(u), v \rangle = 0 \quad \forall v \in H_0^1(\Omega), \tag{3}$$

where

$$t(u, v) = \sum_{i=1}^n \langle u'_{x_i}, v'_{x_i} \rangle$$

is a bilinear form on  $H_0^1(\Omega)$ . Then solutions to (3) are called weak solutions to the Dirichlet problem (1)(2). Since  $t$  is densely defined, symmetric, closed and bounded from below,



there exists a closed linear operator  $T$  on  $L^2(\Omega)$  with the domain  $D(T) \subset H_0^1(\Omega)$  and

$$t(u, v) = \langle Tu, v \rangle \quad \forall u \in D(T), v \in H_0^1(\Omega).$$

The operator  $T$  is positive and invertible, both  $T$  and  $T^{-1} : L^2(\Omega) \rightarrow L^2(\Omega)$  are selfadjoint and closed. Moreover,  $T$  is completely continuous by the Rellich–Kondrashov theorem (comp. [8]).

The spectrum  $\sigma(T)$  of  $T$  consists of a sequence of real eigenvalues with finite multiplicities  $0 < \lambda_1 < \lambda_2 < \dots$ , and  $\lambda_k \rightarrow \infty$  when  $k \rightarrow \infty$ .

Thus the equality

$$t(u, v) = \langle T^{1/2}u, T^{1/2}v \rangle \quad \forall u, v \in H_0^1(\Omega),$$

defines a selfadjoint operator in  $L^2(\Omega)$  such that  $T = (T^{1/2})^2$ .  $T^{1/2}$  is an isomorphism of spaces  $H_0^1(\Omega)$  and  $L^2(\Omega)$  and its inverse is completely continuous. Now we can write our problem (3) in the form

$$\langle T^{1/2}u, T^{1/2}v \rangle - \langle \lambda u, v \rangle + \langle \mathcal{G}(u), v \rangle = 0 \quad \forall v \in H_0^1(\Omega), \tag{4}$$

or, equivalently,

$$\langle w, \zeta \rangle - \langle \lambda T^{-1/2}w, T^{-1/2}\zeta \rangle + \langle T^{-1/2}\mathcal{G}(T^{-1/2}w), \zeta \rangle = 0 \quad \forall \zeta \in L^2(\Omega), \tag{5}$$

where  $w = T^{1/2}u$ ,  $\zeta = T^{1/2}v$ . Since  $T^{1/2}$  is selfadjoint, we obtain an equivalent form

$$\langle w, \zeta \rangle - \langle \lambda T^{-1}w, \zeta \rangle + \langle T^{-1/2}\mathcal{G}(T^{-1/2}w), \zeta \rangle = 0 \quad \forall \zeta \in L^2(\Omega). \tag{6}$$

That is, we have an equation in  $L^2(\Omega)$ :

$$w - \lambda T^{-1}w + f(w) = 0, \tag{7}$$

where  $f : L^2(\Omega) \rightarrow L^2(\Omega)$  is given by  $f(w) = T^{-1/2}\mathcal{G}(T^{-1/2}w)$ . This map is of class  $C^1$  and  $f(w) = o(\|w\|)$ , whenever  $w \rightarrow 0$ ; hence  $w = 0$  is a solution to (7). Bifurcation of nontrivial solutions can happen only for  $\lambda = \lambda_k \in \sigma(T)$ , when  $\text{Ker}(\text{id} - \lambda T^{-1}) \neq 0$ .

Defining a selfadjoint and compact operators  $A_\lambda := \text{id} - (\lambda_k + \lambda)T^{-1}$  for  $\lambda \in \mathbb{R}$  we write (7) in the form

$$A_\lambda w + f(w) = 0.$$

Since  $\lambda_k$  is an isolated eigenvalue of  $T$  with finite multiplicity, we have  $0 < \dim \text{Ker } A_0 < \infty$  and for  $\lambda \neq 0$   $A_\lambda$  is an isomorphism onto  $R(A_\lambda)$  in a small neighbourhood of  $\lambda = 0$ . It is easy to check that  $F : L^4(\Omega) \times \mathbb{R} \rightarrow L^2(\Omega)$  given by  $F(w, \lambda) := A_\lambda w + f(w)$  is a family of gradient vector fields with potentials for the components given by  $a_\lambda(w) = \frac{1}{2}\langle A_\lambda w, w \rangle$ ,  $\varphi(w) = \int_\Omega \gamma(T^{-1/2}w(x)) dx$ , respectively. In this way we obtain a bifurcation problem in the sense of [7], Section 5:

$$F(w, \lambda) = 0. \tag{8}$$

That is, for some interval  $0 \in (\lambda_1, \lambda_2) \subset [\lambda_1, \lambda_2]$  we have  $F(0, \lambda) = 0$  for all  $\lambda$ , and the derivatives with respect to  $\omega$ ,  $D_\omega F(0, \lambda_1)$  and  $D_\omega F(0, \lambda_2)$  are isomorphisms. We can now apply a finite-dimensional reduction:

Assume that we have two Banach spaces embedded continuously in a Hilbert space  $E_1 \subset E_0 \subset H$ . They all can be representations of the group  $G$ .

**THEOREM 7.1** ([7], Theorem 5.1). *Let a  $G$ -equivariant mapping  $F : \Omega_F \rightarrow E_0$  define a bifurcation problem on  $[\lambda_1, \lambda_2]$ . If there exist decompositions*

$$E_1 = V \oplus W_1, \quad E_0 = V \oplus W_0, \quad F(x, y, \lambda) = (f_1(x, y, \lambda), f_2(x, y, \lambda)),$$



such that

$$Df_2(0, \lambda)|_{W_1} : W_1 \approx W_0 \text{ for } \lambda \in [\lambda_1, \lambda_2]$$

then there exist:

- (1) an open  $G$ -invariant subset  $\Omega \subset \Omega_f$ , with  $\{0\} \times [\lambda_1, \lambda_2] \subset \Omega$ ;
- (2) a map  $h : \Omega_h \rightarrow E_0$ , defining a bifurcation problem on  $[\lambda_1, \lambda_2]$

such that

- (a)  $F|_\Omega$  defines a bifurcation problem on  $[\lambda_1, \lambda_2]$  equivalent to that defined by  $h$ ;
- (b)  $h(V \cap \Omega_h) \subset V$  and  $h^{-1}(0) \subset V$ ;
- (c) if  $D_1 f_2(0, 0, \lambda) = 0$  then  $D_1 h(0, 0, \lambda) = D_1 f_1(0, 0, \lambda)$ .

Here the finite-dimensional subspace  $V$  is  $\text{Ker } A_0$ .

REMARK. In fact, in order to apply the above theorem we use a small perturbation argument near  $\lambda = 0$  (comp. [13], Section II.4.2). We omit the details (see also [21]). Observe that the reduction procedure works for equivariant maps.

In a finite-dimensional case  $V = (\mathbb{R}^n, \varphi)$  is an orthogonal representation of a compact Lie group  $G$ , i.e.  $\varphi : G \rightarrow O(n)$  is a group homomorphism. Let  $S(V) := \{x \in V : |x| = 1\}$ , and  $S(\mathbb{R}^n, \epsilon) := \{x \in \mathbb{R}^n : |x| = \epsilon\}$ . We can use the following

LEMMA 7.2 ([7], Lemma 6.2). *Let  $f : \Omega_f \rightarrow \mathbb{R}^n$  be a gradient equivariant map defining a bifurcation problem on  $[-1, 1]$  and  $A_\lambda := D_x f(0, \lambda)$ ,  $\lambda \in [-1, 1]$ . Assume that there is  $C > 0$  such that*

$$\langle A_1(x), x \rangle \geq C|x|^2 \quad \text{for } x \in \mathbb{R}^n$$

and

$$\langle A_{-1}(x), x \rangle \leq -C|x|^2 \quad \text{for } x \in \mathbb{R}^n.$$

Then for a sufficiently small  $\epsilon$  the number of zero  $G$ -orbits of  $f$  in  $S(\mathbb{R}^n, \epsilon) \times (-1, 1)$  is not less than the cup-length of  $S(V)/G$ .

Applying the above results to the trivial group  $G = \{e\}$ , we have the following:

THEOREM 7.3. *For each  $k \in \mathbb{N}$  the point  $(0, \lambda_k)$  is a bifurcation point of (7) of order at least 2, i.e. for  $\lambda = \lambda_k$  there exist at least two solutions on each sufficiently small sphere in  $L^2(\Omega)$ .*

*Proof.* It is enough to observe that the cup-length of a sphere is 2 and use Lemma 7.2. ■

Assume now that our domain is symmetric and the function  $g$  in problem (1) is odd  $g(-x) = -g(x)$ . Then

$$F(-w, \lambda) = -F(w, \lambda).$$

THEOREM 7.4. *If  $g$  is an odd function and  $\Omega$  is a symmetric domain in  $\mathbb{R}^n$  with respect to the antipodal action, then each point  $(0, \lambda_k)$  is a bifurcation point of order  $2l$ , where  $l$  is the multiplicity of  $\lambda_k$ .*

*Proof.* One observes that the reduction procedures preserve the equivariance property. An action of  $\mathbb{Z}_2$  on  $L^2(\Omega) \times \mathbb{R}$  is given by  $-1 \cdot (f(x), \lambda) = (f(-x), \lambda)$ . The main ingredient is that the cup-length of  $S^{l-1}/\mathbb{Z}_2$  is equal to  $l$ . ■





The last two results are not new, in fact. Similar results one can find e.g. in [2]. They are described here as a simple illustration of the technique. More complicated symmetries may be considered (comp. an example in [21]). The author is also convinced that problems with  $p$ -Laplacians can be considered in a similar way. An application to periodic solutions of Hamiltonian systems is given in [7], where a natural action the group  $G = S^1$  on the space of periodic functions is used.

**Acknowledgments.** This research was supported by Polish Ministry of Higher Education Grant nr N N 201 394037. The author is indebted to the anonymous referee for careful reading the manuscript and pointing out many mistakes.

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