

## THE PAIRED-DOMINATION AND THE UPPER PAIRED-DOMINATION NUMBERS OF GRAPHS

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**Abstract.** In this paper we continue the study of paired-domination in graphs. A paired-dominating set, abbreviated PDS, of a graph  $G$  with no isolated vertex is a dominating set of vertices whose induced subgraph has a perfect matching. The paired-domination number of  $G$ , denoted by  $\gamma_p(G)$ , is the minimum cardinality of a PDS of  $G$ . The upper paired-domination number of  $G$ , denoted by  $\Gamma_p(G)$ , is the maximum cardinality of a minimal PDS of  $G$ . Let  $G$  be a connected graph of order  $n \geq 3$ . Haynes and Slater in [*Paired-domination in graphs*, Networks 32 (1998), 199–206], showed that  $\gamma_p(G) \leq n - 1$  and they determine the extremal graphs  $G$  achieving this bound. In this paper we obtain analogous results for  $\Gamma_p(G)$ . Dorbec, Henning and McCoy in [*Upper total domination versus upper paired-domination*, Quaestiones Mathematicae 30 (2007), 1–12] determine  $\Gamma_p(P_n)$ , instead in this paper we determine  $\Gamma_p(C_n)$ . Moreover, we describe some families of graphs  $G$  for which the equality  $\gamma_p(G) = \Gamma_p(G)$  holds.

**Keywords:** paired-domination, paired-domination number, upper paired-domination number.

**Mathematics Subject Classification:** 05C69.

### 1. INTRODUCTION

Domination and its variations in graphs are now well studied. The literature on this subject has been surveyed and detailed in the two books by Haynes et al. [5, 6]. Paired-domination in graphs was introduced by Haynes and Slater [7] as a model for assigning backups to guard for security purposes. This concept of domination is well studied (see [1–4, 8–10]).

Let  $G = (V, E)$  be a graph which is finite, undirected, without loops, multiple edges and isolated vertices. The number of vertices of  $G$  is called the *order* of  $G$  and is denoted by  $n(G)$ . When there is no confusion we use the abbreviation  $n(G) = n$ . Let  $H$  be a connected graph. Then we denote by  $mH$ ,  $m \geq 1$ , the graph consisting of

$m$  components  $H_1, \dots, H_m$ , where  $H_i = H$  for  $i = 1, \dots, m$ . A *matching* in a graph  $G$  is a set of independent edges in  $G$ . A *perfect matching*  $M$  in  $G$  is a matching in  $G$  such that every vertex of  $G$  is incident to an edge of  $M$ . A *paired-dominating set*, abbreviated PDS, of a graph  $G$  is a set  $S = \{u_1, \dots, u_t, v_1, \dots, v_t\}$  of vertices of  $G$  such that every vertex is adjacent to some vertex in  $S$  and the subgraph  $\langle S \rangle$  induced by  $S$  contains a perfect matching  $M = \{e_1, \dots, e_t\}$ , where  $e_i = u_i v_i$  for  $i = 1, \dots, t$ . Two vertices  $u_i$  and  $v_i$  joined by an edge of  $M$  are said to be *paired*. Let  $S_p$  be the set of paired vertices in  $S$ , that is  $S_p = \{\{u_i, v_i\} \mid i = 1, \dots, t\}$ . The *paired-domination number* of  $G$ , denoted by  $\gamma_p(G)$ , is the minimum cardinality of a PDS. A PDS  $S$  of  $G$  is *minimal* if no proper subset of  $S$  is a PDS of  $G$ . The *upper paired-domination number* of  $G$ , denoted by  $\Gamma_p(G)$ , is the maximum cardinality of a minimal PDS. A minimal PDS of  $G$  of cardinality  $\Gamma_p(G)$  we call a  $\Gamma_p(G)$ -set.

## 2. GRAPHS WITH EQUAL $\gamma_p$ AND $\Gamma_p$

The aim of this section is describing graphs  $G$  for which  $\gamma_p(G) = \Gamma_p(G) = n - i$  for  $i = 0, 1, 2$ .

We start from the following statement.

**Observation 2.1.** *For a graph  $G$ ,  $\Gamma_p(G) = n$  if and only if  $G$  is  $mK_2$ .*

*Proof.* Obviously,  $\Gamma_p(mK_2) = 2m = n$ , since for  $G = mK_2$  the unique PDS of  $G$  is  $V(G)$ .

Now, suppose that  $\Gamma_p(G) = n$  and  $G \neq mK_2$ . Then,  $n$  is even and all the vertices of  $G$  are paired in  $S_p$ . Since  $G \neq mK_2$ , without loss of generality we may assume that vertex  $u_j$  is adjacent to vertex  $v_k$ , where  $j \neq k$ . But then the vertices of  $V(G) - \{v_j, u_k\}$  form a paired-dominating set, which is a contradiction with minimality of  $S = V(G)$ .  $\square$

The *subdivided star*  $K_{1,t}^*$  is the graph obtained from a star  $K_{1,t}$  by subdividing every edge once. In [7] we have the following notation and statements. Let  $\mathcal{F}$  be the collection of graphs  $C_3, C_5$  and the subdivided stars  $K_{1,t}^*$ .

**Theorem 2.2** ([7]). *If  $G$  is a connected graph of order  $n \geq 3$ , then  $\gamma_p(G) \leq n - 1$ . Furthermore  $\gamma_p(G) = n - 1$  if and only if  $G \in \mathcal{F}$ .*

We can reformulate Corollary 8 of [7] and then we obtain the following statement.

**Corollary 2.3.** *Let  $G$  be a graph with  $n \geq 3$ . Then  $\gamma_p(G) = n - 1$  if and only if  $G = H \cup rK_2$ , where  $H \in \mathcal{F}$  and  $r \geq 0$ .*

Let  $K_{1,t}^{*\Delta}$  be the graph obtained by attaching zero or more triangles to the central vertex of  $K_{1,t}^*$  (see Figure 1). Now let  $\mathcal{F}^\Delta = \{C_3, C_5, K_{1,t}^{*\Delta}\}$ .

Now we establish a bound on  $\Gamma_p(G)$  for connected graphs  $G$ . Moreover, we determine extremal graphs achieving this bound.

**Theorem 2.4.** *If  $G$  is a connected graph of order  $n \geq 3$ , then  $\Gamma_p(G) \leq n - 1$ . Furthermore,  $\Gamma_p(G) = n - 1$  if and only if  $G \in \mathcal{F}^\Delta$ .*



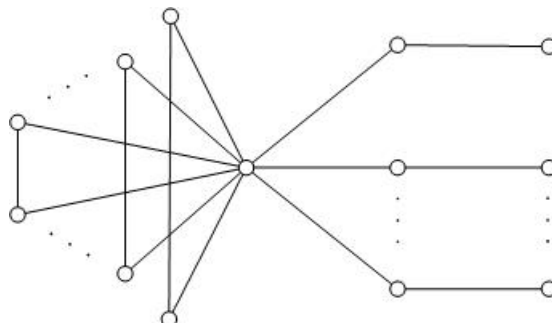


Fig. 1. The graph  $K_{1,t}^{\Delta}$ .

*Proof.* Since  $G$  is a connected graph with  $n \geq 3$ , by Observation 2.1 we have that  $\Gamma_p(G) \leq n - 1$ . It is easy to see that  $\Gamma_p(C_3) = 2$ ,  $\Gamma_p(C_5) = 4$  and  $\Gamma_p(K_{1,t}^{\Delta}) = n - 1$ , and so  $\Gamma_p(G) = n - 1$  for  $G \in \mathcal{F}^{\Delta}$ .

Now assume that  $G$  is a connected graph with  $n \geq 3$  such that  $\Gamma_p(G) = n - 1$ . Let  $S$  be a  $\Gamma_p(G)$ -set and let  $V - S = \{x\}$ . Since  $S$  dominates  $G$ ,  $x$  has at least one neighbour in  $S$ , say  $u_1$ . If  $\Gamma_p(G) = 2$ , then  $G$  is either  $P_3 = K_{1,1}^*$  or  $C_3$ , so  $G \in \mathcal{F} \subseteq \mathcal{F}^{\Delta}$ . Thus we may assume that  $\Gamma_p(G) \geq 4$ . Now we state that  $S - \{u_1, v_1\}$  induces an independent set of edges. Let us assume that there is on the contrary. Then without loss of generality we may suppose that vertex  $u_i$  is adjacent to vertex  $v_k$ , where  $2 \leq i < k$ . It follows that  $S - \{v_i, u_k\}$  is a PDS of  $G$  with  $S_p - \{\{u_i, v_i\}, \{u_k, v_k\}\} \cup \{\{u_i, v_k\}\}$  as a set of paired vertices, that contradicts the minimality of  $S$ . Furthermore, if the pair  $\{u_i, v_i\} \in S_p - \{\{u_1, v_1\}\}$  has a common neighbour in  $S$ , then  $S - \{u_i, v_i\}$  is a PDS, contradicting the minimality of  $S$ . Suppose that  $u_1$  is adjacent to  $u_i$ , where  $i \geq 2$ . Then,  $S_p - \{\{u_1, v_1\}, \{u_i, v_i\}\} \cup \{\{u_1, u_i\}\}$  is a set of paired vertices for a PDS which is  $S - \{v_1, v_i\}$ , again contradicting the minimality of  $S$ . Hence  $N(u_1) = \{x, v_1\}$ . By connectivity, exactly one vertex from each pair  $\{u_i, v_i\} \in S_p - \{u_1, v_1\}$  must be adjacent to  $v_1$  or vertices from  $\{u_i, v_i\}$  must be adjacent to  $x$ .

Now assume that  $v_1$  is adjacent to  $u_i$  for  $i \geq 2$ . If  $N(x) \cap (S - \{u_1, v_i\}) \neq \emptyset$ , then the vertices in the pairs of  $S_p - \{\{u_1, v_1\}, \{u_i, v_i\}\} \cup \{\{u_i, v_1\}\}$  form a PDS of  $G$ , a contradiction. Hence, if  $xv_i \in E(G)$  then  $N(v_1) = \{u_1, u_i\}$  and  $N(x) = \{u_1, v_i\}$  and  $G = C_5$ .

Thus we have the remaining cases:

(1) exactly one vertex from each pair  $\{u_i, v_i\} \in S_p - \{\{u_1, v_1\}\}$  is adjacent to  $v_1$  and we obtain  $G = K_{1,t}^*$

and

(2) at least one vertex from  $\{u_i, v_i\}$  is adjacent to  $x$  and then we obtain  $G = K_{1,t}^{\Delta}$ .

This completes the proof of the theorem.  $\square$

**Corollary 2.5.** *Let  $G$  be a graph with  $n \geq 3$ . Then  $\Gamma_p(G) = n - 1$  if and only if  $G = H \cup rK_2$ , where  $H \in \mathcal{F}^{\Delta}$  and  $r \geq 0$ .*



Now, let us consider the following problem: for which graphs  $G$  the equality  $\gamma_p(G) = \Gamma_p(G)$  holds? In this paper we present a solution of the above problem for large parameters.

By Theorem 6 of [7] and Observation 2.1 we obtain the following statement.

**Fact 2.6.** *Let  $G$  be a graph. Then  $\gamma_p(G) = \Gamma_p(G) = n$  if and only if  $G = mK_2$ .*

Since  $\mathcal{F} \subseteq \mathcal{F}^\Delta$ , by Corollary 2.3 and Corollary 2.5, we obtain the following result.

**Corollary 2.7.** *Let  $G$  be a graph satisfying  $n \geq 3$ . Then  $\gamma_p(G) = \Gamma_p(G) = n - 1$  if and only if  $G = H \cup rK_2$ , where  $H \in \mathcal{F}$  and  $r \geq 0$ .*

Now we determine graphs  $G$  for which  $\gamma_p(G) = \Gamma_p(G) = n - 2$ .

In [10] we showed that only the graphs in family  $\mathcal{G}$  (see Figure 2) are connected and satisfy the condition  $\gamma_p(G) = n - 2$ .

**Theorem 2.8.** *Let  $G$  be a connected graph of order  $n \geq 4$ . Then  $\gamma_p(G) = n - 2$  if and only if  $G \in \mathcal{G}$ .*

**Corollary 2.9.** *If  $G$  is a graph of order  $n \geq 4$ , then  $\gamma_p(G) = n - 2$  if and only if:*

- 1) *exactly two of the components of  $G$  are isomorphic to graphs of the family  $\mathcal{F}$  given in Theorem 2.2 and every other component is  $K_2$  or*
- 2) *exactly one of the components of  $G$  is isomorphic to a graph of the family  $\mathcal{G}$  given in Theorem 2.8 and every other component is  $K_2$ .*

Next, we describe graphs with the paired-domination and the upper paired-domination numbers two less than their order.

**Corollary 2.10.** *If  $G$  is a graph of order  $n \geq 4$ , then  $\gamma_p(G) = \Gamma_p(G) = n - 2$  if and only if  $G$  is a graph given in Theorem 2.8 and Corollary 2.9.*

*Proof.* It follows from the former theorems that the condition  $\gamma_p(G) = n - 2$  holds if and only if  $G \in \mathcal{G}$  or  $G$  is the graph described in Corollary 2.9. It follows the necessity. Now let  $G \in \mathcal{G}$  or  $G$  be a graph from Corollary 2.9. Since  $G$  is a graph of even order, moreover  $\Gamma_p(G) \geq \gamma_p(G)$  and  $G \neq mK_2$ , then by Observation 2.1 we conclude that  $\Gamma_p(G) = \gamma_p(G)$ . But then by Theorem 2.8 and Corollary 2.9 we obtain the sufficiency.  $\square$

### 3. $\Gamma_p$ FOR PATHS AND CYCLES

Dorbec *et al.* [2] established the upper paired-domination number of the path.

**Proposition 3.1.** *For  $n \geq 2$  an integer,*

$$\Gamma_p(P_n) = 8\lfloor(n+1)/11\rfloor + 2\lfloor((n+1) \bmod 11)/3\rfloor.$$

In this paper we determine the upper paired-domination number for the cycle.

**Proposition 3.2.** For  $n \geq 3$  an integer,

$$\Gamma_p(C_n) = 8\lfloor n/11 \rfloor + 2\lfloor (n \bmod 11)/3 \rfloor$$

for  $n \neq 5$  and  $\Gamma_p(C_5) = 4$ .

*Proof.* For  $3 \leq n \leq 12$  we can determine the values of  $\Gamma_p(C_n) = 2, 2, 4, 4, 4, 4, 6, 6, 8, 8$ , respectively. Thus, the statement holds.

For  $n \geq 13$ , let  $f(n) = 8\lfloor (n+1)/11 \rfloor + 2\lfloor ((n+1) \bmod 11)/3 \rfloor$ .

Then we proceed with the following statement.

**Claim 1.** For  $n \geq 3$  an integer,  $f(n-1) \geq 2\lfloor n/3 \rfloor$ .

*Proof of Claim 1.* Let  $n = 33k + r$ , where  $0 \leq r < 33$ . Then  $f(n-1) = 24k + r_1$ ,  $2\lfloor n/3 \rfloor = 22k + r_2$  and  $r_1 \geq r_2$ . Hence we can obtain the desired result.

Now, for the path  $P_n$  of order  $n$ , we construct a special  $\Gamma_p(P_n)$ -set.

**Claim 2.** Let  $P_n$  be the path  $v_1, v_2, \dots, v_n$  of order  $n$ , where  $n \geq 2$  and  $n \neq 4$ . Then there exists a  $\Gamma_p(P_n)$ -set  $S$  such that  $v_1 \in S$ .

*Proof of Claim 2.* Assume that  $v_1, v_2, \dots, v_n$  are consecutive vertices on the path  $P_n$ . We construct a set  $S$  as follows. Let  $S_p = A_n$  be a set of paired vertices in  $S$  for the path  $P_n$ . First we define  $A_n$  for  $2 \leq n \leq 10$ . Let

$$\begin{aligned} A_2 = A_3 &= \{\{v_1, v_2\}\}, & A_4 &= \{\{v_2, v_3\}\}, \\ A_5 &= \{\{v_1, v_2\}, \{v_4, v_5\}\}, & A_6 = A_7 &= \{\{v_1, v_2\}, \{v_5, v_6\}\}, \\ A_8 = A_9 &= \{\{v_1, v_2\}, \{v_3, v_4\}, \{v_7, v_8\}\}, & A_{10} &= A_8 \cup \{\{v_9, v_{10}\}\}. \end{aligned}$$

Now, we determine  $A_n$  for  $n = 11k + r$ , where  $k \geq 1$  and  $0 \leq r < 11$ . At first we define the sets  $B_i$  for  $i \geq 0$  as follows:

$$B_i = \{\{v_{1+11i}, v_{2+11i}\}, \{v_{3+11i}, v_{4+11i}\}, \{v_{7+11i}, v_{8+11i}\}, \{v_{9+11i}, v_{10+11i}\}\}.$$

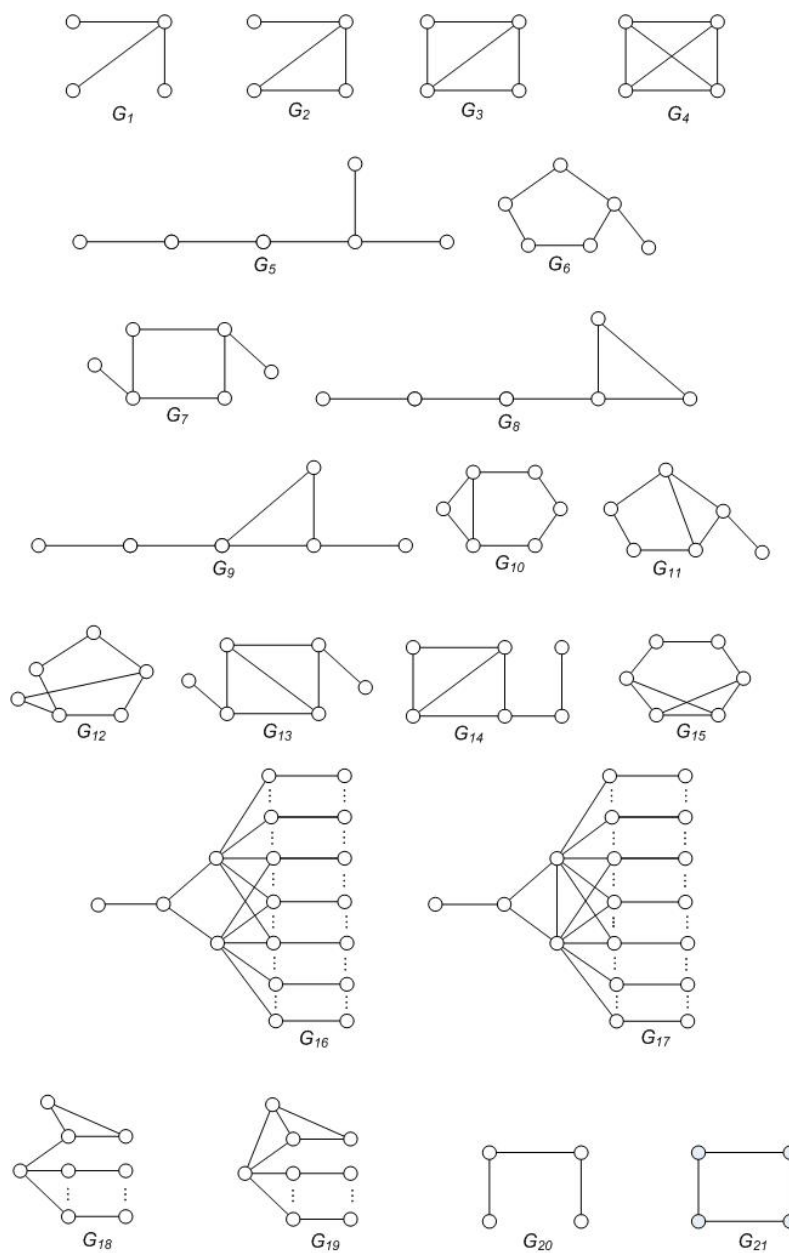
Next, we define  $A_n$  as follows:  $A_n = \bigcup_{i=0}^{k-1} B_i$  for  $r = 0$ ,  $A_n = \bigcup_{i=0}^{k-1} B_i \cup A_r$  for  $r \geq 2$  and

$$A_n = \bigcup_{i=0}^{k-1} B_i - \{\{v_{11k-2}, v_{11k-1}\}\} \cup \{\{v_{11k}, v_{11k+1}\}\} \quad \text{for } r = 1.$$

It is clear that the above set  $S$  is a PDS of  $P_n$ . Now we show the minimality of  $S$ . For this purpose suppose that  $S' \subseteq S$  and  $S' \neq S$ , next consider two possibilities. If  $S' = S - \{v_j, v_{j+1}\}$ , where  $\{v_j, v_{j+1}\} \in S_p = A_n$ , then  $S'$  is not a PDS of  $P_n$ . Now assume that  $\{v_j, v_{j+1}\}, \{v_{j+2}, v_{j+3}\} \in S_p$ . Then  $S' = S - \{v_j, v_{j+3}\}$  with paired vertices  $v_{j+1}$  and  $v_{j+2}$ , is not a PDS of  $P_n$  again. Now we calculate the size of  $S$ . Let  $n = 11k + r$ , where  $k \geq 1$  and  $0 \leq r < 11$ . Then consider the following cases.

*Case A.*  $r = 0$ . Then we have  $|S| = (8/11)n = 8k$ , but on the other hand

$$f(n) = 8\lfloor (11k+1)/11 \rfloor + 2\lfloor ((11k+1) \bmod 11)/3 \rfloor = 8k.$$



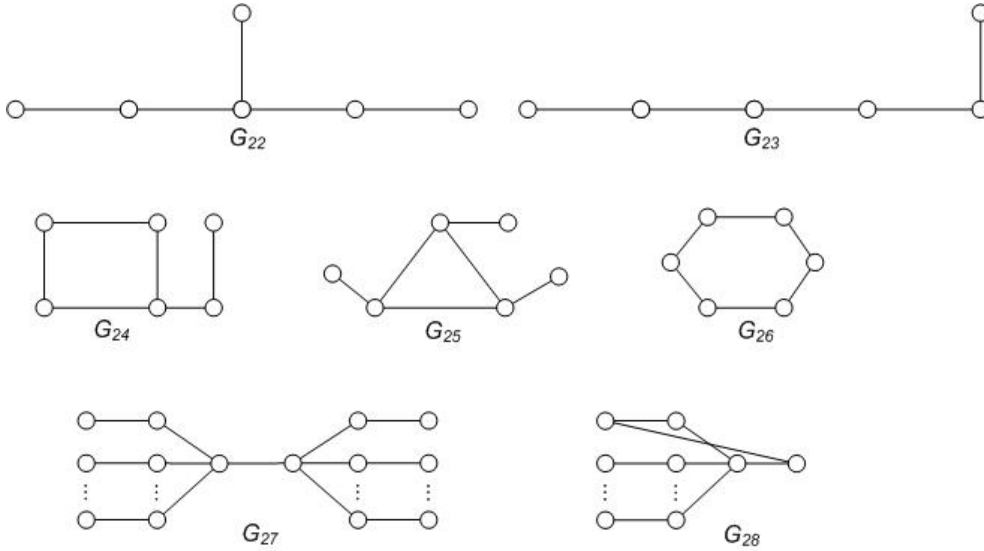


Fig. 2. Graphs in family  $\mathcal{G}$

Case B.  $r \geq 2$ . Now we obtain

$$|S| = 8k + f(r) = 8k + 8\lfloor(r + 1)/11\rfloor + 2\lfloor((r + 1) \bmod 11)/3\rfloor.$$

Case B.1.  $r < 10$ . Then

$$|S| = 8k + 2\lfloor((r + 1) \bmod 11)/3\rfloor = f(n).$$

Case B.2.  $r = 10$ . Then

$$|S| = 8k + f(r) = 8k + 8 = 8\lfloor(n + 1)/11\rfloor = f(n).$$

Case C.  $r = 1$ . In this case we have  $|S| = 8k$ , but on the other hand

$$f(n) = 8\lfloor(11k + 2)/11\rfloor + 2\lfloor((11k + 2) \bmod 11)/3\rfloor = 8k.$$

Thus, in every case we have that  $|S| = f(n)$  and  $S$  is a  $\Gamma_p(P_n)$ -set.

Now let  $v_1, \dots, v_n$  are consecutive vertices on the cycle  $C_n$  and consider the path  $P_{n-1} = C_n - v_n$ . By Claim 2, we conclude that there exists a  $\Gamma_p(P_{n-1})$ -set  $S$  such that  $v_1 \in S$ . It is obvious that  $S$  is a PDS of  $C_n$ . Now suppose that there exists a proper subset  $S'$  of  $S$  such that  $S'$  is a PDS of  $C_n$ . Since  $v_n \notin S'$ , then  $S'$  would be a PDS of  $P_{n-1}$ , contradicting the minimality of  $S$ . Therefore,  $S$  is a minimal PDS of  $C_n$ .

Hence we obtain

$$\Gamma_p(P_{n-1}) \leq \Gamma_p(C_n).$$

Now we show that  $\Gamma_p(C_n) \leq \Gamma_p(P_{n-1})$ .

At first assume that there exists a  $\Gamma_p(C_n)$ -set  $S'$  such that for all vertices  $v_i, v_{i+1}$  paired in  $S'$ ,  $v_{i-1} \notin S'$  and  $v_{i+2} \notin S'$ . Then we have  $\Gamma_p(C_n) \leq 2\lfloor n/3 \rfloor$ . Hence and by Claim 1 we obtain  $\Gamma_p(P_{n-1}) \geq \Gamma_p(C_n)$ .

So we may assume that for every  $\Gamma_p(C_n)$ -set  $S'$  there exist vertices  $v_i, v_{i+1}$  paired in  $S'$  and such that at least one vertex  $v_{i-1}, v_{i+2}$  is in  $S'$ .

Without loss of generality we may assume that vertices  $v_{n-1}, v_n$  are paired in  $S'$  and at least one among vertices  $v_{n-2}, v_1$  is in  $S'$ . It follows from the minimality of  $S'$  that exactly one of  $v_{n-2}, v_1$  belongs to  $S'$ . Let  $v_1 \notin S'$  and  $v_{n-2} \in S'$ . Hence  $v_{n-3} \in S'$ . Note that  $v_{n-4} \notin S'$ , because vertices  $v_{n-4}, v_{n-5}$  would be paired in the opposite case, which contradicts the minimality of  $S'$ .

Now consider the following cases.

*Case 1.*  $v_2 \in S'$ . Then  $v_{n-5} \notin S'$ , because the set  $S' - \{v_{n-3}, v_n\}$  would be a PDS of  $C_n$  in the opposite case, which contradicts the minimality of  $S'$ . Now  $S'$  is a minimal PDS of  $P_{n-1} = C_n - v_1$ . Really, suppose that  $S''$ , where  $S'' \subseteq S'$  and  $S'' \neq S'$ , is a PDS of  $P_{n-1}$ . Then  $S''$  must include vertices  $v_{n-3}, v_{n-2}, v_{n-1}, v_n$ , therefore  $S''$  would be a PDS of  $C_n$ , a contradiction.

*Case 2.*  $v_2 \notin S'$ . Then  $v_3 \in S'$ .

*Case 2.1.*  $v_{n-5} \in S'$ . Then consider the path  $P_{n-1} = C_n - v_{n-4}$ . By reasoning similar to that in Case 1 we conclude that  $S'$  is a minimal PDS of  $P_{n-1}$ .

*Case 2.2.*  $v_{n-5} \notin S'$ . Then  $S'$  is a minimal PDS of  $P_{n-1} = C_n - v_1$ . Really, suppose that  $S''$ , where  $S'' \subseteq S'$  and  $S'' \neq S'$ , is a PDS of  $P_{n-1}$ . Then  $S''$  must include vertices  $v_{n-3}, v_{n-2}, v_{n-1}, v_n$  and  $v_3$ , therefore  $S''$  would be a PDS of  $C_n$ , a contradiction.

In all cases we have that  $S'$  is a minimal PDS of  $P_{n-1}$  and so  $\Gamma_p(C_n) \leq \Gamma_p(P_{n-1})$ .

This completes the proof of the statement.  $\square$

Now let us consider the problem when

$$\gamma_p(G) = \Gamma_p(G)$$

for  $G = P_n$  or  $G = C_n$ .

Since  $\gamma_p(P_n) = \gamma_p(C_n) = 2\lfloor n/4 \rfloor$  (see [7]), by Proposition 3.1 and by Proposition 3.2 one can obtain the following statements.

**Proposition 3.3.**  $\gamma_p(P_n) = \Gamma_p(P_n)$  if and only if  $n = 2, 3, 4, 5, 6, 7$  or  $9$ .

**Proposition 3.4.**  $\gamma_p(C_n) = \Gamma_p(C_n)$  if and only if  $n = 3, 4, 5, 6, 7, 8, 9, 10$  or  $13$ .

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