NOVEL ANALYTIC-NUMERICAL MODEL OF FREE CONVECTION: WITH LEADING EDGE CONSIDERED

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Abstract: A novel solution of the free convection boundary problem is represented in analytical form for velocity and temperature for an isothermal vertical plate, as an example. These fields are built as a Taylor Series in the x coordinate with coefficients as functions of the vertical coordinate (y). We restrict ourselves by cubic approximation for both functions. The basic Navier-Stokes and Fourier-Kirchhoff equations and boundary conditions give links between coefficients and connected with free convection heat transfer phenomenon which define the analytical form of the solution as a function of the Grashof number only. In the solution the non zero velocity of a fluid flow through a leading edge of the plate is taken into account. The solution in the form of velocity and temperature profiles is numerically evaluated and illustrated for air.

Keywords: Navier-Stokes equations, Fourier-Kirchhoff equation, free convective heat transfer, isothermal surface, boundary layer, vertical plate, leading edge

1. Introduction

The conventional boundary layer theory of the fluid flow used to describe free convection assumes zero velocity at the leading edge of a heated plate. More advanced theories of self-similarity also accept this same boundary condition [1–4]. However, the experimental visualization shows distinctly that the fluid is in motion in the vicinity of such an edge, see the left side of Figure 1 prepared on the basis of [5] and [6]. This emerges self-evidently from the application of the mass conservation law in the integral form at the leading edge.

The second inaccuracy of the Prandtl theory based on the zero value of the boundary layer thickness at the leading edge of the plate is that the heat transfer coefficient, which is inversely proportional to the thickness of the boundary layer, becomes infinite. This contradicts the fact that the plate does not transfer heat at the starting edge of the phenomenon. The whole qualitative picture of the phenomenon is well known: the velocity and temperature profiles normal to a vertical plate are reproduced by the theoretical Prandtl and self-similarity concepts, but these profiles change differently in comparison with the manner suggested by isotherm visualization (see *e.g.* [6]) and direct measurements [7, 8]. The same effect is observed in the results of numerical simulations [9]. Obviously, the dependence of isotherms on the vertical coordinate y differs significantly from the power law dependence $\delta \sim y^{1/4}$ of boundary layer theories at the area adjacent to the leading edge (see again the left side of Figure 1. One of the first attempts to build a two-dimensional theory of a fluid flow without the heat transfer account in the vicinity of a flat surface is presented in [10]. One of possible theoretical descriptions of the phenomena is given in [11] where the theory of the leading and trailing edges of laminar free convection is matched to a description of the boundary layer. Such description implies a consideration of the whole picture in three subdomains - incoming, main and outgoing.

In this article we develop a novel theory which would be an alternative to the conventional theories of the boundary layer and self-similarity theories. Compared to the three-domain theory of [11] we restricted ourselves by the main theory only, taking into account the influence of leading and trailing edges by means of the boundary conditions.

We consider a two-dimensional free convective fluid flow in the x,y plane generated by a vertical isothermal plate of height L placed in an undisturbed environment.

Our model is based on an explicit form of the solution of the basic fundamental equations (Navier-Stokes and Fourier-Kirchhoff). By means of the scale analysis of the equations delivered by celebrated authors Lorentz, Prandtl and Schlichting [12] we conventionally neglect the horizontal (normal to the surface of the plate) component of the velocity. Such an assumption automatically excludes the continuity equation in a differential form and one of the Navier-Stokes equations.

In this article we restrict ourselves by a minimal model which however accounts for the principle features of the theory. In our model we left only two basic equations: the vertical component of Navier-Stokes and Fourier-Kirchhoff equation for two variables vertical velocity and temperature. It follows from the second Navier-Stokes equation that the horizontal derivative of pressure in our approximation is zero. Hence, the pressure in the first Navier-Stokes equation is eliminated by differentiating with respect to y. The consequence of this operation arises from the equation order that we take into account in the solution construction.

The algorithm of the solution construction is as follows. First, we expand the basic fields, velocity and temperature in a power series of horizontal variable

x, its substitution into the basic system gives a system of ordinary differential equations for coefficients of the expansion as functions of the vertical variable y. As such a system is generally infinite, we should cut the expansion at some power. The form of such cutting defines the model. A minimum number of terms in the model is determined by the physical conditions on the velocity and temperature profiles. The minimum number of terms is taken to be three: the parabolic part guarantees the existence of a maximum velocity, while the third term changes the sign of the velocity derivative. The temperature behavior of the same order of approximation is defined by the basic system of equations. The first term $(C(y)x)$ in the temperature expansion is linear in x, that accounts for the boundary condition on the plate (isothermal one). The coefficient, denoted as $C(y)$, satisfies an ordinary differential equation of the fourth order. The equation relates to the Mittag-Leffler class [13], its solution is expressed in terms of the Mittag-Leffler special function. We, in this paper, restrict ourselves by the special case of the equation the solutions of which are expressed in terms of elementary functions. The order of the equation implies four boundary conditions at the leading $(y=0)$ and separating edge $(y=L)$ (end of the plate). The differential links other expansion coefficients with C add two integration constants, hence, the necessity for two extra conditions. The formulation of the six boundary conditions is most essential and difficult in such a model.

In the second section we present the basic system in its dimensional and dimensionless forms. By means of cross-differentiation we eliminate the pressure terms and then neglect the horizontal velocity that results in two partial differential equations for the vertical component of velocity and temperature.

In the third section we expand both the velocity and temperature fields into a Taylor series in x and derive ordinary differential equations for the coefficients by direct substitution into the basic system. The minimal (cubic) version is obtained by restricting the infinite system of equations using a special constraint.

The fourth section deals with the formulation of the boundary conditions and its explicit form in terms of the coefficient functions of basic fields. It is important to stress that the set of boundary conditions and conservation laws determine all the necessary parameters including the Grashof and Rayleigh numbers in the stationary regime under consideration.

The fifth section contains the solution $C(y)$ in explicit form via elementary functions and the expressions of constants of integrations in terms of boundary conditions. The solution depends also on parameters of the whole problem: Gr, L, a, l. The parameter l is the width in the x direction of the stream at the leading edge, while a is proportional to the velocity gradient on the plate.

In the sixth section the parameters l and a are expressed in terms of Gr and Pr numbers solving conservation laws of mass and energy equations in integral form. The equations link the field functions of temperature and velocity between leading and separating edges of the plate. As the main result of our study we have expressed the total velocity and temperature fields as a function of Rayleigh Ra and Prandtl Pr numbers.

In the last section we illustrate our results by the plot of $C(y)$ and velocity and temperature profiles for example conditions: air, $L=0.5\,\mathrm{m}$, $\Delta T=10\,\mathrm{K}$.

The solution of the problem allows us to express the heat transfer coefficient k in terms of Rayleigh Ra in the well known form $k\sim C\cdot \mathrm{Ra}^{1/4}$ The numerical value of C differs not significantly from the experimental one due to the simplified version of the Mittag-Leffler equation that we use.

2. The basic equations

Let us consider a two dimensional stationary flow of an incompressible fluid in a gravity field. The flow is generated by convective heat transfer from a solid plate to the fluid. The plate is isothermal and vertical. In the Cartesian coordinates x (horizontal and orthogonal to the plate), y (vertical and tangent to the plate) the Navier-Stokes (NS) system of equations takes the form [2]:

$$\rho\left(W_x\frac{\partial W_y}{\partial x}+W_y\frac{\partial W_y}{\partial y}\right)=g\rho_\infty b(T-T_\infty)-\frac{\partial p}{\partial y}+\rho\nu\left(\frac{\partial^2 W_y}{\partial y^2}+\frac{\partial^2 W_y}{\partial x^2}\right) \tag{1}$$

$$\rho\left(W_x\frac{\partial W_x}{\partial x}+W_y\frac{\partial W_x}{\partial y}\right)=-\frac{\partial p}{\partial x}+\rho\nu\left(\frac{\partial^2 W_x}{\partial y^2}+\frac{\partial^2 W_x}{\partial x^2}\right) \tag{2}$$

In the above equations the pressure terms are divided into two parts $\tilde{p}=p_0+p$. The first one is the hydrostatic one which is equal to the mass force $-g\rho_\infty$, where

$$\rho=\rho_\infty(1-b(T-T_\infty)) \tag{3}$$

is the density of a liquid in the non-disturbed area where the temperature is T_∞. The second one is the extra pressure denoted by $-\nabla p$. The part of gravity force $gb(T-T_\infty)$ arises from the dependence of the extra density on temperature, b is the coefficient of thermal expansion of the fluid. In the case of gases $b=-\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p=\frac{1}{T_\infty}$. The last terms of the above equations represent friction forces with the kinematic coefficient of viscosity ν.

The mass continuity equation under the conditions of the natural convection of an incompressible fluid in the steady state [3] takes the form:

$$\frac{\partial W_x}{\partial x}+\frac{\partial W_y}{\partial y}=0. \tag{4}$$

The temperature dynamics is described by the stationary Fourier-Kirchhoff (FK) equation:

$$W_x\frac{\partial T}{\partial x}+W_y\frac{\partial T}{\partial y}=\kappa\left(\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial x^2}\right) \tag{5}$$

where W_x and W_y are the components of the fluid velocity $\overline{W}$, T is the temperature, p is the pressure disturbances and κ is the thermal diffusivity.

From the clarity of further transformations we use the same scale L along both variables x and y. We will return to the eventual difference between characteristic scales in different directions while the solution analysis is to be provided. After introducing nontraditional variables:

$$\begin{gathered} x'=x/L, \quad y'=y/L, \quad T'=(T-T_w)/(T_w-T_\infty), \\ p'=p/p_\infty, \quad W'_x=W_x/W_o, W'_y=W_y/W_o \end{gathered} \tag{6}$$

we obtain the Boussinesq approximation (in all terms apart from the buoyancy we write $\rho\approx\rho_\infty$).

$$W'_x\frac{\partial W'_y}{\partial x'}+W'_y\frac{\partial W'_y}{\partial y'}=\frac{gb(T_w-T_\infty)L}{W_o^2}(T'+1)-\frac{p_\infty}{\rho_\infty W_o^2}\frac{\partial p'}{\partial y'}+\nu'\left(\frac{\partial^2 W'_y}{\partial y'^2}+\frac{\partial^2 W'_y}{\partial x'^2}\right) \tag{7}$$

$$W'_x\frac{\partial W'_x}{\partial x'}+W'_y\frac{\partial W'_x}{\partial y'}=-\frac{p_\infty}{\rho_\infty W_o^2}\frac{\partial p'}{\partial x'}+\nu'\left(\frac{\partial^2 W'_x}{\partial y'^2}+\frac{\partial^2 W'_x}{\partial x'^2}\right) \tag{8}$$

and the FK equation is written as

$$W'_x\frac{\partial T'}{\partial x'}+W'_y\frac{\partial T'}{\partial y'}=\kappa'\left(\frac{\partial^2 T'}{\partial y'^2}+\frac{\partial^2 T'}{\partial x'^2}\right) \tag{9}$$

where $\frac{\nu}{LW_o}=\nu'$, $\frac{\kappa}{LW_o}=\kappa'$, L is a characteristic linear dimension and W_0 is a characteristic velocity. If

$$W_o=\frac{\nu}{L}, \tag{10}$$

then $\kappa'=\mathrm{Pr}$, $\nu'=1$ and $\frac{gb(T_w-T_\infty)L}{W_o^2}=\mathrm{Gr}$, is the Grashof number, which after plugging (10) takes the form:

$$\mathrm{Gr}=\frac{gb(T_w-T_\infty)L^3}{\nu^2} \tag{11}$$

After cross differentiation of equations (7) and (8) we have:

$$\begin{gathered} \frac{\partial}{\partial x'}\left[W'_x\frac{\partial W'_y}{\partial x'}+W'_y\frac{\partial W'_y}{\partial y'}-\mathrm{Gr}(T'+1)-\left(\frac{\partial^2 W'_y}{\partial y'^2}+\frac{\partial^2 W'_y}{\partial x'^2}\right)\right]= \\ =\frac{\partial}{\partial y'}\left[W'_x\frac{\partial W'_x}{\partial x'}+W'_y\frac{\partial W'_x}{\partial y'}-\left(\frac{\partial^2 W'_x}{\partial y'^2}+\frac{\partial^2 W'_x}{\partial x'^2}\right)\right] \end{gathered} \tag{12}$$

The FK equation rescales as

$$\mathrm{Pr}\left(W'_x\frac{\partial T'}{\partial x'}+W'_y\frac{\partial T'}{\partial y'}\right)=\left(\frac{\partial^2 T'}{\partial y'^2}+\frac{\partial^2 T'}{\partial x'^2}\right) \tag{13}$$

and

$$\rho=\rho_\infty(1-b(T-T_\infty))=\rho_\infty(1-b\Phi(T'+1)). \tag{14}$$

where $\Phi=T_w-T_\infty$

Next, we formulate the problem of free convection over the heated vertical isothermal plate $x=0$, $y\in[0,L)$. In the text below we still use the nondimensional variables but without primes.

In this case we assume that the angle between the plate and the stream line is small, which means that the horizontal component of velocity of the fluid

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can be neglected, and denote the vertical component by $W(y,x)$. In this paper we restrict ourselves to the assumption that $W_x = 0$ and $W_y = W$, that yields

$$\frac{\partial}{\partial x}\left[W\frac{\partial W}{\partial y} - \mathrm{Gr}(T+1) - \left(\frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial x^2}\right)\right] = 0 \tag{15}$$

$$\mathrm{Pr}W\frac{\partial T}{\partial y} = \left(\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial x^2}\right) \tag{16}$$

3. Method of solution and approximations

The aim of this paper is the application of theory to the standard example of a finite vertical plate. Having only two basic functions we consider the power series expansions of the velocity and temperature in Cartesian coordinates:

$$W(x,y) = \gamma(y)x + \alpha(y)x^2 + \beta(y)x^3 + \vartheta(y)x^4 + \ldots \tag{17}$$

$$T(x,y) = C(y)x + A(y)x^2 + B(y)x^3 + F(y)x^4 + \ldots \tag{18}$$

According to the standard boundary conditions on the plate we assume that both the functions tend to zero when $x \to 0$, hence, for the calculation we choose the variable that has a zero value for non-dimensional temperature (6). This means that the value of $T(x,y)$ outside of the convective flow tends to -1. Substituting expressions (17) and (18) into the equations (15) and (16) we take into account the linear independence of monomials x^n, which gives a system of coupled nonlinear equations for the coefficients $\gamma(y)$, $\alpha(y)$, ... and $C(y)$, $A(y)$, ...

Such a system is infinite, hence, for practical purposes we need to choose an appropriate scheme of closed formulation for a finite number of variables. We will restrict ourselves to the fourth-order approximation for both variables, that is, we neglect higher order terms, starting from the fourth one. The area of validity of the approximations is defined by the comparison of terms in expansions (17) and (18). From the relations that appear after the substitution of (17) and (18) into (15). Let us show three first terms of the resulting expansion in (18)

$$\left(\gamma(y)\frac{\partial C(y)}{\partial y} - a\frac{\partial^2 A(y)}{\partial y \partial y} - 12aF(y)\right)x^2 + \left(-a\frac{\partial^2 C(y)}{\partial y \partial y} - 6aB(y)\right)x - 2aA(y) = 0 \tag{19}$$

From the terms x^0 it follows that $A(y) = 0$. The second term gives a link between $C(y)$ and $B(y)$. The third one reduces as $\left(\gamma(y)\frac{\partial C(y)}{\partial y} - 12aF(y)\right)$ and links $F(y)$ and $C(y)$. We, however, restrict ourselves by a third power of the x engineering model that will be considered as a base for analytic-numeric modeling.

Finally, from both equations (15), (16) we obtain a system of equations for the coefficients $B(y)$, $C(y)$, $\alpha(y)$ and $\beta(y)$:

$$6B(y) + \frac{\partial^2 C(y)}{\partial y \partial y} = 0 \tag{20}$$

$$\mathrm{Pr}\alpha(y)\frac{\partial C(y)}{\partial y} - \frac{\partial^2 B(y)}{\partial y \partial y} = 0 \tag{21}$$

$$-6\beta(y) - \mathrm{Gr}C(y) = 0 \tag{22}$$

$$\gamma(y)\frac{\partial\gamma(y)}{\partial y} - \frac{\partial^2\alpha(y)}{\partial y\partial y} = 0 \tag{23}$$

The first two (20), (21) arise from the FK equation and the remaining ones are from the NS one. The system of equations is closed, if $\gamma(y) = \mathrm{const} = \gamma$. This means that the number of equations and the number of unknown functions is the same. Finally, to a first approximation, the velocity and temperature are expressed as:

$$W(x,y) = \gamma x + \alpha(y)x^2 + \beta(y)x^3, \quad T(x,y) = C(y)x + B(y)x^3. \tag{24}$$

From (23) we have

$$\alpha(y) = C_1 y + C_2 \tag{25}$$

From (20) it follows that

$$B(y) = -\frac{1}{6}\frac{\partial^2 C(y)}{\partial y\partial y} \tag{26}$$

hence, (21) goes to:

$$\frac{1}{6}\frac{d^4C(y)}{dy^4} + \mathrm{Pr}(yC_1 + C_2)\frac{dC(y)}{dy} = 0 \tag{27}$$

which is the Mittag Loeffler equation [13]. The equation is solved in terms of either power series [13] or by means of Laplace transformation [14]. We however neglect the term yC_1 (see (38)) that results in the equation with constant coefficient (47) which is Rayleigh number $\mathrm{Ra} = \mathrm{PrGr}$. Equation (22) reads:

$$\beta(y) = -\frac{\mathrm{Gr}}{6}C(y) \tag{28}$$

which results in

$$W(x,y) = \gamma x + (C_1 y + C_2)x^2 - \frac{\mathrm{Gr}}{6}C(y)x^3, \qquad T(x,y) = C(y)x - \frac{1}{6}\frac{d^2C(y)}{dy^2}x^3 \tag{29}$$

The form of the equation (27) indicates that for a unique solution, we need four boundary conditions for the given parameters C_1 and C_2. Apart from such conditions we should also have values for γ and Gr. Hence, for the explicit determination of $W(x,y)$ and $T(x,y)$ we need seven conditions, compared with our previous paper [15, 16], where we used the local Grashof number.

4. Analysis of problem formulation and boundary conditions for temperature and velocity

4.1. On conservation laws

In looking for the boundary conditions, let us apply the laws of conservation of mass, momentum and energy to a control volume V. The mass conservation law in a steady state has the form:

$$\int_S \rho\vec{W}\cdot\vec{n}dS = 0 \tag{30}$$

where: S is the sum of all lateral surfaces $S=\sum_{i=1}^{6}\Sigma_i$ (Figure 1). On the right side of a vertical plate the grey color indicates the solution stability area $y\in(Y_0,Y_L)$ ($C(y)\approx\text{const}$, see the equation $C(y)$ (48)). The left side of Figure 1 represents interferometric study results from [5, 6] and two example isotherm curves ($\delta\sim y^{1/4}$) of the conventional boundary layer theory.

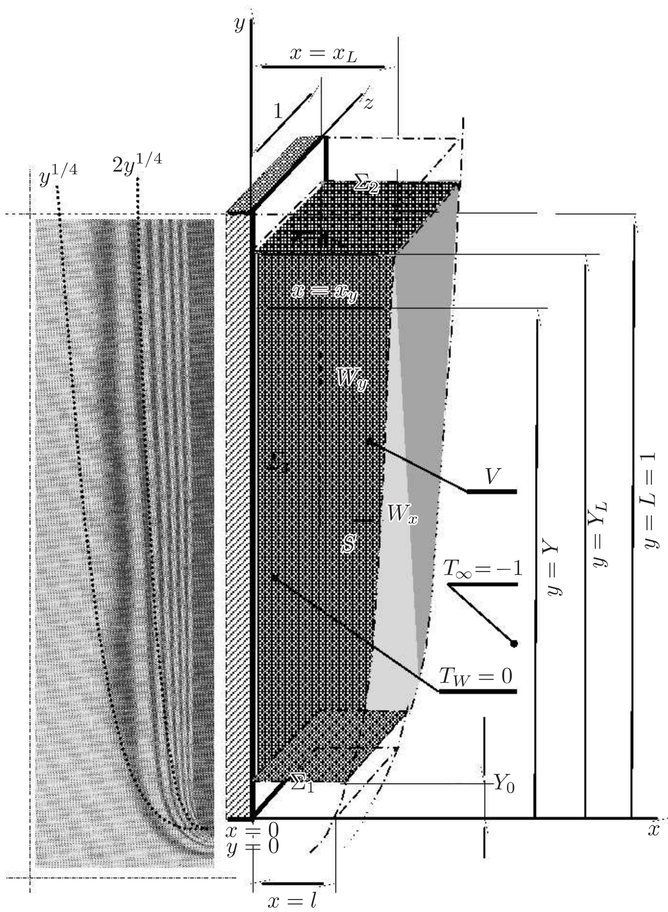

Figure 1. Result of interferometric study and examplary curves of conventional boundary layer theory (left side); coordinate system and notations for our model of twodimensional convective fluid flow from isothermal vertical plate (right side)

Let us bear in mind that the integral form of the law of conservation of mass (30) is formulated by the division of surface Σ into just two lower Σ_1 and upper Σ_2 boundaries. The horizontal mass flux is neglected, due to assumption that $W_x=0$. According to our main assumption about the two-dimensionality of the stream, we neglect the dependence of variables on the z coordinate. We also introduce the parameter l as the width of the incoming flow ($y=0$, $W(l,0)=0$), and hence, we restrict our theory by $x\in[0,l]$ at the leading edge $y=0$.

Hence the condition of total mass conservation is as follows:

$$\int_{\Sigma_1} \rho \vec{W} \cdot \vec{n} dS = \int_{\Sigma_2} \rho \vec{W} \cdot \vec{n} dS \tag{31}$$

where the flow from below Σ_1 is approximately the product of density at temperature $T = -1$ and the velocity of the incoming flow in the interval $x \in [0,l]$. We follow the idea of the velocity field continuity at $y = 0$, hence $W(x,0) = \gamma x + \alpha(0)x^2 + \beta(0)x^3$.

The next boundary condition is connected with the conservation of energy in a control volume V (area S with unit width see Figure 1), which arises from the FK equation (5) by integration over the volume.

$$\mathrm{Pr}\int_V \left(W\frac{\partial T}{\partial y}\right) dV = \int_V \left(\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial x^2}\right) dV = \int_S (\mathrm{grad} T)\vec{n} dS \tag{32}$$

The left-hand side of the energy conservation equation (32) is transformed in a similar way by applying the identity $\mathrm{div}(T\vec{W}) = T\,\mathrm{div}\vec{W} + \vec{W}\cdot\nabla T$ and (4). According to our assumptions, we are left with the flows across Σ_1, Σ_2, Σ_3 (Figure 1) and on the basis of the homogeneity of the problem. The horizontal energy flux from the control volume V of the fluid to the surrounding fluid is neglected due to the small temperature difference on the control volume boundary.

4.2. Boundary conditions for temperature and velocity

The temperatures in the vicinity of the boundary edge point $y = 0$, $x \in [0,l]$ are taken to have a value of -1 (temperature of the incoming bottom flow). In the dimensional form the interval under consideration has the characteristic width of the incoming flow l which we identify with a parameter we used when dimensional variables were introduced (6).

For a stationary process, the edge condition may be considered as the initial one for a Cauchy problem along y. Having a power series approximation (in our case two terms (29) of such conditions, we choose the coefficients $B(0)$ and $C(0)$ of the series on the basis of the following conditions:

The first boundary condition is given: $\left[\left(x^3B(0) + C(0)x\right)\right]_{x=l} = B(0)l^3 + C(0)l = -1$, then $B(0) = \frac{-1-C(0)l}{l^3}$. By the definition of l: $W(l,0) = \gamma l + C_2 l^2 - \frac{\mathrm{Gr}}{6}C(0)l^3 = 0$, then we have the expressions

$$C(0) = \frac{1}{l^2\mathrm{Gr}}(6\gamma + 6lC_2) \tag{33}$$

$$B(0) = -\frac{6\gamma + 6lC_2 + l\mathrm{Gr}}{l^4\mathrm{Gr}} \tag{34}$$

The temperature gradient values dT/dx on the plate decrease as y increases. At the upper trailing edge $y = L = 1$ we impose the condition $\partial T/\partial x|_{x=0} = 0$ because the fluid loses contact with the plate (see for example isotherms in numerical modeling for a finite vertical plate [9]. This yields the third boundary condition at $y=1$ (29)

$$C(1)=0 \tag{35}$$

The phenomenon of free convective heat transfer from an isothermal vertical plate ($T=0$) implies that temperature gradient on the plate is negative ($C<0$) and decreases along y ($\partial C/\partial y<0$). It is also known that the velocity profile has a maximum at the distance $x_m>0$. The extrema for the curve are defined by a derivative of $W(x,y)$ as a function of x. Hence, the relation $\frac{dW}{dx}=\gamma+2\alpha x+3\beta x^2=0$ indicates that for $\alpha<0$, $\beta>0$ and $\gamma>0$ we have two extrema points

$$x_m=-\frac{\alpha}{3\beta}-\sqrt{\frac{\alpha^2}{9\beta^2}-\frac{\gamma}{3\beta}} \quad \text{and} \quad x_0(y)=-\frac{\alpha}{3\beta}+\sqrt{\frac{\alpha^2}{9\beta^2}-\frac{\gamma}{3\beta}} \tag{36}$$

if $\frac{\alpha^2}{9\beta}-\frac{\gamma}{3}>0$.

The notations are chosen to mark the maximum position point as x_m while the minimum one is $x_0(y)>x_m$.

In the exceptional case of $\beta(y=1)=0$, the expression simplifies to

$$x_{mL}=-\frac{\gamma}{2\alpha(1)} \tag{37}$$

which is positive for $\alpha<0$. The second extreme does not exist now (see Figure 2).

We can choose the value $W(x_0,y)=0$ taking x_0 as the conditional boundary of the upward stream. Hence, we define $x_L=2x_{mL}=-\frac{\gamma}{\alpha(1)}$.

At the starting horizontal edge of the vertical plate the vertical velocity component of the incoming flow (29) varies slowly, hence, we assume that

$$C_1=0 \tag{38}$$

hence

$$x_L=-\frac{\gamma}{C_2} \tag{39}$$

and

$$W(x,y)=\gamma x+C_2x^2-\frac{\mathrm{Gr}}{6}C(y)x^3 \tag{40}$$

The extrema of the velocity profile (36) after accounting for (38) and (28) are transformed as, for the maximum: $x_m(y)=\frac{2C_2}{\mathrm{Gr}C(y)}-\sqrt{\frac{C_2^2}{\left(\frac{\mathrm{Gr}}{2}C(y)\right)^2}+\frac{2\gamma}{\mathrm{Gr}C(y)}}$ and the minimum: $x_0(y)=\frac{2C_2}{\mathrm{Gr}C(y)}+\sqrt{\frac{{C_2}^2}{\left(\frac{\mathrm{Gr}}{2}C(y)\right)^2}+\frac{2\gamma}{\mathrm{Gr}C(y)}}$. The following identity $x_0^2=\frac{2}{\mathrm{Gr}C(y)}(\gamma+2x_0C_2)$ holds for: $\gamma+2x_0C_2<0$.

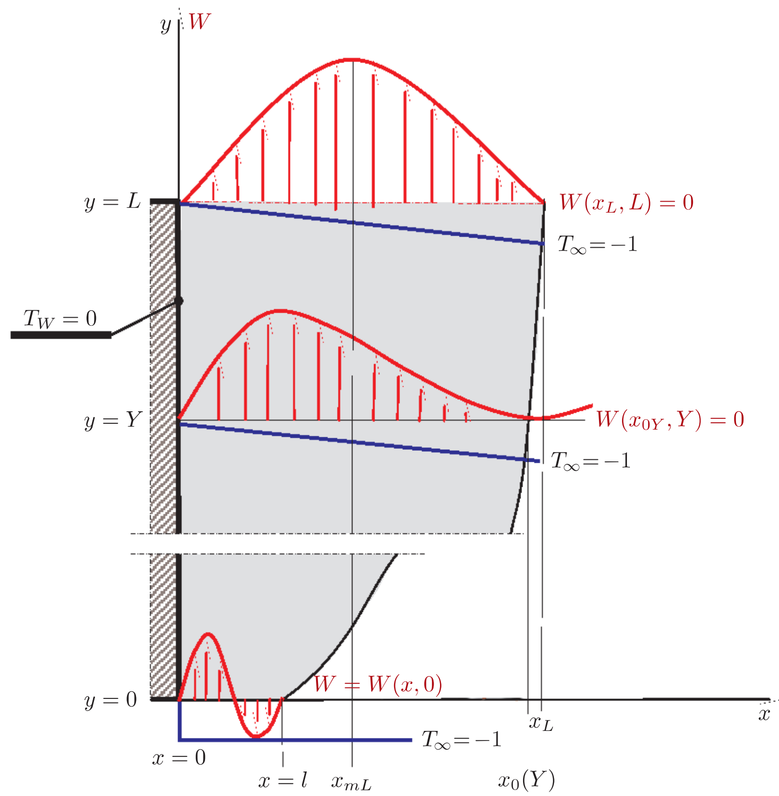

Figure 2. Distributions of velocity and temperature of natural convective fluid flow on characteristic levels of vertical plate $y=0$, Y and L.

Suppose there exists a level $y=Y$ at which

$$W(x_0(Y),Y)=0 \tag{41}$$

where $x_0(Y) \equiv x_{0Y}$ denotes the boundary layer thickness analogue. The equation (41) is solved with respect to $C(Y)$, which gives:

$$C(Y) = -\frac{3}{2\gamma}\frac{C_2^2}{\mathrm{Gr}} \tag{42}$$

as a function of the problem parameters. Then, plugging (42) for the expression for x_{0Y} yields

$$x_{0Y} = -2\frac{\gamma}{C_2} \tag{43}$$

Let us return to the expression for the temperature (29) with neglecting the last term in temperature (the possibility of such assumption will be explained below) on the level Y and substitute (42) and (43) into it equalizing to the surrounding temperature ($T=-1$).

$$T(x=x_{0Y},Y) = C(Y)x_{0Y} = -1 \tag{44}$$

we have:

$$C_2 = -\frac{\mathrm{Gr}}{3}, \quad x_{0Y} = 6\frac{\gamma}{\mathrm{Gr}} = 6a, \quad x_L = \frac{3\gamma}{\mathrm{Gr}} = 3a, \quad C(Y) = -\frac{1}{6a} \tag{45}$$

where:

$$a = \frac{\gamma}{\mathrm{Gr}} \tag{46}$$

5. Equation for $C(y)$

From the equation (27), after plugging C_2 (45) and taking into account $C_1 = 0$ (38), we have

$$\frac{1}{2}\frac{d^4C(y)}{dy^4} - \mathrm{PrGr}\frac{dC(y)}{dy} = 0 \tag{47}$$

The equation has been studied recently [15, 16] where the solution is given by:

$$C(y) = A_0 + A_1\exp(sy) + \exp\left(-\frac{sy}{2}\right)\left[B_1\cos\left(\tfrac{\sqrt{3}}{2}sy\right) + B_2\sin\left(\tfrac{\sqrt{3}}{2}sy\right)\right] \tag{48}$$

where

$$s = \sqrt[3]{2\mathrm{PrGr}} \tag{49}$$

is expressed via the Rayleigh number

$$\mathrm{Ra} = \mathrm{GrPr} \tag{50}$$

We have also the boundary conditions: (59)–(35) that after substitution of C_2 and $\gamma = a\mathrm{Gr}$ from (45)–(46) give:

$$\begin{gathered}
C(0) = A_0 + A_1 + B_1 = \frac{6a-2l}{l^2} \\
C''(0) = -6B(0) = \tfrac{1}{2}s^2\left(2A_1 - B_1 - \sqrt{3}B_2\right) = -6\left(\frac{1}{l^3} - 6\frac{a}{l^4}\right) \\
C(1) = A_0 + e^{-\frac{1}{2}s}\left(B_1\cos\tfrac{1}{2}\sqrt{3}s + B_2\sin\tfrac{1}{2}\sqrt{3}s\right) + A_1e^s = 0 \\
C(Y) = A_0 + A_1\exp(sY) + \exp\left(-\frac{sY}{2}\right)\left[B_1\cos\left(\tfrac{\sqrt{3}}{2}sY\right) + B_2\sin\left(\tfrac{\sqrt{3}}{2}sY\right)\right] = -\frac{1}{6a}
\end{gathered} \tag{51}$$

The solution of the system results in a rather big expression for A_1 as a function of A_0 which we skip in this text, going straight on to the following approximation.

The explicit form of the equation (48) shows that the last three terms behave exponentially as a function of sy. This means that there are three different domains of the fluid flow structure. The first is the starting one where all the terms are significant. The separating edge is characterized by the first two terms, while the medium domain is described by the first one only. We choose the parameter

$y = Y$ such that it belongs to that medium range (Y_0, Y_L). In such conditions we have

$$A_0 = C(Y) = \frac{-1}{6a\left(-\frac{1}{e^s}e^{Ys}+1\right)} \tag{52}$$

Solving the system with respect to: A_0, A_1, B_1, B_2 we have approximately $e^s \gg e^{Ys} \gg 1$

$$\begin{aligned} A_0 &\approx \frac{-1}{6a\left(-\frac{1}{e^s}e^{Ys}+1\right)} \approx -\frac{1}{6a} \\ A_1 &\approx -\frac{A_0}{e^s} \approx \frac{1}{6ae^s} \\ B_1 &= -A_0 + \frac{1}{l^2}(6a-2l) \approx \frac{1}{6a} + \frac{1}{l^2}(6a-2l) \\ B_2 &= \frac{1}{3}\sqrt{3}\left(A_0 + \frac{1}{s^2}\left(\frac{12}{l^3} - 72\frac{a}{l^4}\right) - \frac{1}{l^2}(6a-2l)\right) \\ &\approx \frac{1}{\sqrt{3}}\left(-\frac{1}{6a} + \frac{12}{s^2 l^3} - 72\frac{a}{s^2 l^4} - \frac{6a}{l^2} + \frac{2}{l}\right) \end{aligned} \tag{53}$$

Plugging the results to (48) yields the explicit expression for $C(y)$.

$$\begin{aligned} C(y) = &-\frac{1}{6a} + \frac{1}{6ae^s}e^{sy} + e^{-\frac{1}{2}sy}\left(\cos\frac{\sqrt{3}sy}{2}\right)\left(\frac{6a-2l}{l^2} + \frac{1}{6a}\right) + \\ &-\frac{e^{-\frac{1}{2}sy}}{\sqrt{3}}\left(\sin\frac{\sqrt{3}sy}{2}\right)\left(\frac{6a-2l}{l^2} - \frac{1}{s^2}\left(\frac{12}{l^3} - 72\frac{a}{l^4}\right) + \frac{1}{6a}\right) \end{aligned} \tag{54}$$

6. Application of conservation laws

6.1. Mass conservation law

On the left-hand side of the equation (31) taking into account the relations (29), (25) and (28) for the incoming flow we have:

$$\int_{\Sigma_1} \rho \vec{W} \cdot \vec{n} dS = \rho_\infty \int_0^l \left(\gamma x + \alpha(0)x^2 + \beta(0)x^3\right) dx = \rho_\infty \left(\tfrac{1}{4}\beta(0)l^4 + \tfrac{1}{3}\alpha(0)l^3 + \tfrac{1}{2}\gamma l^2\right) \tag{55}$$

and the outgoing flow Σ_2 is expressed similarly:

$$\int_{\Sigma_2} \rho \vec{W} \cdot \vec{n} dS = \rho_\infty \int_0^{x_L} \left(\gamma x + (C_1 + C_2)x^2 - \frac{\mathrm{Gr}}{6}C(1)x^3\right) dx = \tfrac{1}{24}\rho_\infty x_L^2 (12\gamma + 8C_2 x_L) \tag{56}$$

where the conditions $C_1 = 0$ and $C(1) = 0$ (38), (35) are taken into account.

The mass conservation law (31) yields

$$\tfrac{1}{2}l^2\gamma - \tfrac{1}{24}x_L^2(12\gamma + 8C_2 x_L) + \tfrac{1}{3}l^3 C_2 - \tfrac{1}{24}l^4 \mathrm{Gr}\, C(0) = 0 \tag{57}$$

Plugging the relations $\gamma = a\mathrm{Gr}$, $C_2 = -\frac{\mathrm{Gr}}{3}$, $x_L = 3a$ and $C(0) = \frac{6a-2l}{l^2}$ into the mass conservation law (30) we have $-\frac{1}{36}\mathrm{Gr}(3a-l)\left(6al + 18a^2 - l^2\right) = 0$

The algebraic equation of the third order for the parameter $a>0$ has two positive solutions

$$a=l/3, \quad a=\frac{1}{6}l\left(\sqrt{3}-1\right) \tag{58}$$

Let us analyze the case $a=l/3$. It gives the trivial solution $C(0)=0$, while the other positive value $a=\frac{1}{6}l\left(\sqrt{3}-1\right)$ gives

$$B(0)=\frac{-\sqrt{3}+2}{l^3}, \quad C(0)=\frac{\sqrt{3}-3}{l}, \quad a=\frac{1}{6}l\left(\sqrt{3}-1\right) \tag{59}$$

6.2. Energy conservation law

The energy conservation equation (32) is transformed in a similar way, we are left with the flows across Σ_1, Σ_2, Σ_3 (Figure 1) and on the basis of the homogeneity of the problem with respect to coordinate z we have:

$$\int_0^1 \left.\frac{\partial T}{\partial x}\right|_{x=0} dy+\mathrm{Pr}\left(-\int_0^l T(x,0)W(x,0)dx+\int_0^{x_L} T(x,1)W(x,1)dx\right)=0 \tag{60}$$

To link the incoming fluid temperature $T=-1$ (from the bottom edge flow) we put $T(x,0)=-1$ at $y=0$ and the outgoing fluid (see (29)) which we take at the stability end level $y=Y$ results in:

$$\begin{aligned}\int_0^1 \left.\frac{\partial T}{\partial x}\right|_{x=0} dy+\mathrm{Pr}\Bigg(-\int_0^l(-1)\left(\gamma x+C_2x^2-\frac{\mathrm{Gr}}{6}C(0)x^3\right)dx+\\ +\int_0^{x_Y}(C(Y)x)\left(\gamma x+C_2x^2-\frac{\mathrm{Gr}}{6}C(Y)x^3\right)dx\Bigg)=0\end{aligned} \tag{61}$$

The equation (27) is an ordinary differential equation of the fourth order, hence its solution needs four constants of integration. These constants depend on two parameters C_1 and C_2, which enter the coefficients of Equation (27). The function $C(y)$ defines the other functions $\beta(y)$ and $B(y)$ via the above relations. This means that we have six constants determining the solution of the problem, but we also need six corresponding boundary conditions.

Now, we can return to the energy conservation equation (32) plugging the boundary conditions for the domain restricted by the plate in interval $(0,y=Y)$. This simplifies the expression for the integral along the plate surface (heat transfer from the plate in this interval). Consequently, we change x_L to x_{0Y} and neglect the integrand oscillations in the vicinity of $y=0$. It follows from the analysis of the behavior of function $C(y)$ that in the vicinity of $y=Y$ the second derivative $C''(y)$ and therefore $B(Y)$ is very small. In the same approximation $\int_0^1 C(y)dy=-\frac{1}{6a}$, $\gamma=a\mathrm{Gr}$, $C_2=-\frac{\mathrm{Gr}}{3}$, $x_{0Y}=6a$, $C(0)=\frac{6a-2l}{l^2}$ and $C(Y)=-\frac{1}{6a}$ we have:

$$-\mathrm{Pr}l^4\mathrm{Gr}\left(4\sqrt{3}-7\right)-180=0 \tag{62}$$

It allows us to link the Rayleigh number Ra with the width of the incoming bottom flow l at the leading edge.

$$\mathrm{Ra}=\mathrm{GrPr}=\frac{720\sqrt{3}+1260}{l^4} \tag{63}$$

We consider the parameter l as the basic characteristic scale of the phenomenon in our theory which is entered in the expressions of velocity and temperature.

$$W(x,y)=-\frac{1}{6}x\mathrm{Gr}\left(2x+x^2C(y)-\left(\sqrt{3}-1\right)\sqrt[4]{-\frac{180}{\mathrm{Ra}\left(4\sqrt{3}-7\right)}}\right) \tag{64}$$

$$T(x,y)=C(y)x-\frac{1}{6}\frac{d^2C(y)}{dy^2}x^3 \tag{65}$$

where the functional parameter $C(y)$ is given by (54). Plugging (59) yields:

$$\begin{aligned}C(y)=\left(\sqrt{3}+1\right)\frac{e^{s(y-1)}-1}{2l}+e^{-\frac{1}{2}sy}\left(\frac{3\sqrt{3}-5}{2l}\cos\frac{\sqrt{3}sy}{2}+\right.\\ \left.-\left(\frac{12\left(5-3\sqrt{3}\right)}{\sqrt{3}l^2s^2}-4+\frac{7}{\sqrt{3}}\right)\sin\frac{\sqrt{3}sy}{2}\right)\end{aligned} \tag{66}$$

The formulas show dependence on the only parameter $\mathrm{Ra}=\frac{bg\Phi L^3}{\nu^2}\mathrm{Pr}$ via explicit expressions for s (49) and l see (63).

7. Numerics

After substitution of the expression for $s=\sqrt[3]{2\mathrm{Ra}}$ and l

$$l=\left(\frac{180}{\mathrm{Ra}\left(7-4\sqrt{3}\right)}\right)^{\frac{1}{4}} \tag{67}$$

we have approximate formulas which define the expression for $C(y)$ as the function of parameters Ra (49) via the plate height L. The numerical value for the parameter as a function of Ra is used in further calculations. We choose the following set of data for air and our setup: $L=0.5\,\mathrm{m}$, $\nu=16\cdot10^{-6}\,\mathrm{m^2/s}$, $b=\frac{1}{303}\,\mathrm{K^{-1}}$, $g=10\,\mathrm{m/s^2}$, $T_w=40°\mathrm{C}$, $T_\infty=20°\mathrm{C}$, $\Phi=20\,\mathrm{K}$, $T=T'(T_w-T_\infty)+T_w$ [°C], $\mathrm{Pr}=0.7$. It gives: $\mathrm{Ra}=2.2561\cdot10^8$, $l=5.7737\cdot10^{-2}$, $s=767.0$, $W_o=\frac{\nu}{L}=3.2\cdot10^{-5}\,\mathrm{m/s}$, $a=7.0444\cdot10^{-3}$ and $x_{0Y}=4.2266\cdot10^{-2}$.

We plot the dependence $C(y)$ for the parameter values indicated above.

From the plot it is seen that the value of C is almost constant in the wide range of y, $y\in(Y_0=0.005,Y_L=0.98)$.

In the studies complete range of heights, the general expression of velocity is given by (64). Plugging numerical values for Ra one can plot the profiles at arbitrary y. See for example the conclusion.

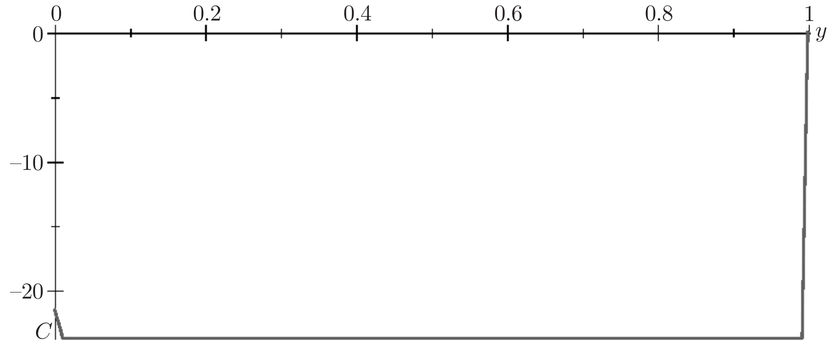

Figure 3. Dependence of C on y which corresponds to the choice of parameters given above

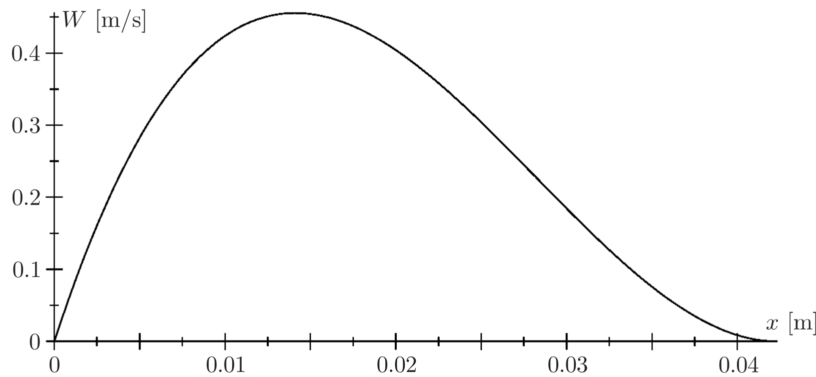

Figure 4. Vertical component of velocity profile in dimensional units at the range of approximately constant $C(y)$

The velocity profile $W(x,Y)$ at the middle level $Y=0.5$ is defined by (64) and the values of parameters (45):

$$W(x,Y)=x\mathrm{Ra}\frac{\left(x-l\left(\sqrt{3}-1\right)\right)^2}{6l\mathrm{Pr}\left(\sqrt{3}-1\right)}\cdot W_0=40600x(x-0.042)^2\left[\frac{\mathrm{m}}{\mathrm{s}}\right] \qquad (68)$$

In fact the values of the vertical velocity component are equal to $W_y=W_0W'_y=\frac{\nu}{L}W'_y$ (see (6) and (10)), where $W'_y\equiv W$ is shown on the plot. The amplitude value of the velocity is $3.2\cdot10^{-5}\,\frac{\mathrm{m}}{\mathrm{s}}\cdot14000=0.448\,\frac{\mathrm{m}}{\mathrm{s}}$.

The dimensional temperature (in °C units) profile at the level $Y=0.5$ is defined by (65) and the values of parameters (45) $T(x,Y)=(T_w-T_\infty)\big(x^3B(Y)+C(Y)x\big)+T_w=40.0-473x$. Compare with experimental data from [8]

8. Conclusions

The results of our theoretical study and numerical modeling show the reasonable behavior of the basic field profiles. Additionally our solution demonstrates a description of a real phenomenon at the leading edge vicinity. It is characterized by the new parameter l which is the width of the stream of the fluid inflowing from the bottom. This as well as other parameters depend on one convectional

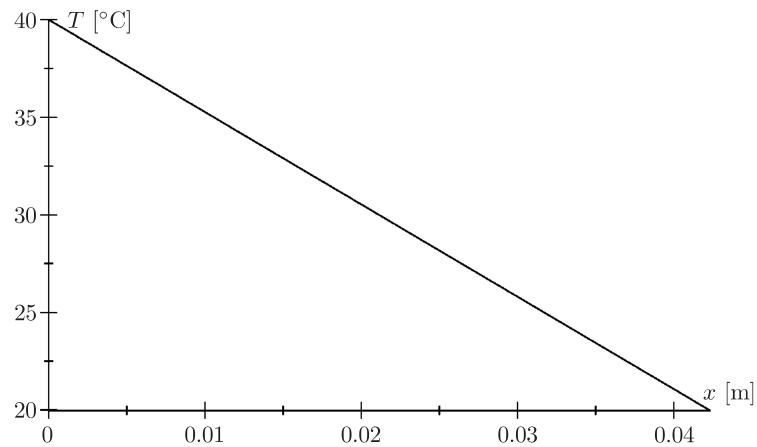

Figure 5. The temperature profile in Celsius units at the range of approximately constant $C(y)$

number Ra (the Rayleigh number). This parameter, absent in the conventional boundary layer theory, is visible and can be measured experimentally.

Our explicit solution form and parameter values estimation allows us to conclude that:

1. A set of boundary conditions is found which yields a complete set of solution parameters such as functions of Ra. These conditions of formulation were inspired by the visualization data [5, 6] and numerical simulations [9].
2. The velocity values of the fluid flow at the leading edge of the plate are nonzero at the interval $(0,l)$, that improves the existing theory of the free convection, similar to [11] but the self-similarity background of their theory complicates extraction of parameters from formalism.
3. The streamlines and isotherms of the flow are almost parallel to the vertical heating plate surface ($C(y) \approx \text{const}$, in the domain of stability). The conventional dependence differs from experiment at the vicinity of the leading edge (see Figure 1). It follows from the analysis of the explicit expression for $C(y)$ that the essential difference with the expression (68) takes place for the small interval in the vicinity of $y = 0$. To show this difference we have chosen for plotting the profile of velocity (68) at values of: $y = 0$.

It shows the change of the leading edge flow velocity direction that represents the vortex (Figure 6). It means that the contribution of the horizontal motion is comparable with the vertical one in this region. We will study this phenomenon in the nearest future on the basis of the analytical expressions introduced in this article.

We would stress again that the three-term model we present here has the engineering character of approximations. However, it includes direct possibilities for development by simply taking the next terms of expansions into account. Modification of the boundary conditions which would improve the description of regimes at both ends of the y-dependence is also possible.

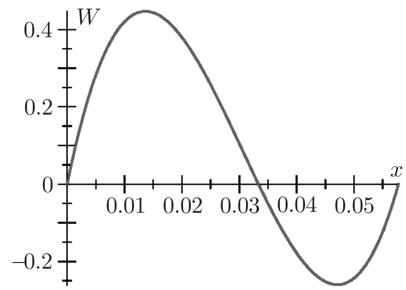

Figure 6. Development of velocity profile at the leading edge $y = 0$

Nevertheless, in this simple model we observe some important characteristic features of a real convection phenomenon, like the almost parallel streamlines and isotherms in the stability region (as, for example in visualizations of the interferometric study from [6]). This follows from the behavior of the functional parameter $C(y)$ inside the domain and the small contribution of the cubic term in the expression for temperature (29). One of the important practical conclusions is the well known form of the heat transfer coefficient k in terms of Rayleigh Ra ($k \sim C \cdot \mathrm{Ra}^{1/4}$) that is experimentally verified. The slight discrepancy in the numerical value of C may be corrected by means of taking into account all the terms of the Mittag-Leffler equation for the basic function $C(y)$.

Our theory may be directly applied to a more advanced fluid description, see for example [17] and [18].

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