Interval incidence coloring of bipartite graphs☆

Robert Janczewski *, Anna Małafiejska, Michał Małafiejski

Department of Algorithms and Systems Modelling, Gdańsk University of Technology, Narutowicza 11/12, Gdańsk, Poland

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ABSTRACT

In this paper[1] we study the problem of interval incidence coloring of bipartite graphs. We show the upper bound for interval incidence coloring number (χ_{ii}) for bipartite graphs $\chi_{ii} \leq 2\Delta$, and we prove that $\chi_{ii} = 2\Delta$ holds for regular bipartite graphs. We solve this problem for subcubic bipartite graphs, i.e. we fully characterize the subcubic graphs that admit 4, 5 or 6 coloring, and we construct a linear time exact algorithm for subcubic bipartite graphs. We also study the problem for bipartite graphs with $\Delta = 4$ and we show that 5-coloring is easy and 6-coloring is hard ($\mathcal{NP}$-complete). Moreover, we construct an $O(n\Delta^{3.5} \log \Delta)$ time optimal algorithm for trees.

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1. Introduction

1.1. Problem definition

In the following we consider connected simple nonempty graphs only, and we use standard notations of the graph theory. For a given simple graph $G = (V, E)$, we define an *incidence* as a pair (v, e), where vertex $v \in V$ is one of the endpoints of edge $e \in E$. The set of all incidences of G will be denoted[2] by I, thus $I := \{(v, e): v \in V \land e \in E \land v \in e\}$. We say that two incidences (v, e) and (w, f) are *adjacent* if and only if one of the following holds: (1) $v = w$ and $e \neq f$; (2) $e = f$ and $v \neq w$; (3) $e = \{v, w\}, f = \{w, u\}$ and $v \neq u$.

By an *incidence coloring* of G we mean a function $c: I \to \mathbb{N}$ such that $c(v, e) \neq c(w, f)$ for any adjacent incidences (v, e) and (w, f). The *incidence coloring number* of G, denoted by χ_i, is the smallest number of colors in an incidence coloring of G.

A finite nonempty set $A \subseteq \mathbb{N}$ is an *interval* if and only if it contains all integers between $\min A$ and $\max A$. For a given incidence coloring c of graph G and $v \in V$ let $A_c(v) := \{c(v, e): v \in e \land e \in E\}$. By an *interval incidence coloring* of graph G we mean an incidence coloring c of G such that for each vertex $v \in V$ the set $A_c(v)$ is an interval. By an *interval incidence k-coloring* we mean a coloring using all colors from the set $\{1, \ldots, k\}$. The *interval incidence coloring number* of G, denoted by $\chi_{ii}(G)$, is the smallest number of colors in an interval incidence coloring of G. An interval incidence coloring of G using χ_{ii} colors is said to be *minimal*. We say that $v \in V$ is *minimal* if $\min A_c(v) = \min c(I)$, and we say that $v \in V$ is *maximal* if $\max A_c(v) = \max c(I)$.

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* Corresponding author. Tel.: +48 58 347 10 64.
  E-mail addresses: skalar@eti.pg.gda.pl (R. Janczewski), aniam@kaims.pl (A. Małafiejska), mima@kaims.pl (M. Małafiejski).

1 The extended abstract of this paper appeared on PPAM 2009, LNCS 6067 (2010) 11–20.

2 To simplify notation, we write I instead of $I(G)$ whenever G is clear from the context. The same rule applies to other parameters of G appearing in the paper.

1.2. Background, previous results and our contribution

In this paper we consider a restriction of the incidence graph coloring problem, in which the colors of incidences at a vertex form an interval. The considerations in this paper are motivated by the multicasting communication in a multifiber WDM (*wavelength-division multiplexing*) all-optical star network [2,4,5].

Alon et al. [1] defined a problem of partitioning a graph into a minimal number of star forests as a model of the possible fastest exchanging messages in a network with the assumption of the ability of sending messages to all neighbors at the same time and blocking receiving more than one message (a dual model is also possible). Brualdi and Massey [6] formulated a model of *incidence graph coloring* with references to some other models of graph coloring, such as strong edge and vertex coloring of graphs. Guiduli [14] observed that the problem of incidence graph coloring is a special case of the problem of partitioning a symmetric digraph into directed star forests. See [7,9,8,19,20] for more information about the incidence coloring of graphs.

The *interval edge coloring* of graphs was proposed by Asratian and Kamalian [3] who analyzed the complexity and the basic properties of interval edge coloring, defined also as consecutive coloring. A detailed review of the interval edge coloring problem is described in [11,13]. This problem has applications in open-shop scheduling, especially in dedicated systems without breaks [12].

In [16] the authors introduced the concept of *interval incidence coloring* modeling message passing in networks, and in [17] the authors studied the applications in a model of one-multicast transmission per node in multifiber WDM networks. In [18] the authors proved some lower and upper bounds for the *interval incidence coloring number* (χ_{ii}), and determined the exact values of χ_{ii} for selected classes of graphs: paths, cycles, stars, wheels, fans, necklaces, complete graphs and complete k-partite graphs. In [18] the authors also studied the complexity of the interval incidence coloring problem for subcubic graphs for which they showed that the problem whether $\chi_{ii} \leq 4$ is easy, and $\chi_{ii} \leq 5$ is $\mathcal{NP}$-complete.

In this paper, we study the problem of interval incidence coloring for bipartite graphs. In Section 2 we obtain an upper bound for the interval incidence coloring number of bipartite graphs, namely $\chi_{ii} \leq 2\Delta$, and we prove that $\chi_{ii} = 2\Delta$ holds for regular bipartite graphs. In Section 3 we construct a linear time exact algorithm for subcubic bipartite graphs. In Section 4 we study the problem for bipartite graphs with $\Delta = 4$ and we show that 5-coloring is easy and 6-coloring is hard ($\mathcal{NP}$-complete). In Section 5 we construct a polynomial time exact algorithm for trees. Moreover, we fully characterize all bipartite graphs that admit 4-colorings (Section 3.1) and 5-colorings (Sections 3.2 and 4.1).

2. Bounds on χ_{ii} for bipartite graphs

In this section, we construct some lower and upper bounds on interval incidence coloring number for bipartite graphs. Observe that $\chi_i \leq \chi_{ii}$ and hence any lower bound for χ_i is a lower bound for χ_{ii}.

Theorem 1. *For any nonempty bipartite graph G we have*

$$\Delta + 1 \leq \chi_i \leq \chi_{ii} \leq 2\Delta.$$

Proof. It is easy to prove that the first two inequalities hold for all nonempty graphs and therefore we omit this part of the proof.

To prove the right-hand side inequality, we divide the vertex set into 2 independent sets denoted by V_1 and V_2. We create a coloring c from I to $\mathbb{N}$ in the following way: if $v \in V_1$, then we assign colors to incidences at vertex v (i.e. of form (v, e)) in such a way that $A_c(v) = \{1, \ldots, \deg v\}$, and if $v \in V_2$, then we assign $A_c(v) = \{\Delta + 1, \ldots, \Delta + \deg v\}$. Hence we have $A_c(v) \cap A_c(w) = \emptyset$ for any $v \in V_1$ and $w \in V_2$, thus c is an interval incidence 2Δ-coloring of G. □

Let G be any regular bipartite graph of degree Δ. It is easy to observe, that in any interval incidence χ_{ii}-coloring c of G, there is at least one vertex v that is minimal, i.e. $\min A_c(v) = 1$. Moreover, there is a vertex u adjacent to v such that $c(v, \{v, u\}) = \Delta$, hence $\min A_c(u) > \Delta$, and by Theorem 1 we have

Theorem 2. $\chi_{ii}(G) = 2\Delta$ *for any regular bipartite graph G.* □

3. Polynomial time algorithm for subcubic bipartite graphs

In this section, we focus on the interval incidence coloring problem for subcubic bipartite graphs. Observe, that $\chi_{ii}(P_2) = 2$ and $\chi_{ii}(P_3) = \chi_{ii}(P_4) = 3$, where P_n is a n-vertex path. If G is a path with at least 5 vertices or a cycle, then it is easy to observe that $\chi_{ii}(G) = 4$. In the following, let G be a bipartite graph with $\Delta(G) = 3$. By Theorem 1 the interval incidence chromatic number $\chi_{ii}(G)$ is between 4 and 6. We construct an efficient algorithm for coloring such graphs with minimum number of colors (i.e. using 4, 5 or 6 colors).

3.1. Interval incidence 4-coloring of bipartite graphs with $\Delta = 3$

Lemma 1. *If $\chi_{ii}(G) = 4$ then*
 (i) *each vertex v of degree 3 has at most one neighbor of degree 3,*

(ii) *each vertex v of degree 3 has at least one neighbor of degree* 1,

(iii) *no two vertices of degree 3 have a common neighbor of degree* 2.

Proof. Suppose that $\chi_{ii}(G) = 4$ and let c be any interval incidence 4-coloring of G. Property (i) follows from the fact that in the set $\{1, 2, 3, 4\}$ one can find only two intervals of length 3, i.e. $\{1, 2, 3\}$ and $\{2, 3, 4\}$, and in each of them there is only one element not belonging to the other, which are needed for the incidences on edges between two vertices of degree 3. To prove property (ii) observe that if $A_c(v) = \{1, 2, 3\}$ ($A_c(v) = \{2, 3, 4\}$, respectively) then vertex u for which incidence $(v, \{v, u\})$ is colored with 3 (with 2, respectively) must be a leaf. Now, we prove property (iii). For $v \neq u$ let $\deg(v) = \deg(u) = 3$. Suppose conversely that there is vertex x of degree 2 that is a common neighbor of v and u. Obviously, $c(x, \{x, v\}) \notin \{2, 3\}$ and $c(x, \{x, u\}) \notin \{2, 3\}$. Because $\deg(x) = 2$, $A_c(x)$ is $\{1, 2\}$ or $\{2, 3\}$ or $\{3, 4\}$, hence $2 \in A_c(x)$ or $3 \in A_c(x)$, a contradiction. □

Lemma 2. *If G satisfies properties* (i)–(iii), *then $\chi_{ii}(G) = 4$.*

Proof. Suppose that G satisfies properties (i)–(iii). Because G is bipartite, there is a proper vertex coloring p of G that uses two colors a and b. Now, we construct an interval incidence coloring q of the graph G. Let us assign a set of colors to each vertex v as follows:

- if $p(v) = a$, then $1 \in A_q(v)$ (i.e. $A_q(v) \in \{\{1\}, \{1, 2\}, \{1, 2, 3\}\}$),
- if $p(v) = b$, then $4 \in A_q(v)$ (i.e. $A_q(v) \in \{\{4\}, \{3, 4\}, \{2, 3, 4\}\}$).

Let $\{u, v\} \in E(G)$. From the properties of the graph G and the coloring p we can distribute colors from the sets defined above among incidences as follows:

- let $\deg(u) = 3$ and $\deg(v) = 3$, if $p(u) = a$ and $p(v) = b$, then $q(u, \{u, v\}) = 1$ and $q(v, \{v, u\}) = 4$;
- let $\deg(u) = 2$ and $\deg(v) = 3$, if $p(u) = a$ ($p(u) = b$) and $p(v) = b$ ($p(v) = a$), then $q(u, \{u, v\}) = 1$ ($q(u, \{u, v\}) = 4$) and $q(v, \{v, u\}) \geq 3$ ($q(v, \{v, u\}) \leq 2$);
- let $\deg(u) = 2$ and $\deg(v) = 2$, if $p(u) = a$ and $p(v) = b$, then $q(u, \{u, v\}) \leq 2$ and $q(v, \{v, u\}) \geq 3$;
- let $\deg(u) = 1$ and $\deg(v) \leq 3$, if $p(u) = a$ ($p(u) = b$) and $p(v) = b$ ($p(v) = a$), then $q(u, \{u, v\}) = 1$ ($q(u, \{u, v\}) = 4$) and $q(v, \{v, u\}) = 2$ or $q(v, \{v, u\}) = 3$.

The above method guarantees that $q(u, \{u, v\}) \neq q(v, \{v, u\})$ for all adjacent vertices u, v (if $\deg(u) = \deg(v) = 3$ then it follows from property (i), if $\deg(u) = 2$ and $\deg(v) = 3$—from property (iii), $\deg(u) = \deg(v) = 2$—it follows obviously from the degree, $\deg(u) = 1$ and $\deg(v) \leq 3$—from property (ii)). Therefore q is a proper interval incidence 4-coloring of the graph G. □

By Lemmas 1 and 2 the problem of interval incidence 4-coloring for subcubic bipartite graphs is equivalent to verifying properties (i)–(iii), which can be done in linear time.

3.2. Interval incidence 5-coloring of bipartite graphs with $\Delta = 3$

By Theorem 1 for bipartite subcubic graphs we have $4 \leq \chi_{ii} \leq 6$. Obviously, every interval of length 3 included in $\{1, 2, 3, 4, 5\}$ contains 3, hence

Lemma 3. *If $\chi_{ii}(G) = 5$ then*

(iv) *each vertex v of degree 3 has at most two neighbors of degree* 3. □

Lemma 4. *If G satisfies property* (iv), *then $\chi_{ii}(G) \leq 5$.*

Proof. Suppose that G satisfies property (iv). Because G is bipartite, there is a coloring p of G that uses two colors a and b. Now, we construct an interval incidence coloring q of the graph G. Let us assign a set of colors to each vertex v as follows:

(c_1) if $p(v) = a$, then $1 \in A_q(v)$ (i.e. $A_q(v) \in \{\{1\}, \{1, 2\}, \{1, 2, 3\}\}$),
(c_2) if $p(v) = b$, then $5 \in A_q(v)$ (i.e. $A_q(v) \in \{\{5\}, \{4, 5\}, \{3, 4, 5\}\}$).

Let $\{u, v\} \in E(G)$. If $\deg(u) = \deg(v) = 3$, then we put colors satisfying $q(u, \{u, v\}) \leq 2$ and $q(v, \{v, u\}) \geq 4$ (it can be done since G satisfies property (iv)). If $\deg(u) \leq 3$ and $\deg(v) \leq 2$, then by properties (c_1) and (c_2) we have $A_q(v) \cap A_q(u) = \emptyset$. Hence, the rest of colors from sets $A_q(v)$ can be distributed in any way resulting in a proper interval incidence coloring using at most 5 colors. □

If $\chi_{ii}(G) > 4$, then by Lemmas 3 and 4 the problem of interval incidence 5-coloring is equivalent to verifying property (iv), which can be done in linear time.

3.3. Algorithm for interval incidence coloring of subcubic bipartite graphs

Suppose that G is a connected subcubic bipartite graph. If $\Delta(G) = 2$, then G is a cycle or a path, and this was discussed above. If $\Delta(G) = 3$, then we use the following algorithm that runs in linear time and is optimal.

Step 1: if G satisfies properties (i)–(iii), then by Lemma 2 color it with 4 colors.
Step 2: otherwise, by Lemma 1 we have $\chi_{ii}(G) > 4$, so if G satisfies property (iv), then by Lemma 4 color it with 5 colors.
Step 3: otherwise, by Lemma 3 we have $\chi_{ii}(G) > 5$, and by Theorem 1 it can be colored with 6 colors.

Theorem 3. *Finding a minimal interval incidence coloring of a subcubic bipartite graph can be done in linear time.* □

4. Coloring of bipartite graphs with $\Delta = 4$: 5 is easy, 6 is hard

In this section, we study the problem of interval incidence coloring for bipartite graphs with $\Delta = 4$. We prove that 5-coloring is easy, i.e. we fully characterize all graphs that admit a 5-coloring, and we prove that 6-coloring is hard ($\mathcal{NP}$-complete). In the following, let G be a bipartite graph with $\Delta(G) = 4$.

4.1. 5-coloring of bipartite graphs with $\Delta = 4$ in linear time

Lemma 5. *If $\chi_{ii}(G) = 5$ then*

  (i) *each vertex has at most one neighbor of degree 4;*
 (ii) *each vertex of degree at least 3 has at most two neighbors of degree at least 3;*
(iii) *each vertex of degree 4 has at least one neighbor of degree 1.*

Proof. Suppose that $\Delta(G) = 4$ and $\chi_{ii}(G) = 5$ and let c be any interval incidence 5-coloring of G. Observe, that for each vertex v of degree four we have $A_c(v) = \{1, 2, 3, 4\}$ or $A_c(v) = \{2, 3, 4, 5\}$. Hence if there is a vertex u with two neighbors of degree four, then $1 \in A_c(u)$ and $5 \in A_c(u)$, thus $A_c(u)$ is not an interval, a contradiction, so the property (i) follows. To prove property (ii) observe that $3 \in A_c(v)$ for each vertex v of degree at least three. Suppose conversely, that some vertex v of degree at least three has three neighbors of degree at least three. If $\deg(v) = 3$, then there is a vertex u of degree at least three adjacent to v, such that $c(v, \{v, u\}) = 3$, which contradicts $3 \in A_c(u)$. Let $\deg(v) = 4$ and assume that $A_c(v) = \{1, 2, 3, 4\}$ ($\{2, 3, 4, 5\}$ analogously). As before, there is no vertex u of degree at least three adjacent to v such that $c(v, \{v, u\}) = 3$, hence there is a vertex u of degree at least three adjacent to v such that $c(v, \{v, u\}) = 4$, but this implies that $A_c(u) = \{1, 2, 3\} \subset A_c(v)$, a contradiction. Now, we prove (iii). Let $\deg(v) = 4$ and assume that $A_c(v) = \{1, 2, 3, 4\}$ ($\{2, 3, 4, 5\}$ analogously). Suppose conversely that each neighbor of v has degree at least two. Observe that for each vertex u adjacent to v we have $c(u, \{u, v\}) = 5$. Consider vertex u adjacent to v such that $c(v, \{v, u\}) = 4$, hence $5 \in A_c(u)$, $4 \notin A_c(u)$ and $\deg(u) \geq 2$, a contradiction. □

Lemma 6. *If G satisfies properties (i)–(iii), then $\chi_{ii}(G) = 5$.*

Proof. Suppose that G satisfies properties (i)–(iii). Because G is bipartite, there is a coloring p of G that uses two colors a and b. We construct an interval incidence coloring q of graph G. Let us assign a set of colors to each vertex v as follows:

(c_1) if $p(v) = a$, then $1 \in A_q(v)$ (i.e. $A_q(v) \in \{\{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$);
(c_2) if $p(v) = b$, then $5 \in A_q(v)$ (i.e. $A_q(v) \in \{\{5\}, \{4, 5\}, \{3, 4, 5\}, \{2, 3, 4, 5\}\}$).

Let $v \in V(G)$ and assume that $p(v) = a$ ($p(v) = b$ analogously). Now, let $u_1, \ldots, u_{\deg(v)}$ be a sequence of vertices adjacent to v ordered with non-increasing degrees, then by properties (i)–(iii) we have $\deg(u_i) \leq 5 - i$ (for $i = 1, \ldots, \deg(v)$), hence let $c(v, \{v, u_i\}) = i$, for $i = 1, \ldots, \deg(v)$. Because $p(u_i) = b$, then $5 \in A_q(u_i)$, and by $\deg(u_i) \leq 5 - i$ we have $c(v, \{v, u_i\}) = i \leq 5 - \deg(u_i) < \min A_q(u_i)$, hence q is an interval incidence coloring of G using at most 5 colors. □

By Lemmas 5 and 6 the problem of interval incidence 5-coloring for bipartite graphs with $\Delta = 4$ is equivalent to verifying properties (i)–(iii), which can be done in linear time.

4.2. 6-coloring of bipartite graphs with $\Delta = 4$ is $\mathcal{NP}$-complete

Let us denote by $\overline{3SAT}$ the restriction of the classical $3SAT$ problem, defined as follows: $\overline{3SAT}$ is the problem of satisfiability of a given CNF formula with 2 or 3 literals in each clause and satisfying the condition that for any variable x the total number of clauses with literals x or $\neg x$ is not more than 3. Moreover, we may assume that for each variable both x and $\neg x$ appear in the formula. This problem is known to be $\mathcal{NP}$-complete [10].

Theorem 4. *The problem of deciding whether $\chi_{ii} \leq 6$ is $\mathcal{NP}$-complete for bipartite graphs with maximum degree $\Delta = 4$.*

Proof. We construct a polynomial time reduction from $\overline{3SAT}$ to the problem of interval incidence 6-coloring. For a given formula $\phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ of $\overline{3SAT}$, we construct a bipartite graph $G(\phi)$ with $\Delta = 4$ as follows:

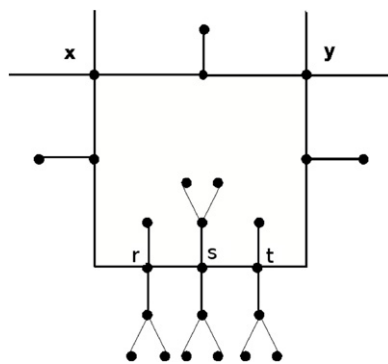● each clause $\{x \vee y\}$ is represented by a gadget T_2 (Fig. 1);

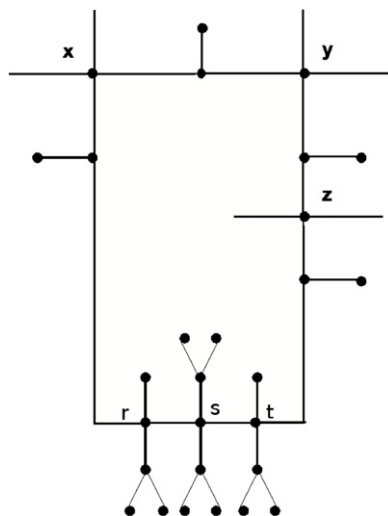**Fig. 1.** Gadget T_2 for clause $\{x \vee y\}$.

Fig. 2. Gadget T_3 for clause $\{x \vee y \vee z\}$.

- each clause $\{x \vee y \vee z\}$ is represented by a gadget T_3 (Fig. 2);
- each literal is represented by vertex x or y in T_2, or x, y or z in T_3;
- for each variable x we join each literal x with $\neg x$ by a gadget D (Fig. 3);
- let x be any variable, then if in formula ϕ there is
  – one literal x and two literals $\neg x$, then we attach a pendant vertex to each vertex representing $\neg x$;
  – two literals x and one literal $\neg x$, then we attach a pendant vertex to each vertex representing x;
  – one literal x and one literal $\neg x$, then we attach a pendant vertex to each vertex representing x and $\neg x$.

Note that in fact we take isomorphic copies of graphs T_2, T_3 and D for each clause or pair $\{x, \neg x\}$ of literals. Because each gadget is bipartite, and each two vertices representing literals x and $\neg x$ belong to the same partition, it is easy to observe that $G(\phi)$ is bipartite. Because for any variable x the total number of clauses with literals x or $\neg x$ is not more than 3, the graph $G(\phi)$ has maximum degree equal to 4. Moreover, each vertex representing a literal is of degree 4.

In the following we use the simplified notation G instead of $G(\phi)$. Let c be any interval incidence 6-coloring of G such that incidences at pendant vertices are colored with 1 or 6 (by recoloring it is always possible to get such a coloring). We define an auxiliary vertex labeling p of G with labels a, a' and b. For each vertex v we define $p(v)$ as follows: if $1 \in A_c(v)$, then $p(v) = a$, if $6 \in A_c(v)$, then $p(v) = a'$, otherwise $p(v) = b$. If $\deg(v) = 4$, then labels a, a', b correspond to sets $\{1, 2, 3, 4\}$, $\{3, 4, 5, 6\}$, $\{2, 3, 4, 5\}$, respectively. Thus we get an easy

Proposition 7. *For every vertex v of degree 4 we have $\{3, 4\} \subset A_c(v)$.*   □

Lemma 8. *The labeling p is a vertex coloring of $G(\phi)$ and the following holds:*

(i) *each vertex of degree ≤ 2 is assigned label a or a';*
(ii) *each vertex of degree 3 with two neighbors of degree 4 is assigned label a or a';*
(iii) *each vertex of degree ≥ 2 has at most one neighbor of degree 4 that is assigned label b;*
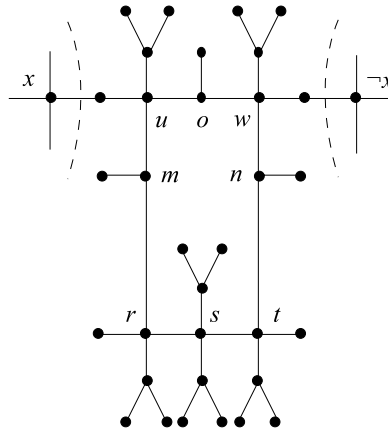
Fig. 3. Gadget D joining any pair of literals: x and $\neg x$.

(iv) *each vertex of degree 4 with neighbors, each of degree at least 3, must be labeled with b;*

(v) *for each vertex v of degree 4 with neighbor set: two vertices u_1 and u_2 of degree 4, and two vertices w_1 and w_2 of degree 3, we have:*

    (a) *u_1 is labeled with a, and u_2 is labeled with a', or conversely;*

    (b) *w_1 is labeled with a, and w_2 is labeled with a', or conversely.*

Proof. Because there are no two adjacent vertices, each of degree two or three, it is easy to observe that any two adjacent vertices have different labels, thus p is a vertex coloring. Property (i) obviously holds for pendant vertices (by assumption). Because each vertex v of degree 2 has two neighbors of degree 4, by Proposition 7 we have $A_c(v) = \{1, 2\}$ or $\{5, 6\}$, thus we get property (i). Suppose that vertex v of degree 3 has two neighbors of degree 4. Analogously, by Proposition 7 we have $\{1, 2\} \subset A_c(v)$ or $\{5, 6\} \subset A_c(v)$, hence property (ii) follows. To prove property (iii) suppose conversely that two vertices of degree 4 adjacent to vertex v ($\deg(v) \geq 2$) have label b, i.e. the color set of the incidences is $\{2, 3, 4, 5\}$. Then $1 \in A_c(v)$ and $6 \in A_c(v)$, a contradiction, thus property (iii) holds. To prove property (iv) suppose conversely, that vertex v of degree 4 is assigned label a (a' analogously), i.e. $A_c(v) = \{1, 2, 3, 4\}$. Let u be the vertex adjacent to v, such that $c(v, \{v, u\}) = 4$. By assumption $\deg(u) \geq 3$, so $A_c(u) = \{1, 2, 3\} \subset A_c(v)$, a contradiction, hence vertex v is labeled with b. Now, we prove property (v). By property (iv) vertex v is labeled with b ($A_c(v) = \{2, 3, 4, 5\}$). By Proposition 7 there is $c(v, \{v, u_1\}) = 2$ and $c(v, \{v, u_2\}) = 5$, or conversely. But this implies $c(u_1, \{u_1, v\}) = 1$ and $c(u_2, \{u_2, v\}) = 6$ (or conversely), hence we get property (a). Observe that two remaining colors 3 and 4 from $A_c(v)$ are used to color incidences $(v, \{v, w_1\})$ and $(v, \{v, w_2\})$, hence property (b) follows. □

As a consequence of Lemma 8 we have the two following lemmas.

Lemma 9. *For a gadget D joining two literals x and $\neg x$ we have:*

  (i) *exactly one vertex among u and w is labeled with b;*

 (ii) *if vertex u (w) is labeled with b, then vertex o is labeled with a and vertex m (n) is labeled with a', or conversely;*

(iii) *if vertex x ($\neg x$) is labeled with b, then vertex $\neg x$ (x) is labeled with a or a'.*

Proof. Consider gadget D shown in Fig. 3. Suppose that vertices u and w are labeled with a or a'. By Lemma 8(ii) vertices m, n and o are labeled with a or a'. By Lemma 8(iv) vertex s is labeled with b, and by Lemma 8(v) vertex r is labeled with a and vertex t is labeled with a', or conversely. Hence there is a path of even length (i.e. even number of edges) consisting of vertices labeled with a and a' only, with end vertices (r and t) labeled with a and a', a contradiction. By Lemma 8(iii) at least one from vertices u and w must be labeled with a or a', thus the property (i) follows. Suppose that vertex u is labeled with b. By Lemma 8(iv) vertex s is labeled with b. By Lemma 8(ii) vertices m, n and o are labeled with a or a'. Let us assume that r is labeled with a and t is labeled with a', then m is labeled with a' and n is labeled with a. Observe that o is labeled with a, otherwise vertex w has two neighbors labeled with a and a' (which would imply that w is labeled b, contradicting Lemma 8(iii) at vertex o), thus property (ii) follows. If x is labeled with b, then by Lemma 8(iii) u is labeled with a or a', hence by property (i) vertex w is labeled with b and by Lemma 8(iii) vertex $\neg x$ is labeled with a or a', thus we get (iii). □

Lemma 10. *In each gadget representing a clause there is at least one vertex representing a literal that is assigned label b, more precisely:*

 (i) *in each gadget T_2 exactly one vertex from $\{x, y\}$ is labeled with b;*

(ii) *in each gadget T_3 at least one vertex from $\{x, y, z\}$ is labeled with b.*

Proof. Consider graph T_2 shown in Fig. 1. Suppose conversely that x and y are not labeled with b. By Lemma 8(iv) the only one vertex (say s) of degree 4, with two neighbors of degree 3 and two neighbors of degree 4 must be labeled with b. By assumption and Lemma 8(ii) each vertex of degree 3 from the cycle of even length containing vertex s is labeled with a or a'. By Lemma 8(v) both neighbors of s from this cycle are labeled with different labels (a or a'), hence there is a path of even length consisting of vertices labeled with a and a', with end vertices labeled with a and a', a contradiction. Thus vertex x or y is labeled with b. Moreover, by Lemma 8(iii) exactly one from them is labeled with b.

Consider gadget T_3 shown in Fig. 2. Analogously like in the proof for gadget T_2, at least one of vertices from set $\{x, y, z\}$ must be labeled with b. $\square$

Now, we show that having a proper labeling (with a, a', b) of vertices of $G(\phi)$ one can construct an interval incidence 6-coloring of $G(\phi)$.

Lemma 11. *Consider any labeling p of the graph $G(\phi)$ satisfying all properties from Lemma 8 and the following condition:*

(1) *any gadget T_2 or T_3 has exactly one vertex corresponding to a literal (i.e. x or y in T_2, and x or y or z in T_3) that is labeled with b.*

Then there is an interval incidence 6-coloring c of $G(\phi)$.

Proof. We construct an interval incidence 6-coloring c satisfying the property: for any vertex v we have $1 \in A_c(v)$ iff $p(v) = a$ and $6 \in A_c(v)$ iff $p(v) = a'$.

Case 1: by Lemma 8(i) we can color the incidence at a vertex of degree 1 with color 1 (if it is labeled with a) or 6 (if it is labeled with a');

Case 2: take any vertex v of degree 2, by Lemma 8(i) we can assume that it is labeled with a (a' analogously). By Lemma 8(iii) the neighbors of v must be labeled both with a' or one with a' and the second one with b, hence it is easy to color both incidences at v with colors from $\{1, 2\}$;

Case 3: let v be a vertex of degree 3, then, by construction of $G(\phi)$, one of the following holds:

(x_1) v has two neighbors of degree 4 and one neighbor of degree 1;
(x_2) v has one neighbor of degree 4 and two neighbors of degree 1.

In the first case (x_1) by Lemma 8(ii) vertex v is labeled with a' (a analogously). By Lemma 8(i) and (iii) the sequence of labels assigned to all neighbors of v is a, a, b or a, a, a. Hence it is easy to color all incidences at vertex v with colors 4, 5, 6, starting from $c(v, \{v, u\}) = 6$ (if $p(u) = b$). In the second case (x_2) if vertex v is labeled with a or a', we color analogously. If vertex v is labeled with b, then, if the vertex u of degree 4 adjacent to v is labeled with a (a' analogously), we color $c(v, \{v, u\}) = 5$, and the rest of the incidences of v with 3 and 4;

Case 4: let v be a vertex of degree 4, hence, by construction of $G(\phi)$, one of the following holds:

(y_1) v has two neighbors of degree 4 and two neighbors of degree 3;
(y_2) v has one neighbor of degree 4, two neighbors of degree 3 and one neighbor of degree 1;
(y_3) v has three neighbors of degree 3 and one neighbor of degree 2;
(y_4) v has two neighbors of degree 3, one neighbor of degree 2 and one neighbor of degree 1 or 2.

In the case (y_1) by Lemma 8(iv) vertex v (corresponding to vertex s in all figures) is labeled with b. Let u_1, u_2, w_1, w_2 be a sequence of vertices adjacent to v ordered with non-increasing degrees. By Lemma 8(v) we have $p(u_1) = a$ and $p(u_2) = a'$ (or conversely), and $p(w_1) = a$ and $p(w_2) = a'$ (or conversely), hence we can color the incidences as follows: $c(v, \{v, u_1\}) = 5, c(v, \{v, u_2\}) = 2, c(v, \{v, w_1\}) = 4$ and $c(v, \{v, w_2\}) = 3$.

In the case (y_2) the considered vertex v (corresponding to s or t in all figures) is adjacent to a vertex u of degree 4 described in the case (y_1), hence it is labeled with a or a', and u is labeled with b. Let us assume that v is labeled with a (a' analogously). One vertex (say w_1) of degree 3 adjacent to v fulfills (x_1) hence by Lemma 8(ii) it is labeled with a', the second one (w_2) is labeled with a' or b, and by Lemma 8(i) the pendant vertex (x) is labeled with a'. Hence we color the incidences as follows: $c(v, \{v, u\}) = 1, c(v, \{v, w_1\}) = 3, c(v, \{v, w_2\}) = 2$ and $c(v, \{v, x\}) = 4$. The resulting coloring is feasible with the coloring of w_2 in both cases (w_2 colored a' or b).

In the case (y_3) the considered vertex v is equal to one of the vertices u or w from gadget D shown in Fig. 3, let us assume that $v = u$. If v is labeled with a (a' analogously) then it is easy to color all incidences starting from $c(v, \{v, z\}) = 4$, where z is the only vertex of degree 2 adjacent to v. If v is labeled with b, then by Lemma 9(ii) m is labeled with a and o is labeled with a', or conversely. We color then $c(v, \{v, o\}) = 2$ and $c(v, \{v, m\}) = 5$. If the third neighbor of degree 3 of v, say g, is labeled with a', then $c(v, \{v, g\}) = 3$, otherwise $c(v, \{v, g\}) = 4$. Incidence $(v, \{v, h\})$, where vertex h has degree 2, can be colored with 3 or 4 by Lemma 8(i).

In the case (y_4) the considered vertex v corresponds to one of the literals (in T_2 or T_3). By Lemma 8(i) and (ii) all neighbors of v are labeled with a or a'. If v is labeled with a (a' analogously), then it is easy to color all incidences starting from $c(v, \{v, z\}) = 4$, where z is any neighbor (of v) of degree at most 2. Let v be labeled with b. Consider the only cycle in our gadget. By Lemma 8(ii) all vertices of degree 3 in the cycle are labeled with a or a'. Moreover, vertices r and t have labels a, a' or a', a (by Lemma 8(v)). Hence, by assumption (1) and by Lemma 8(iv), there are exactly 2 vertices labeled with b in the cycle (s is one of them). Since the cycle is even, both neighbors of v of degree 3 are labeled with different labels a and a'. So, we color the corresponding incidences with color 5 and 2, respectively. We color the rest of the incidences easily with 3 and 4. $\square$

Now, we will show that the formula ϕ is satisfiable if and only if the graph $G(\phi)$ admits an interval incidence 6-coloring.

($\Rightarrow$) Assume that the formula ϕ is satisfiable and let w be an assignment of values TRUE and FALSE to each variable such that $\phi(w)$ is TRUE, i.e. each clause contains at least one literal x for which $w(x) =$ TRUE. We construct a labeling p as follows: for each clause take exactly one literal with value TRUE and label a corresponding vertex in the graph T_2 (x or y) or T_3 (x, y or z) with b. This labeling satisfies condition (1) from Lemma 11. Now, take a gadget T_3 representing a clause with three literals (T_2 analogously), and label the only vertex of degree 4 with all neighbors of degree at least 3 (according to Lemma 8(iv)) with b and label its neighbors with a and a', according to Lemma 8(v). Because there are only two vertices from the cycle labeled with b, then one can easily color the rest of the vertices from the even cycle with colors a and a'. The remaining (uncolored) part of a gadget forms a forest and it can be easily colored with 2 labels: a and a'. It is easy to observe that this labeling satisfies all the conditions from Lemma 8. Now, we label vertices of each gadget D. Observe that one of the following holds:

(1) both x and $\neg x$ are labeled with a or both are labeled with a';
(2) x is labeled with a and $\neg x$ is labeled with a', or conversely;
(3) x is labeled with b and $\neg x$ is labeled with a or a', or conversely.

In the case (1) assume that x and $\neg x$ are labeled with a. We label vertices of D as follows: vertices s and w with b; vertices t, m and o with a'; and vertices r, n and u with a. It is easy to observe that one can label the rest of a gadget D with a and a', according to Lemma 8(i)–(iii) and (v).

In the case (2) assume that x is labeled with a and $\neg x$ is labeled with a'. We label vertices of D in the same manner as in the case (1).

In the case (3) assume that x is labeled with b and $\neg x$ is labeled with a. We label vertices of D in the same manner as in the case (1).

Now, observe that the constructed labeling fulfills all properties from Lemma 8 and condition (1) from Lemma 11. By Lemma 11 there is an interval incidence 6-coloring of $G(\phi)$.

($\Leftarrow$) Assume that there is an interval incidence 6-coloring c of graph $G(\phi)$. For each vertex v we define $p(v)$ as follows: $p(v) = a$ if $1 \in A_c(v)$, $p(v) = a'$ if $6 \in A_c(v)$ and $p(v) = b$ otherwise. From each clause corresponding with gadget T_2 or T_3 by Lemma 10 we can choose a vertex labeled with b, and we put TRUE value for a literal corresponding to this vertex. By Lemma 9 this assignment is legal, hence the value of the formula is TRUE.   □

5. Polynomial time algorithm for trees

In this section, we construct an $O(n\Delta^{3.5} \log \Delta)$ time algorithm for the optimal interval incidence coloring of trees, using a bottom-up technique and an algorithm for finding min-weight perfect matchings in complete bipartite graphs.

5.1. Sketch of the algorithm

We color the incidences of a given tree T using the bottom-up technique in accordance to the defined orientation of T. We orient all edges of T in an *in-tree* manner with a leaf as root, i.e. we choose any leaf r as root and we orient the edges of the tree towards the root r. As a result, for each vertex $v \in V(T) \setminus \{r\}$, there is exactly one oriented edge outcoming from v (towards r), let us denote this edge by $e_v = \{v, u\}$. By T_v we denote the subtree of T rooted at v and consisting of all (oriented) edges that lead to vertex v. By T_v^* we mean the tree obtained from T_v by adding the edge e_v, i.e. $T_v^* = T_v \cup e_v$. Let $p = \deg(v) - 1$ and $N_v^- = \{v_1, \ldots, v_p\}$ be the set of vertices adjacent to v and different from u.

Now, for each vertex v we define two matrices, namely A_v and B_v.

(A_v) : for each a, b, let $A_v[a, b]$ be the maximum color used in a minimal interval incidence coloring of T_v^* assuming that the incidence (v, e_v) is colored with a and incidence (u, e_v) is colored with b. If $A_v[a, b]$ cannot be legally defined we initially set the value as ∞;

(B_v) : for each c and l, let $B_v[c, l]$ be the maximum color used in a minimal interval incidence coloring of T_v^* assuming that the incidence (v, e_v) is colored with $l + c$ and the minimal color of an incidence at vertex v is l, i.e. the set of colors of the incidences at vertex v is $I_v(l) = \{l, \ldots, l + \deg(v) - 1\}$.

By Theorem 1 for each tree we have $\chi_{ii} \leq 2\Delta$, hence we may assume that $a, b \in \{1, \ldots, 2\Delta\}$, $c \in \{0, \ldots, \deg(v) - 1\}$ and $l \in \{1, \ldots, 2\Delta - \deg(v) + 1\}$. Therefore the size of both matrices can be bounded by 2Δ.

The main idea of our algorithm uses a *bottom-up* technique, as follows:

1. starting from leaves go towards the root r;
2. traversing tree T for each vertex v:
   (i) construct matrix B_v using $A_{v_1}, \ldots, A_{v_p}$;
   (ii) construct matrix A_v from B_v.
3. return $\chi_{ii}(T) = \min_{i,j=1,\ldots,2\Delta} A_s[i, j]$, where s is the only neighbor of r.

We show that the time complexity of the above algorithm can be bounded by $O(\sum_{v \in V(T)} (\deg(v)\Delta^{3.5} \log \Delta)) = O(n\Delta^{3.5} \log \Delta)$. Constructing an interval incidence coloring of T with $\chi_{ii}(T)$ colors is possible by using additional data

structures for remembering the chosen pairs of colors for adjacent incidences while building matrices and does not change the time complexity of the algorithm.

5.2. Constructing matrix B_v

Consider $v \in V(T)$, and let $l \in \{1, \ldots, 2\Delta - p + 1\}$ and $c \in \{0, \ldots, p - 1\}$, where $p = \deg(v) - 1$. Let u be the neighbor of v that is closer to vertex r than v, and let $\{v_1, \ldots, v_p\}$ be the set of neighbors of v for which we calculated matrices $A_{v_1}, \ldots, A_{v_p}$. If $\deg(v) = 1$ then $c = 0$ and we put $B_v[0, l] = \max\{l, 2\}$. If $\deg(v) > 1$ then we build a weighted complete bipartite graph $H_v(c, l)$ with partitions N_v^- and $I_v(l) \setminus \{l + c\}$, and weight function $w_l: E(H_v(c, l)) \to \mathbb{N}$ defined as follows: $w_l(\{x, y\}) = \min_{x \in \{1, \ldots, 2\Delta\} \setminus I_v(l)} A_{v_i}[x, y]$, which corresponds to the minimal interval incidence coloring of $T_{v_i}^*$, with precoloring of incidences $(v_i, \{v_i, v\})$ and $(v, \{v, v_i\})$ with colors x and y, respectively. Observe that $|N_v^-| = |I_v(l) \setminus \{l + c\}| = p$.

Now, we show how to efficiently calculate values of w_l for fixed y and for all l satisfying $y \in I_v(l)$. For vertex v_i let us define the auxiliary matrices $W_{v_i}^-[l, y]$ and $W_{v_i}^+[l, y]$ as follows:

- $W_{v_i}^-[l, y] = \min_{x \in \{1, \ldots, l-1\}} A_{v_i}[x, y]$;
- $W_{v_i}^+[l, y] = \min_{x \in \{l+\deg(v), \ldots, 2\Delta\}} A_{v_i}[x, y]$.

Obviously we have $w_l(\{v_i, y\}) = \min\{W_{v_i}^-[l, y], W_{v_i}^+[l, y]\}$. To speed up the calculations let us observe the following easy recursive properties:

- $W_{v_i}^-[l + 1, y] = \min\{W_{v_i}^-[l, y], A_{v_i}[l, y]\}$, for $l > 1$; and $W_{v_i}^-[1, y] = \infty$;
- $W_{v_i}^+[l - 1, y] = \min\{W_{v_i}^+[l, y], A_{v_i}[l, y]\}$, for $l < 2\Delta$; and $W_{v_i}^+[2\Delta, y] = \infty$.

Hence, we can construct matrices $W_{v_i}^-$ and $W_{v_i}^+$ in advance, so the time complexity of constructing the graph $H_v(c, l)$ can be bounded by $O(\Delta \deg(v))$.

Let M be a min-weight perfect matching of weight m in the graph $H_v(c, l)$, i.e. the maximum weight in a perfect matching is minimized. Let us color the incidence $(v, \{v, u\})$ with $l + c$, and the rest of incidences at vertex v according to the perfect matching, i.e. if $\{v_i, y\}$ belongs to the perfect matching, then color incidence $(v, \{v, v_i\})$ with y. Let us color the incidence $(u, \{u, v\})$ with the minimum possible color. Because the perfect matching M is min-weight, it guarantees a minimal coloring of $T_v^* = T_{v_1}^* \cup \cdots \cup T_{v_p}^* \cup \{v, u\}$, under the assumption that the minimum color at v is l and that the incidence $(v, \{v, u\})$ is colored with $l + c$. Thus we can put $B_v[c, l] = \min\{m, l + c\}$.

Using Hopcroft–Karp algorithm [15] for finding a perfect matching in bipartite graphs (it runs in $O(|E|\sqrt{|V|})$ time) we get an algorithm for finding a min-weight perfect matching in the graph $H_v(c, l)$ in $O(\deg^{2.5}(v) \log \Delta)$ time, just using bisection:

1. let $H_v^d(c, l)$ be the subgraph of $H_v(c, l)$ that contains all edges e of $H_v(c, l)$ with weight $w_l(e) \leq d$;
2. use bisection on the interval $\{1, \ldots, 2\Delta\}$ to find the minimal d such that $H_v^d(c, l)$ has a perfect matching.

Hence, the time complexity of constructing the matrix B_v can be bounded by $O(\deg(v)\Delta^{3.5} \log \Delta)$.

5.3. Constructing matrix A_v

By definition of A_v and B_v observe that the following holds

- $A_v[a, b] = \min\{\infty\} \cup \{\max\{B_v[c, l], b\}: a = l + c \wedge b \notin I_v(l)\}$.

Because $I_v(l)$ is an interval of integers this computation can be done in $O(\deg(v))$ time, hence constructing matrix A_v can be done in $O(\deg(v)\Delta^2)$ time. Thus the complexity of the algorithm is $O(\sum_{v \in V(T)}(\deg(v)\Delta^{3.5} \log \Delta))$ which gives $O(n\Delta^{3.5} \log \Delta)$.

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