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The cohomological span of $\mathcal{LS}$-Conley index

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Abstract

In this paper we introduce a new homotopy invariant – the cohomological span of $\mathcal{LS}$-Conley index. We prove the theorems on the existence of critical points for a class of strongly indefinite functionals with the gradient of the form $Lx + K(x)$, where L is bounded linear and K is completely continuous. We give examples of Hamiltonian systems for which our methods give better results than the Morse inequalities. We also give a formula for the $\mathcal{LS}$-index of an isolated critical point, which is an extension of the classical Dancer theorem for the case of $\mathcal{LS}$-index.

MSC: 37B30; 58E05

Keywords: Conley index in Hilbert spaces; Strongly indefinite operator; Critical point theory

1. Introduction

The purpose of this paper is to present a new homotopy invariant, the cohomological span of $\mathcal{LS}$-Conley index, and to show how this invariant can be applied to obtain existence and multiplicity results for strongly indefinite functionals.

The existence problem for solutions of differential equation can be restated in the terms of existence problems for critical points of functionals defined on Hilbert spaces. This approach leads to consider strongly indefinite functionals, i.e. both stable and unstable manifolds at a

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critical point are of infinite dimension. There are many methods developed to deal with such a functionals including both variational and topological methods (e.g. [1–7]).

In this paper we are going to work with $\mathcal{LS}$-Conley index – the extension of classical Conley index for Hilbert spaces presented by K. Gęba, M. Izydorek and A. Pruszko in [8]. Further development of this theory was given by Izydorek in [9]. He defined the cohomology groups of $\mathcal{LS}$-index and gave the examples how this theory can be applied for the existence problems for periodic solution of Hamiltonian systems. Applications to PDE were given later by M. Izydorek and K. Rybakowski in [10].

In 1984, E. Dancer proved in [11] that if p is a critical point of $F:\mathbb{R}^n \to \mathbb{R}$ then the classical Conley index of $\{p\}$ has the homotopy type of a k-fold suspension of some space connected with null space of $\nabla^2 F(p)$. In this paper we prove a similar statement for $\mathcal{LS}$-index (Theorem 3.8). This is a crucial observation for further results when we deal with cohomological span.

Let X be an isolating neighborhood. The cohomological span of $\mathcal{LS}$-index $h_{\mathcal{LS}}(X)$ is defined to be a pair

$$\Gamma(h_{\mathcal{LS}}(X)) = (\underline{\gamma}(h_{\mathcal{LS}}(X)), \overline{\gamma}(h_{\mathcal{LS}}(X))),$$

where $\underline{\gamma}(h_{\mathcal{LS}}(X))$ and $\overline{\gamma}(h_{\mathcal{LS}}(X))$ stand for the numbers of the first and the last nontrivial cohomology groups of $\mathcal{LS}$-index $h_{\mathcal{LS}}(X)$. In some cases, if $\operatorname{Inv} X$ is an isolated critical point then the difference $|\Gamma(X)| = \overline{\gamma}(h_{\mathcal{LS}}(X)) - \underline{\gamma}(h_{\mathcal{LS}}(X))$ can be globally bounded by some constant M. Hence, if for some set X we have $|\Gamma(X)| \geq M$ then X does not contain only critical point (see Corollary 4.8). This observation, combined with examining certain long exact sequences, gives us the existence results. Given examples show that in some situations our theory allows to find more critical points than another widely-applied method, i.e. examining Morse inequalities.

To be more precise, assume that S is an isolated invariant set and $S_1 \subset S$ is an isolated critical point. Suppose that Morse polynomials of S_1 and S are $P(t, S_1) = t^0 + t^1$ and $P(t, S) = t^m$, $m \geq M$. In this case, the Morse equation

$$t^0 + t^1 = t^m + (1+t)Q(t)$$

is not satisfied. Therefore, there exists at least one critical point in S, say p, different to S_1. Letting $P(t, \{p\}) = t^m$ and $Q(t) = t^0$ we obtain equality $t^0 + t^1 + t^m = t^m + (1+t)Q(t)$, which is true for all t.

On the other hand we have $|\Gamma(S_1)| = 2$, $|\Gamma(S)| = 1$, $\underline{\gamma}(S) - \overline{\gamma}(S_1) = m - 1$ and

$$|\Gamma(S)| + |\Gamma(S_1)| + \underline{\gamma}(S) - \overline{\gamma}(S_1) - 2 = m \geq M.$$

Theorem 4.16 follows that $S \setminus S_1$ contains at least two critical points. This result cannot be obtained by examining Morse equation (cf. Section 5).

This paper is organized as follows. In Section 2 we recall basic definitions and facts about the $\mathcal{LS}$-Conley index theory. We refer the reader to [8,9] for more details. In the next section we prove the formula for $\mathcal{LS}$-index of an isolated critical point. In Section 4 we introduce the cohomological span and next we prove the theorems concerning the existence of critical points. Finally, in Section 5 we give simple examples to show how our theory works when asymptotically linear Hamiltonian systems are considered.

2. $\mathcal{LS}$-Conley index

To make this paper self-contained we briefly recall basic facts from $\mathcal{LS}$-Conley index theory. The main references for this section are [8,9].

2.1. The $\mathcal{LS}$-index

Let E be a real Hilbert space and $L: E \to E$ be a linear bounded operator.

Definition 2.1. We say that a pair (E, L) is a base pair if

- (E_1) L gives a splitting $E = \bigoplus_{n=0}^{\infty} E_k$ onto finite dimensional, mutually orthogonal L-invariant subspaces,
- (E_2) $\sigma_0(L) = \sigma(L) \cap i\mathbb{R}$ is isolated in $\sigma(L)$,
- (E_3) $L(E_0) \subset E_0$, E_0 is a subspace corresponding to $\sigma_0(L)$, and $L(E_k) = E_k$ for all $k > 0$.

It follows easily that if (E, L) is a base pair then L is a Fredholm map of index 0. Let $E^n = \bigoplus_{k=0}^{n} E_k$. We will denote by P^n (resp. P_n) the orthogonal projection from E onto E^n (resp. E_n).

Let Λ be a compact space. A continuous map $f: E \times \Lambda \to E$ is completely continuous if it maps bounded sets to relatively compact sets.

Definition 2.2. A family of flows $\eta: \mathbb{R} \times E \times \Lambda \to E$ is a family of $\mathcal{LS}$-flows if there exists a completely continuous map $U: \mathbb{R} \times E \times \Lambda \to E$ such that $\eta(t, x, \lambda) = e^{tL}x + U(t, x\lambda)$.

Definition 2.3. A family of maps $f: E \times \Lambda \to E$ is a family of $\mathcal{LS}$-fields if there exists a completely continuous and locally Lipschitz continuous map $K: E \times \Lambda \to E$ such that $f(x, \lambda) = Lx + K(x, \lambda)$.

If $\Lambda = \{\lambda_0\}$ then we drop Λ out from notation and we call η an $\mathcal{LS}$-flow and f an $\mathcal{LS}$-field. If $\eta: \mathbb{R} \times E \times \Lambda \to E$ is a family of $\mathcal{LS}$-flows and $X \subset E$ then we define

$$\mathrm{Inv}(X \times \Lambda, \eta) = \{(x, \lambda) \in X \times \Lambda : \eta(t, x, \lambda) \in X,\ t \in \mathbb{R}\}.$$

The set $\mathrm{Inv}(X \times \Lambda, \eta)$ is the maximal η-invariant subset of X.

Theorem 2.4. *(See 2.3, [8].) Let η be a family of $\mathcal{LS}$-flows. If $X \subset E$ is closed and bounded then $S = \mathrm{Inv}(X \times \Lambda)$ is a compact subset of $X \times \Lambda$.*

Let $f: E \to E$ be a vector field. We say that f generates a (local) flow η if

$$\dot{\eta} = -f \circ \eta,\ \ \eta(0, x) = x.$$

We say that an $\mathcal{LS}$-field f is subquadratic if there exist $a, b > 0$ such that

$$|\langle K(x), x\rangle| \le a\|x\|^2 + b,\ \text{ for all } x \in E.$$

Theorem 2.5. *(See 2.6, [8].) If f is a subquadratic $\mathcal{LS}$-field then f generates an $\mathcal{LS}$-flow.*

Definition 2.6. We say that a closed and bounded subset $X \subset E$ is an isolating neighborhood for η if $\mathrm{Inv}(X, \eta) \subset \mathrm{int}(X)$.

To simplify notation we write $\mathrm{Inv}(X, f)$ or $\mathrm{Inv}(X)$ instead of $\mathrm{Inv}(X, \eta)$. Let $\nu\colon \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ be a fixed map and define $\rho\colon \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ by setting

$$\rho(0) = 0 \text{ and } \rho(n) = \sum_{i=0}^{n-1} \nu(i), \ n \geq 1.$$

Let $\{\mathcal{E}_n\}_{n=n(\mathcal{E})}^{\infty}$ be a sequence of compact pointed spaces and let $\left\{\varepsilon_n\colon S^{\nu(n)}\mathcal{E}_n \to \mathcal{E}_{n+1}\right\}_{n=n(\mathcal{E})}^{\infty}$ be a sequence of maps.

Definition 2.7. We say that a pair $\mathcal{E} = \left(\{\mathcal{E}_n\}_{n=n(\mathcal{E})}^{\infty}, \{\varepsilon_n\}_{n=n(\mathcal{E})}^{\infty}\right)$ is a spectrum if there exists $n_0 \geq n(\mathcal{E})$ such that ε_n is a homotopy equivalence for all $n \geq n_0$.

One can define a notion of maps of spectra, homotopy of spectra, homotopy type of spectra, etc. For fixed ν we denote by $\mathrm{Spec}(\nu)$ the category of spectra. We refer the reader to [8,9] for more details. In order to define the homotopy type $[\mathcal{E}]$ of spectra $\mathcal{E} \in \mathrm{Spec}(\nu)$ one only needs a sequence $\{\mathcal{E}_n\}$ such that $S^{\nu(n)}\mathcal{E}_n \simeq \mathcal{E}_{n+1}$ for sufficiently large n. The homotopy type $[\mathcal{E}]$ is uniquely determined by $\{\mathcal{E}_n\}$.

Given two spectra $\mathcal{E}, \mathcal{E}' \in \mathrm{Spec}(\nu)$ their wedge $\mathcal{E} \vee \mathcal{E}'$ is defined to be $\mathcal{E}^w \in \mathrm{Spec}(\nu)$ such that $\mathcal{E}_n^w = \mathcal{E}_n \vee \mathcal{E}'_n$ and

$$\varepsilon_n^w\colon S^{\nu(n)}(\mathcal{E}_n \vee \mathcal{E}'_n) \to S^{\nu(n)}\mathcal{E}_n \vee S^{\nu(n)}\mathcal{E}'_n \to \mathcal{E}_{n+1} \vee \mathcal{E}'_{n+1} = \mathcal{E}_{n+1}^w.$$

For $\mathcal{E} \in \mathrm{Spec}(\nu)$ and $\mathcal{E}' \in \mathrm{Spec}(\mu)$ we define smash product $\mathcal{E} \wedge \mathcal{E}'$ to be $\mathcal{E}^s \in \mathrm{Spec}(\nu + \mu)$ such that $\mathcal{E}_n^s = \mathcal{E}_n \wedge \mathcal{E}'_n$ and

$$\varepsilon_n^s\colon S^{\nu(n)+\mu(n)}(\mathcal{E}_n \wedge \mathcal{E}'_n) \to S^{\nu(n)}\mathcal{E}_n \wedge S^{\mu(n)}\mathcal{E}'_n \to \mathcal{E}_{n+1} \wedge \mathcal{E}'_{n+1} = \mathcal{E}_{n+1}^s.$$

Let $\mathcal{O} = \left(\{\mathcal{O}_n\}_{n=0}^{\infty}, \{o_n\}_{n=0}^{\infty}\right)$ be a spectrum such that $\mathcal{O}_n$ consists only of the base point, o_n maps the base point into the base point. We call $\mathcal{O}$ the trivial spectrum.

The spectrum $\mathcal{E}$ has the homotopy type of $\mathcal{O}$ if $\mathcal{E}_n$ is homotopy equivalent to one-point space for n sufficiently large. Obviously, the trivial spectrum is an object of $\mathrm{Spec}(\nu)$ for arbitrary ν.

Definition 2.8. Let $\nu_0(n) = 0$. The spectrum $\mathcal{E}$ is an object of $\mathrm{Spec}(\nu_0)$, if there exists $n_0 \geq n(\mathcal{E})$ such that $\mathcal{E}_{n+1} \simeq \mathcal{E}_n$ provided $n \geq n_0$.

Fix $k \in \mathbb{Z}$. Assume that there exists $n \in \mathbb{N} \cup \{0\}$ such that $k + \rho(n) \geq 0$. Let $n(\mathcal{S}^k)$ be the smallest number for which the above inequality holds. Define

$$\mathcal{S}^k = \left(\{S^{k+\rho(n)}\}_{n=n(\mathcal{S}^k)}^{\infty}, \{\sigma_n^k\}_{n=n(\mathcal{S}^k)}^{\infty}\right),$$

where $\sigma_n^k = id_{S^{k+\rho(n+1)}}$.

Definition 2.9. We call $\mathcal{S}^k$ the k-dimensional sphere in the category Spec(ν).

If $k \geq 0$ then $n(\mathcal{S}^k) = 0$. Hence, if $k \geq 0$ then $\mathcal{S}^k$ is an object of Spec(ν) for arbitrary ν. Let $\mathcal{E} \in$ Spec(ν). Note that

$$[\mathcal{E}] = [\mathcal{S}^k] \Longleftrightarrow \exists_{n_0 \geq \max\{n(\mathcal{E}), n(\mathcal{S}^k)\}} \forall_{n \geq n_0} \mathcal{E}_n \simeq S^{k+\rho(n)}$$

If $\mathcal{E} \in$ Spec(ν) and $\mathcal{S}^1 \in$ Spec(ν_0) then the spectrum $\mathcal{S}^1 \wedge \mathcal{E} \in$ Spec(ν) has a homotopy type of the suspension $S\mathcal{E}$ introduced in [8].

Let $m^-(A)$ denote the Morse index of a linear map $A: E \to E$, i.e. the maximal dimension of a subspace of E on which $\langle Ax, x\rangle$ is negative definite.

Set $\nu(n) = \dim E^-_{n+1} = m^-(L_{|E_{n+1}})$. Let $f: E \to E$ be an $\mathcal{LS}$-field. For simplicity we assume that f is globally subquadratic. Let η denote the $\mathcal{LS}$-flow generated by f and let $X \subset E$ be an isolating neighborhood for η. Define

$$f_n: E^n \to E^n, \; f_n(x) = Lx + P^n K(x)$$

Let η_n be the flow induced by f_n.

Lemma 2.10. *(See 4.1, [8].) There exists $n_0 \in \mathbb{N} \cup \{0\}$ such that if $n \geq n_0$ then $X^n = X \cap E^n$ is an isolating neighborhood for the flow η_n.*

It follows that there exists an index pair (Y^n, Z^n) in X^n and the (classical) Conley index of Inv(X^n, η_n) is the homotopy type of the pointed space Y^n/Z^n. The sequence $\{\mathcal{E}_n\}_{n=n_0}^{\infty} = \{Y^n/Z^n\}_{n=n_0}^{\infty}$ uniquely determines the homotopy type of spectrum $\mathcal{E}$.

Definition 2.11. Let η be an $\mathcal{LS}$-flow generated by an $\mathcal{LS}$-field f and let $X \subset E$ be an isolated neighborhood for η. Define $h_{\mathcal{LS}}(X, \eta) = [\mathcal{E}]$. We call $h_{\mathcal{LS}}(X, \eta)$ the $\mathcal{LS}$-Conley index of X with respect to η.

Note that $h(X^n, \eta_n) = [Y^n/Z^n]$. The following theorem gives the basic properties of the $\mathcal{LS}$-index.

Theorem 2.12. *(See 4.4, 4.5, [8].)*

1. *(nontriviality) Let η be an $\mathcal{LS}$-flow and $X \subset E$ be an isolating neighborhood for η. If $h_{\mathcal{LS}}(X, \eta) \neq [\mathcal{O}]$ then* Inv$(X, \eta) \neq \emptyset$.
2. *(continuation) Let Λ be a compact, connected and locally contractible metric space. Assume that $\eta: \mathbb{R} \times E \times \Lambda \to E$ is a family of $\mathcal{LS}$-flows. Let X be an isolating neighborhood for a flow $\eta_\lambda = \eta(\cdot, \cdot, \lambda)$ for some $\lambda \in \Lambda$. Then there is a compact neighborhood $U_\lambda \subset \Lambda$, $\lambda \in U_\lambda$ such that*

$$h_{\mathcal{LS}}(X, \eta_\mu) = h_{\mathcal{LS}}(X, \eta_\nu), \; \textit{for all } \mu, \nu \in U_\lambda.$$

If E is finite dimensional and $L \equiv 0$ then (E, L) is a base pair and every locally Lipschitz map $f: E \to E$ is an $\mathcal{LS}$-field. In this case an $\mathcal{LS}$-index is represented by a spectrum in Spec(ν_0) and $h_{\mathcal{LS}}(X, f)$ is equal to $h(X, f)$, the classical Conley index.

By abuse of notation, we write $h_{\mathcal{LS}}(\mathrm{Inv}(X), f)$ instead of $h_{\mathcal{LS}}(X, \eta)$, provided η is the flow generated by the $\mathcal{LS}$-field f and X is an isolating neighborhood for η.

2.2. Cohomological $\mathcal{LS}$-Conley index

The main reference for this section is [9]. Throughout and subsequently $\check{H}$ denotes the (reduced) Čech cohomology with coefficients in $\mathbb{Z}$. Let $\mathcal{E} = \left(\{\mathcal{E}_n\}_{n=n(\mathcal{E})}^{\infty}, \{\varepsilon_n\}_{n=n(\mathcal{E})}^{\infty}\right)$ be a spectrum in $\mathrm{Spec}(\nu)$. For a fixed $q \in \mathbb{Z}$ consider a sequence of cohomology groups

$$\check{H}^{q+\rho(n)}(\mathcal{E}_n), \ n \geq n(\mathcal{E}).$$

Define a sequence of homomorphisms h_n

$$h_n\colon \check{H}^{q+\rho(n+1)}(\mathcal{E}_{n+1}) \xrightarrow{\varepsilon_n^{q+\rho(n+1)}} \check{H}^{q+\rho(n+1)}(S^{\nu(n)}\mathcal{E}_n) \xrightarrow{(S^*)^{\nu(n)}} \check{H}^{q+\rho(n)}(\mathcal{E}_n)$$

where S^* denotes the suspension isomorphism.

Definition 2.13. The q-th cohomology group of a spectrum $\mathcal{E}$ is defined to be $H^q(\mathcal{E}) = \varprojlim\{\check{H}^{q+\rho(n)}(\mathcal{E}_n), h_n\}$.

Since $\mathcal{E}_{n+1} \simeq S^{\nu(n)}\mathcal{E}_n$ for sufficiently large n, we see that $h_n\colon \check{H}^{q+\rho(n+1)}(\mathcal{E}_{n+1}) \to \check{H}^{q+\rho(n)}(\mathcal{E}_n)$ is an isomorphism. The sequence of groups $\check{H}^{q+\rho(n)}(\mathcal{E}_n)$ stabilizes and there is $n_0 \in \mathbb{N}$ such that $H^q(\mathcal{E}) \simeq \check{H}^{q+\rho(n)}(\mathcal{E}_n)$ for all $n \geq n_0$. Notice that cohomology groups of $\mathcal{E}$ may be nonzero for both positive and negative integers (see [9]).

We say that the spectrum $\mathcal{E}$ is of finite type if $H^*(\mathcal{E})$ is finitely generated and almost all groups are trivial.

Clearly, for the trivial spectrum $\mathcal{O}$ we have $H^q(\mathcal{O}) = 0$, for all $q \in \mathbb{Z}$. If $\mathcal{S}^k$ is the k-dimensional sphere in $\mathrm{Spec}(\nu)$ then

$$H^q(\mathcal{S}^k) \simeq \check{H}^{q+\rho(n)}(\mathcal{S}^k_n) \simeq \check{H}^{q+\rho(n)}(S^{k+\rho(n)}),$$

for all $n \geq n(\mathcal{S}^k)$. Hence, $H^q(\mathcal{S}^k) = \mathbb{Z}$ if $q = k$ and $H^q(\mathcal{S}^k) = 0$ if $q \neq k$.

Proposition 2.14. *If $\mathcal{S}^k \in \mathrm{Spec}(\nu)$ and $\mathcal{E}_0 \in \mathrm{Spec}(\nu_0)$ then $H^q(\mathcal{S}^k \wedge \mathcal{E}_0) = H^{q-k}(\mathcal{E}_0)$.*

Proof. Let $\mathcal{E}_n = S^{\rho(n)} \wedge \mathcal{E}_{0,n}$. Then $\mathcal{E} = \{\mathcal{E}_n\} \in \mathrm{Spec}(\nu)$. Moreover $H^q(\mathcal{E}) \simeq \check{H}^{q+\rho(n)}(S^{\rho(n)} \wedge \mathcal{E}_{0,n}) \simeq \check{H}^q(\mathcal{E}_{0,n})$. For n sufficiently large we have

$$H^q(\mathcal{S}^k \wedge \mathcal{E}_0) \simeq \check{H}^{q+\rho(n)}(S^{k+\rho(n)} \wedge \mathcal{E}_{0,n}) \simeq \check{H}^{q+\rho(n)-k}(S^{\rho(n)} \wedge \mathcal{E}_{0,n}) \simeq \check{H}^{q-k}(\mathcal{E}_{0,n})$$

and $\check{H}^{q-k}(\mathcal{E}_{0,n}) \simeq H^{q-k}(\mathcal{E}_0)$. □

For the pair of spectra $(\mathcal{E}, \mathcal{A})$ we define the q-th cohomology group $H^q(\mathcal{E}/\mathcal{A})$ as an inverse limit of the $\{\check{H}(\mathcal{E}_n/\mathcal{A}_n), \gamma_n\}$ and the q-th cohomology group of $(\mathcal{E}, \mathcal{A})$ as $H^q(\mathcal{E}, \mathcal{A}) = H^q(\mathcal{E}/\mathcal{A})$.

There is a long exact sequence

$$\dots \xrightarrow{\delta^{q-1}} H^q(\mathcal{E},\mathcal{A}) \longrightarrow H^q(\mathcal{E}) \longrightarrow H^q(\mathcal{A}) \xrightarrow{\delta^q} \dots$$

Assume that all groups in the above sequence are of finite rank and trivial for all q less than some fixed $q_0 \in \mathbb{Z}$. Denote by $r^q(\mathcal{E})$ the rank of $H^q(\mathcal{E})$ and by $d^q(\mathcal{E},\mathcal{A})$ the rank of $\operatorname{im}\{\delta^q : H^q(\mathcal{A}) \to H^{q+1}(\mathcal{E},\mathcal{A})\}$. Let

$$P(t,\mathcal{E}) = \sum_{q\in\mathbb{Z}} r^q(\mathcal{E})t^q,$$

$$Q(t,\mathcal{E},\mathcal{A}) = \sum_{q\in\mathbb{Z}} d^q(\mathcal{E},\mathcal{A})t^q.$$

Lemma 2.15. *(See 3.5, [9].)*

$$P(t,\mathcal{E}/\mathcal{A}) + P(t,\mathcal{A}) = P(t,\mathcal{E}) + (1+t)Q(t,\mathcal{E},\mathcal{A})$$

Since the $P(t,\mathcal{E})$ and $Q(t,\mathcal{E},\mathcal{A})$ are well defined for the homotopy type of spectra, we can substitute $\mathcal{E}, \mathcal{A}, \mathcal{E}/\mathcal{A}$ by its homotopy type (see [9]).

2.3. Morse inequalities

Let η be an $\mathcal{LS}$-flow generated by an $\mathcal{LS}$-field f and assume that $X \subset E$ is an isolating neighborhood for η. Set $S = \operatorname{Inv}(X,\eta)$.

Definition 2.16. The finite collection $\{M(\pi) : \pi \in \Pi\}$ of compact invariant sets in S is said to be a Morse decomposition of S if there exists an ordering $\pi_1,\dots,\pi_n$ of Π such that for every $x \in S \setminus \bigcup_{\pi\in\Pi} M(\pi)$ there exists indices $1 \le i < j \le n$ such that $\omega(x) \in M(\pi_i)$, $\alpha(x) \in M(\pi_j)$. The sets $M(\pi_i)$ are called Morse sets.

We allow the situation that a Morse decomposition consists of finite number of empty sets. The index of $M(\pi) = \emptyset$ can appear in any place in the ordering $\pi_1,\dots,\pi_n$. We adopt the convention that the empty set is an isolating neighborhood of the empty set and the $\mathcal{LS}$-index of the empty set is a homotopy type of the trivial spectrum.

Theorem 2.17. *(See 4.2, [9].) Let $\{M(\pi) : \pi \in \Pi\}$ be a Morse decomposition of S. There are closed subsets $X_1,\dots,X_n = X = X_0^*,\dots,X_{n-1}^*$ of X such that:*

1. *X_i, X_j^* are isolating neighborhoods for η,*
2. $\operatorname{Inv}(X_i \cap X_{i-1}^*, \eta) = M(\pi_i)$, $i = 1,\dots,n$.

If one of the sets $M(\pi_i)$, $M(\pi_{i+1})$ is empty, then we can assume that $X_i = X_{i+1}$, $X_{i-1}^* = X_i^*$ and $X_i \cap X_{i-1}^* = \emptyset$.

Let (A, A^*) be an attractor–repeler pair in S. Since A and A^* are isolated and invariant, there are isolating neighborhoods X_A and X_A^* for η such that $A = \operatorname{Inv}(X_A,\eta)$ and $A^* = \operatorname{Inv}(X_A^*,\eta)$.

Theorem 2.18. *(See 4.3, [9].) There exist spectra $\mathcal{E}_A$, $\mathcal{E}_{A^*}$ and $\mathcal{E}_S$ representing $\mathcal{LS}$-Conley indices of X_A, X_A^* and X_S such that the sequence*

$$\dots \xrightarrow{\delta^{q-1}} H^q(\mathcal{E}_{A^*}) \longrightarrow H^q(\mathcal{E}_S) \longrightarrow H^q(\mathcal{E}_A) \xrightarrow{\delta^q} \dots$$

is exact.

From the above theorem and Lemma 2.15 we obtain:

$$P(t, [\mathcal{E}_{A^*}]) + P(t, [\mathcal{E}_A]) = P(t, [\mathcal{E}_S]) + (1+t)Q(t).$$

Let $\{M(\pi_i): \pi = 1 \dots n\}$ be a Morse decomposition of S. Denote by $[\mathcal{E}_{M(\pi_i)}]$ the $\mathcal{LS}$-index of $\operatorname{Inv}(X_i \cap X_{i-1}^*)$. From 2.17 we have an attractor–repeler pair

$$\big(\operatorname{Inv}(X_{i-1}, \eta), \operatorname{Inv}(X_i \cap X_{i-1}^*, \eta)\big)$$

in $\operatorname{Inv}(X_i, \eta)$ and

$$P\Big(t, \big[\mathcal{E}_{X_{i-1}}\big]\Big) + P\Big(t, \big[\mathcal{E}_{M(\pi_i)}\big]\Big) = P\Big(t, \big[\mathcal{E}_{X_i}\big]\Big) + (1+t)Q_i(t).$$

Adding these equations we obtain the following theorem.

Theorem 2.19. *(See Morse inequalities, 4.7, [9].) Under the above assumptions,*

$$\sum_{i=1}^{n} P\Big(t, \big[\mathcal{E}_{M(\pi_i)}\big]\Big) = P\Big(t, \big[\mathcal{E}_S\big]\Big) + (1+t)Q(t),$$

where $Q(t) = \sum_{i=2}^{n} Q_i(t)$.

Remark 2.20. Recently, an alternative approach of computing cohomological Conley index was introduced by M. Starostka in [12], where author defines an index pair directly in a Hilbert space.

3. Dancer theorem for $\mathcal{LS}$-index

3.1. E-Morse index

Let (E, L) be a base pair. Set $\nu(n) = \dim E_{n+1}^-$ and define $\rho(n)$ as in the previous section. Assume that $A: E \to E$ is a linear and self-adjoint $\mathcal{LS}$-field. Set $X_1 = \operatorname{im} A$ and $X_2 = \ker A$. We will denote by Q the orthogonal projection from E onto X_2, and by T^n the orthogonal projection from X_1 onto $X_1 \cap E^n$, $n \in \mathbb{N} \cup \{0\}$. Since A is Fredholm, $E = X_1 \oplus X_2$. For $x \in E$ we will write $x = x_1 + x_2$ where $x_1 \in X_1$, $x_2 \in X_2$.

The following lemma is due to W. Kryszewski and A. Szulkin. For the proof we refer the reader to [13] (pp. 3198–3200).

Lemma 3.1.

1. *There exists* $n_B \in \mathbb{N} \cup \{0\}$ *such that for all* $n \geq n_B$, $P^n_{|X_2}: X_2 \to P^n X_2$ *is a linear isomorphism.*
2. *For all* $n \in \mathbb{N} \cup \{0\}$ *the spaces* $X_1 \cap E^n$ *and* $P^n X_2$ *are orthogonal. Moreover,* $E^n = (X_1 \cap E^n) \oplus P^n X_2$.
3. *For all* $n \in \mathbb{N} \cup \{0\}$ *the spaces* $X_1 \cap E_n$ *and* $P_n X_2$ *are orthogonal. Moreover,* $E_n = (X_1 \cap E_n) \oplus P_n X_2$.
4. $P^n_{|X_1} - T^n \to 0$ *in* $\mathcal{L}(X_1, E)$.
5. *There exist* $c > 0$ *and* $n_0 > 0$ *such that* $\|T^n A x_1\| \geq c\|x_1\|$, *for all* $n \geq n_0$ *and* $x_1 \in X_1 \cap E^n$.

Definition 3.2. Bounded linear operator $A: E \to E$ is said to be hyperbolic if

$$\inf\{|x - \lambda| : x \in i\mathbb{R}, \lambda \in \sigma(A)\} > 0.$$

The set of all hyperbolic operators is an open subset of the space of linear bounded operators (see [14]).

We denote by $i^0(A)$ the nullity of A, i.e. the dimension of $\ker A$.

Definition 3.3. The E-Morse index of linear self-adjoint $\mathcal{LS}$-field A is defined to be

$$i^-(A) = \lim_{n\to\infty} \left(m^-(T^n A_{|X_1 \cap E^n}) - \rho(n)\right).$$

Definitions of $i^-(A)$ is similar to definition of the $M^-_{\mathcal{E}}$ given by Kryszewski and Szulkin. In case of self-adjoint $\mathcal{LS}$-fields these indexes are equal (see formula (5.1) in [13] and following remarks).

Lemma 3.4.

$$\exists_{n_A \in \mathbb{N}\cup\{0\}} \forall_{n \geq n_A}\ m^-(T^{n+1} A_{|X_1 \cap E^{n+1}}) = m^-(T^n A_{|X_1 \cap E^n}) + \nu(n)$$

Proof. The idea of the proof is adapted from [13]. Let $K = A - L$. Since A is an $\mathcal{LS}$-field, K is compact. Note that $P^{n+1} X_2 = P^n X_2 \oplus P_{n+1} X_2$. Hence $\dim P_{n+1} X_2 = \dim P^{n+1} X_2 - \dim P^n X_2 = 0$, $n \geq n_B$ by Lemma 3.1. This gives $E_{n+1} = (X_1 \cap E_{n+1}) \oplus P_{n+1} X_2 = X_1 \cap E_{n+1}$ and

$$X_1 \cap E^{n+1} = (X_1 \cap E^n) \oplus E_{n+1}.$$

Set $y + z \in (X_1 \cap E^n) \oplus E_{n+1}$. For $n \geq n_B$ define $D^n: [0, 1] \times X_1 \cap E^{n+1} \to X_1 \cap E^{n+1}$ by:

$$D^n_t(y + z) = (1 - t)T^{n+1} A(y + z) + t(T^n Ay + Lz).$$

For $n \geq n_B$, $t \in [0, 1]$ and $y + z \in X_1 \cap E^{n+1}$, $\|y + z\| \leq 1$, we get:

$$\begin{aligned}
&D_0^n(y+z) - D_t^n(y+z)\\
&\quad = t(T^{n+1}A(y+z) - T^nAy - Lz)\\
&\quad = t\left(-(P^{n+1}-T^{n+1})A(y+z) + (P^n - T^n)Ay + P^{n+1}A(y+z) - P^nAy - Lz\right).
\end{aligned}$$

Since $y, z \in E^{n+1}$, $L(E_n) = E_n$ and $z = P_{n+1}z$,

$$\begin{aligned}
P^{n+1}A(y+z) - P^nAy - Lz &= P^{n+1}L(y+z) + P^{n+1}K(y+z) - P^nLy - P^nKy - Lz\\
&= P^{n+1}K(y+z) - P^nKy = P_{n+1}Ky + P^{n+1}KP_{n+1}z.
\end{aligned}$$

It follows that

$$\|D_0^n(y+z) - D_t^n(y+z)\| \le \|P^{n+1} - T^{n+1}\|\|A\| + \|P^n - T^n\|\|A\| + \|P_{n+1}K\| + \|KP_{n+1}\|.$$

Note that $\|P_{n+1}K\| \to 0$, $\|KP_{n+1}\| \to 0$ and $P^n - T^n \to 0$ uniformly on X_1. Finally,

$$\|D_0^n - D_t^n\| \to 0,\ n \to \infty.$$

From Lemma 3.1 it follows that, for large n, the operator $D_0^n = T^{n+1}A_{|X_1\cap E^{n+1}}$ is hyperbolic. Since hyperbolic operators form an open set, there exists $n_A \in \mathbb{N}\cup\{0\}$ such that, for all $n \ge n_A$ and $t \in [0,1]$ the operator D_t^n is hyperbolic. This shows that $m^-(D_t^n)$ is constant. Thus

$$m^-(T^{n+1}A_{|X_1\cap E^{n+1}}) = m^-(T^nA_{|X_1\cap E^n} + L_{|E_{n+1}}) = m^-(T^nA_{|X_1\cap E^n}) + \nu(n),$$

which completes the proof. □

The next theorem follows directly from Lemma 3.4.

Theorem 3.5. *The E-Morse index $i^-(A)$ is finite. Moreover,*

$$i^-(A) = m^-(T^nA_{|X_1\cap E^n}) - \rho(n),\ \textit{for all } n \ge n_A.$$

The index $i^-(A)$ can take any integer values. Let $\{e_n\}$ be an arbitrary basis of E. Set

$$L(e_{2n-1}) = -e_{2n-1},\ L(e_{2n}) = e_{2n}$$

and set E_n to be the space spanned by e_{2n-1}, e_{2n}. Note that (E, L) is a base pair and $\nu(n) = 1$, $\rho(n) = n$.

1. If $A_1 = L$ then $i^0(A_1) = 0$, $i^-(A_1) = 0$.
2. If $A_2 = L + id_{E^n}$ then $i^0(A_2) = \frac{1}{2}\dim E^n$, $i^-(A_2) = -n$.
3. If $A_3 = L \pm 2id_{E^n}$ then $i^0(A_3) = 0$, $i^-(A_3) = \mp n$.

Observe that $\{0\} \subset E$ is an isolated invariant set for the flows generated by A_j, $j = 1,2,3$. Moreover $h_{\mathcal{LS}}(\{0\}, A_j)$ has the homotopy type of $i^-(A_j)$-dimensional sphere in the category Spec(ν).

3.2. $\mathcal{LS}$-index of an isolated critical point

Let $\Phi: E \to \mathbb{R}$. We assume that

(D_1) $\Phi \in C^2(E, \mathbb{R})$,
(D_2) $\nabla\Phi(x) = Lx + K(x)$, where $K \in C^1(E, E)$ is a completely continuous function,
(D_3) $p \in E$ is an isolated critical point of Φ.

Without loss of generality, we can assume $p = 0$. The Hessian $A = \nabla^2\Phi(0) = L + DK(0)$ is a self-adjoint and Fredholm map as a sum of Fredholm and compact map. Since A is Fredholm, $E = X_1 \oplus X_2$.

Suppose $i^0(A) \neq 0$. According to the splitting lemma (Theorem 8.3, [15]), there exist $r > 0$, a local homeomorphism $h: B(0, r) \to E$ and a map $u: B(0, r) \cap X_2 \to X_1$, $u \in C^1$ such that $h(0) = 0$, $u(0) = 0$, $Du(0) = 0$ and

$$\Phi(h(x)) = \frac{1}{2}\langle Ax_1, x_1\rangle + \Phi(x_2 + u(x_2)),$$

where $x = x_1 + x_2 \in B(0, r)$. Moreover, $(I - Q)\nabla\Phi(x_2 + u(x_2)) = 0$, provided $\|x_2\| < r$.

Set $B(x_2) = \Phi(x_2 + u(x_2))$. The map B is C^2 and $0 \in X_2$ is an isolated critical point of B. Define $\Psi: B(0, r) \to \mathbb{R}$ by $\Psi(x_1 + x_2) = \frac{1}{2}\langle Ax_1, x_1\rangle + B(x_2)$. Then

$$\nabla\Psi(x) = Ax_1 + \nabla B(x_2) = Lx + (-Lx_2 + DK(0)x_1 + \nabla B(x_2)),$$

where $-Lx_2 + DK(0)x_1 + \nabla B(x_2)$ is compact. It follows easily that 0 is an isolated critical point of Ψ. Chose $r_0 < r$ such that $X = B(0, r_0)$ is an isolating neighborhood for the flows generated by $\nabla\Phi$ and $\nabla\Psi$ with $\{0\} = \mathrm{Inv}(X, \nabla\Phi) = \mathrm{Inv}(X, \nabla\Psi)$. We can extend the field $\nabla\Psi$ to E by setting

$$\nabla\Psi(x) = Lx + \mu(x)\,(-Lx_2 + DK(0)x_1 + \nabla B(x_2)),$$

where $\mu: E \to \mathbb{R}$ is a smooth function, equals 1 on X and vanishing outside $B(0, r)$. Obviously, $\nabla\Psi$ is a subquadratic $\mathcal{LS}$-vector field.

Lemma 3.6.

$$h_{\mathcal{LS}}(X, \nabla\Psi) = h_{\mathcal{LS}}(X, \nabla\Phi)$$

Proof. Define $H: X \times [0, 1] \to \mathbb{R}$ by

$$\begin{aligned} H(x, \lambda) = {}& \frac{1}{2}\langle Ax_1, x_1\rangle + \frac{1}{2}\lambda(2 - \lambda)\langle Au(x_2), u(x_2)\rangle \\ & + \lambda G(x_2 + u(x_2)) + (1 - \lambda)G(x_1 + x_2 + \lambda u(x_2)), \end{aligned}$$

where $G(v) = \Phi(v) - \frac{1}{2}\langle Av, v\rangle$. This family has been introduced by Dancer in [11]. One can prove that

1. $\nabla H(x,0) = \nabla\Phi(x)$ and $\nabla H(x,1) = \nabla\Psi(x)$,
2. ∇H is a continuous family of $\mathcal{LS}$-fields,
3. $\mathrm{Inv}(X \times [0,1], \nabla H) = \{0\} \times [0,1]$.

For more details we refer the reader to [13]. Now the conclusion follows from Theorem 2.12. □

Set $Y = X \cap X_2 \subset X_2$. The set Y is an isolating neighborhood for the flow η_B generated by ∇B. Choose an index pair (N_B, L_B) in Y.

Definition 3.7. We will denote by $\mathcal{B}$ the spectrum in the category $\mathrm{Spec}(\nu_0)$ such that $n(\mathcal{B}) = 0$ and $\mathcal{B}_0 = N_B/L_B$.

The homotopy type of $\mathcal{B}$ is independent of the choice of an index pair (N_B, L_B). Let $(P^n_{|X_2})_*\nabla B(z) = P^n_{|X_2}\nabla B\left((P^n_{|X_2})^{-1}z\right)$ and let $(P^n_{|X_2})_*\eta_B$ denote the flow induced by $(P^n_{|X_2})_*\nabla B(z)$ on $P^n X_2$. Then $P^n_{|X_2}(Y) \subset P^n X_2$ is an isolated neighborhood for $(P^n_{|X_2})_*\eta_B$ and

$$h\left(P^n_{|X_2}(Y), (P^n_{|X_2})_*\eta_B\right) = h(Y, \eta_B),$$

provided $n \geq n_B$. If $i^0(A) = 0$ then $X_2 = \{0\}$ and A is an isomorphism. We can set $B = 0$ and $\mathcal{B} = \mathcal{O}$. Now, we can formulate the main theorem of this section.

Theorem 3.8. *If $\Phi: E \to \mathbb{R}$ satisfies assumptions (D_1)–(D_3) then*

$$h_{\mathcal{LS}}(\{p\}, \nabla\Phi) = \left[\mathcal{S}^{i^-(A)} \wedge \mathcal{B}\right].$$

Proof. It is sufficient to prove that $h_{\mathcal{LS}}(X, \nabla\Psi) = \left[\mathcal{S}^{i^-(A)} \wedge \mathcal{B}\right]$. Let $[\mathcal{E}] = h_{\mathcal{LS}}(X, \nabla\Psi)$, $\mathcal{E} \in \mathrm{Spec}(\nu)$. We can assume that $X \cap E^n$ is an isolated neighborhood for the flow generated by $\nabla\Psi$, provided $n \geq n(\mathcal{E})$. Assume that $n \geq \max\{n_A, n_B, n(\mathcal{E})\}$.

If $i^0(A) = 0$ then $X_1 = E$, $X_2 = 0$, $B = 0$ and $T^n = P^n$. From Lemmas 3.1 and 3.4 we get

$$\mathcal{E}_n \simeq S^{m^-(P^n A_{|E^n})} = S^{i^-(A)+\rho(n)}.$$

Thus $[\mathcal{E}] = [\mathcal{S}^{i^-(A)}]$. Let $i^0(A) \neq 0$. It follows from Lemma 3.1 that $E^n = (X_1 \cap E^n) \oplus P^n X_2$ and $P^n X_2 \simeq X_2$. For $x_1 + P^n x_2 \in X^n = X \cap ((X_1 \cap E^n) \oplus P^n X_2)$ we obtain

$$\begin{aligned}
\nabla\Psi_n(x_1 + P^n x_2) &= P^n A(x_1 + P^n x_2) + P^n \nabla B(Q(x_1 + P^n x_2)) \\
&= T^n A x_1 + (P^n_{|X_2})_* \nabla B(P^n x_2) \\
&\quad + (P^n - T^n)A x_1 + P^n A P^n x_2 + P^n \left(\nabla B(Q P^n x_2) - \nabla B(x_2)\right).
\end{aligned}$$

Again, by Lemma 3.1 $\sup\{\|(P^n - T^n)Ax_1\| : x_1 \in X^n\} \to 0$.

Note that $(P^n_{|X_2})^{-1}(X \cap P^n X_2) \subset X_2$ is compact. Since $P^n \to I$ uniformly on compact subsets of X_2,

$$\sup\{\|P^n A P^n x_2\| : P^n x_2 \in X^n\} \to 0$$

and

$$\sup\{\|P^n\left(\nabla B(QP^n x_2) - \nabla B(x_2)\right)\|: P^n x_2 \in X^n\}$$
$$\leq L_B \sup\{\|QP^n x_2 - x_2\|: P^n x_2 \in X^n\} \to 0,$$

where L_B is the Lipschitz constant for ∇B. It follows that

$$\sup\{\|\nabla\Phi_n(x_1 + P^n x_2) - T^n A_{|X_1\cap E^n} x_1 - (P^n_{|X_2})_* \nabla B(P^n x_2)\|: x_1 + P^n x_2 \in X^n\} \to 0.$$

For n sufficiently large, $X \cap E^n$ is an isolated neighborhood for the flow generated by $T^n A_{|X_1\cap E^n} + (P^n_{|X_2})_* \nabla B$ and

$$\mathcal{E}_n \simeq h(X^n, \nabla\Psi_n) = h\left(X^n, T^n A_{|X_1\cap E^n} + (P^n_{|X_2})_* \nabla B\right).$$

By Lemma 3.1, for sufficiently large n, $X \cap X_1 \cap E^n$ is an isolated neighborhood for the flow generated by $T^n A_{|X_1\cap E^n}$. Furthermore,

$$h(X \cap X_1 \cap E^n, T^n A_{|X_1\cap E^n}) = [S^{m^-(T^n A_{|X_1\cap E^n})}]$$

From Lemma 3.4 it follows that the sequence $\{S^{m^-(T^n A_{|X_1\cap E^n})}\}$ defines a spectrum in $\mathrm{Spec}(\nu)$ and its homotopy type is equal to $[\mathcal{S}^{i^-(A)}]$. Note that

$$h\left(X \cap P^n X_2, (P^n_{|X_2})_* \nabla B\right) \simeq \mathcal{B}_0.$$

Since $X_1 \cap E^n$ and $P^n X_2$ are orthogonal,

$$h\left(X^n, T^n A_{|X_1\cap E^n} + (P^n_{|X_2})_* \nabla B\right) \simeq S^{i^-(A)+\rho(n)} \wedge \mathcal{B}_0.$$

Hence $h_{\mathcal{LS}}(X, \nabla\Psi) = [\mathcal{S}^{i^-(A)} \wedge \mathcal{B}]$. □

4. The cohomological span

Let $\mathcal{E} \in \mathrm{Spec}(\nu)$ be a spectrum of finite type.

Definition 4.1. The cohomological span of spectrum $\mathcal{E}$ is defined to be a pair

$$\Gamma(\mathcal{E}) = \left(\underline{\gamma}(\mathcal{E}), \overline{\gamma}(\mathcal{E})\right),$$

where

$$\underline{\gamma}(\mathcal{E}) = \min\{q \in \mathbb{Z}: H^q(\mathcal{E}) \neq 0\} \text{ and } \overline{\gamma}(\mathcal{E}) = \max\{q \in \mathbb{Z}: H^q(\mathcal{E}) \neq 0\}.$$

If all groups $H^q(\mathcal{E})$ are trivial then we set $\Gamma(\mathcal{E}) = \emptyset$.

Definition 4.2. The length of $\Gamma(\mathcal{E}) \neq \emptyset$ is defined to be $|\Gamma(\mathcal{E})| = \overline{\gamma}(\mathcal{E}) - \underline{\gamma}(\mathcal{E}) + 1$. For $\Gamma(\mathcal{E}) = \emptyset$ we set $|\Gamma(\mathcal{E})| = 0$.

Given two spectra $\mathcal{E}_1, \mathcal{E}_1 \in \mathrm{Spec}(\nu)$ such that $\Gamma(\mathcal{E}_1), \Gamma(\mathcal{E}_2) \neq \emptyset$.

Definition 4.3.

$$\Gamma(\mathcal{E}_1) \prec \Gamma(\mathcal{E}_2) \Longleftrightarrow \underline{\gamma}(\mathcal{E}_1) \leq \underline{\gamma}(\mathcal{E}_2) \wedge \overline{\gamma}(\mathcal{E}_1) \leq \overline{\gamma}(\mathcal{E}_2).$$

As a special case of the above, we define:

$$\Gamma(\mathcal{E}_1) < \Gamma(\mathcal{E}_2) \Longleftrightarrow \overline{\gamma}(\mathcal{E}_1) \leq \underline{\gamma}(\mathcal{E}_2)$$

Definition 4.4. The gap between spans $\Gamma(\mathcal{E}_1) \prec \Gamma(\mathcal{E}_2)$ is defined to be $g(\mathcal{E}_1, \mathcal{E}_2) = \underline{\gamma}(\mathcal{E}_2) - \overline{\gamma}(\mathcal{E}_1)$.

Suppose that $\Phi\colon E \to \mathbb{R}$ satisfies (D_1)–(D_3) and let the notation be as in the preceding section. Recall from Theorem 3.8 that $h_{\mathcal{LS}}(X, \nabla\Phi) = \left[\mathcal{S}^{i^-(A)} \wedge \mathcal{B}\right]$, where $\mathcal{B} \in \mathrm{Spec}(\nu_0)$. It follows easily that

$$\Gamma(h_{\mathcal{LS}}(X, \nabla\Phi)) = \begin{cases} (i^-(A), i^-(A)) & \text{if } i^0(A) = 0, \\ \text{either } (i^-(A) + \underline{\gamma}(\mathcal{B}), i^-(A) + \overline{\gamma}(\mathcal{B})) \text{ or } \emptyset & \text{if } i^0(A) > 0. \end{cases}$$

Moreover, if $i^0(A) > 0$ then $|\Gamma(h_{\mathcal{LS}}(X, \nabla\Phi))| = |\Gamma(\mathcal{B})|$ and $H^{q+i^-(A)}(h_{\mathcal{LS}}(X, \nabla\Phi)) = H^q(\mathcal{B})$, for all $q \in \mathbb{Z}$. Note that $q \notin [0, i^0(A)]$ implies $H^q(\mathcal{B}) = 0$. Next proposition is analogous to Theorem 4 in [11].

Proposition 4.5. *Assume that p is a degenerate critical point of Φ and $p_2 = Qp$. Then*

$$\Gamma(h_{\mathcal{LS}}(X, \nabla\Phi)) = \begin{cases} (i^-(A), i^-(A)) & \text{if } p_2 \text{ is a local minimum of } B, \\ (i^-(A) + i^0(A), i^-(A) + i^0(A)) & \text{if } p_2 \text{ is a local maximum of } B. \end{cases}$$

If p_2 is neither a maximum nor minimum of B then either $\Gamma(h_{\mathcal{LS}}(X, \nabla\Phi)) = \emptyset$ or

$$i^-(A) + 1 \leq \underline{\gamma}(h_{\mathcal{LS}}(X, \nabla\Phi)) \leq \overline{\gamma}(h_{\mathcal{LS}}(X, \nabla\Phi)) \leq i^-(A) + i^0(A) - 1.$$

Corollary 4.6.

1. *If $i^0(A) = 0, 1, 2$ then $0 \leq |\Gamma(h_{\mathcal{LS}}(X, \nabla\Phi))| \leq 1$.*
2. *If $i^0(A) > 2$ then $0 \leq |\Gamma(h_{\mathcal{LS}}(X, \nabla\Phi))| \leq i^0(A) - 1$.*

Remark 4.7. M. Styborski proved in [14] (with a slightly different assumptions on L) the following formula relating $\mathcal{LS}$-index to the Leray–Schauder degree:

$$\chi(h_{\mathcal{LS}}(X, \nabla\Phi)) = \deg_{\mathcal{LS}}(\nabla\Phi, X, 0),$$

where $\chi(h_{\mathcal{LS}}(X,\nabla\Phi)) = \sum_{q\in\mathbb{Z}}(-1)^q \operatorname{rank}\big(H^q(h_{\mathcal{LS}}(X,\nabla\Phi))\big)$. Clearly, if $i^0(A)=0,1,2$ and $\Gamma(h_{\mathcal{LS}}(X,\nabla\Phi))\neq\emptyset$ then

$$\operatorname{rank} H^q(h_{\mathcal{LS}}(X,\nabla\Phi)) = (-1)^q \deg_{\mathcal{LS}}(\nabla\Phi, \operatorname{int} X, 0),$$

where $q=\underline{\gamma}(h_{\mathcal{LS}}(X,\nabla\Phi))$ (see also [11]).

Set $S=\operatorname{Inv}(X,\nabla\Phi)$. In what follows, to simplify the notation, we will write $CH^q(S)$ or $CH^q(X)$ instead of $H^q(h_{\mathcal{LS}}(X,\nabla\Phi))$ and $\Gamma(S)$ or $\Gamma(X)$ instead of $\Gamma(h_{\mathcal{LS}}(X,\nabla\Phi))$.

4.1. Existence of the critical points

Suppose that $\Phi\colon E\to\mathbb{R}$ satisfies the following assumptions:

(Φ_1) $\Phi\in C^2(E,\mathbb{R})$,
(Φ_2) $\nabla\Phi(x)=Lx+K(x)$, $K\in C^1(E,E)$ is completely continuous,
(Φ_3) X is an isolating neighborhood for the flow generated by $\nabla\Phi$, $S=\operatorname{Inv}(X)$,
(Φ_4) $\operatorname{Crit}(\Phi)\cap S$ is finite,
(Φ_5) there exists a constant $M=M(\Phi,S)\geq 0$ such that

$$M\geq \dim\ker\nabla^2\Phi(p),\ \text{ for all } p\in\operatorname{Crit}(\Phi)\cap S,$$

(Φ_6) there are $K\geq 1$ critical points $S_1,\dots,S_K\subset S$ of Φ and

$$h_{\mathcal{LS}}(S,\nabla\Phi)\neq\bigvee_K h_{\mathcal{LS}}(S_k,\nabla\Phi).$$

By (Φ_4) every critical point of Φ is isolated. It follows from (Φ_5) that for every critical point $p\in\operatorname{Crit}(\Phi)$ dimension of $\ker\nabla^2\Phi(p)$ is finite. Assumption (Φ_6) implies that the set $S'=S\setminus(S_1\cup\ldots\cup S_K)$ is nonempty. Since we deal with the gradient flow, the set S' contains only critical points and connecting orbits. In particular, if S' is not a one point set and $S'\cap\operatorname{Crit}(\Phi)\neq\emptyset$ then S' contains at least two critical points.

Let $\tilde X$ be an isolating neighborhood for the flow generated by $\nabla\Phi$. Directly from Corollary 4.6 we get

Lemma 4.8. *If* $\operatorname{Inv}(\tilde X)$ *is one point set then* $|\Gamma(\tilde X)|\leq 1$, *provided* $M=0,1,2$, *and* $|\Gamma(\tilde X)|\leq M-1$, *provided* $M>2$.

Assume that $\operatorname{Crit}(\Phi)\cap S=\{p_1,\dots,p_N\}$. Clearly, there exists $1\leq i_k\leq N$ such that $S_k=\{p_{i_k}\}$. Without loss of generality we can assume that $\Phi(p_i)\leq\Phi(p_j)$ for $i<j$. Note that the family $\big\{\{p_i\}\colon i=1\dots N\big\}$ is a Morse decomposition of S. Define a new family $\{M_k\colon k=1,\dots 2K+1\}$ of Morse sets as follows

$$M_1=\bigcup_{i=1}^{i_1-1}\{p_i\}\cup\bigcup_{0<i<j}^{i_1-1} C\Big(\{p_j\},\{p_i\}\Big),$$
$$M_{2k}=\{p_{i_k}\}=S_k,$$

$$M_{2k+1} = \bigcup_{i=i_k+1}^{i_{k+1}-1} \{p_i\} \cup \bigcup_{i_k<i<j}^{i_{k+1}-1} C\Big(\{p_j\},\{p_i\}\Big),$$

$$M_{2K+1} = \bigcup_{i=i_K+1}^{N} \{p_i\} \cup \bigcup_{i_K<i<j}^{N} C\Big(\{p_j\},\{p_i\}\Big).$$

Here $C(\{p\},\{q\})$ stands for the set of connecting orbits between $\{p\}$ and $\{q\}$. It is possible that all of the sets $M_1, M_3, \dots, M_{2K+1}$ are empty. In this case, the only critical points in S are $S_1, \dots, S_K$. However, if $M_{2k+1} \neq \emptyset$ then M_{2k+1} contains at least one critical point and this point is different from known critical points $S_1, \dots, S_K$.

It is clear that the family $\{M_k : k = 1, \dots, 2K+1\}$ is a Morse decomposition of S. Note that if $p \in M_{2k+1}$ then $\Phi(M_{2k}) \le \Phi(p) \le \Phi(M_{2k+2})$.

Remark 4.9. The above construction is a special case of more general procedure described in Section 2.2 of [16].

Applying Theorem 2.17 to $\{M_k : k = 1, \dots, 2K+1\}$ we get isolating neighborhoods $X_1, \dots, X_{2K+1} = X$ such that for each $k = 1, \dots, 2K$, pair $(\mathrm{Inv}(X_k), M_{k+1})$ is an attractor–repeler pair in X_{k+1}. Note that $M_1 = \mathrm{Inv}(X_1)$ and $S = \mathrm{Inv}(X_{2K+1})$.

If for some k the set M_{2k+1} is empty then we have $CH^q(M_{2k+1}) = 0$ and $CH^q(X_{2k+1}) \simeq CH^q(X_{2k})$, for all $q \in \mathbb{Z}$. We obtain the long exact sequence

$$\cdots \xrightarrow{\delta^{q-1}} 0 \longrightarrow CH^q(X_{2k+1}) \to CH^q(X_{2k}) \longrightarrow 0 \xrightarrow{\delta^{q-1}} \cdots$$

If the set M_{2k+1} is non-empty then Theorem 2.18 gives us the long exact sequence for $(\mathrm{Inv}(X_k), M_{k+1})$. Combining the above we get the following diagram of long exact sequences:

$$\begin{array}{c}
\cdots \xrightarrow{\delta^{q-1}} CH^q(S_1) \longrightarrow CH^q(X_2) \longrightarrow CH^q(M_1) \xrightarrow{\delta^q} \cdots \\
\vdots \\
\cdots \xrightarrow{\delta^{q-1}} CH^q(S_k) \longrightarrow CH^q(X_{2k}) \longrightarrow CH^q(X_{2k-1}) \xrightarrow{\delta^q} \cdots \\
\| \\
\cdots \xrightarrow{\delta^{q-1}} CH^q(M_{2k+1}) \to CH^q(X_{2k+1}) \longrightarrow CH^q(X_{2k}) \xrightarrow{\delta^q} \cdots \\
\vdots \\
\cdots \xrightarrow{\delta^{q-1}} CH^q(M_{2K+1}) \longrightarrow CH^q(S) \longrightarrow CH^q(X_{2K}) \xrightarrow{\delta^q} \cdots
\end{array} \tag{1}$$

If for some $k = 0, \dots, K$ we have $\Gamma(M_{2k+1}) \neq \emptyset$ then there is $q \in \mathbb{Z}$ such that $CH^q(M_{2k+1})$ is nontrivial. Then, the set M_{2k+1} is non-empty and contains at least one critical point.

Lemma 4.10. *Assume that $\Gamma(M_{2k+1}) \neq \emptyset$ for some $0 \le k \le K$.*

1. *If $M = 0, 1, 2$ then the set M_{2k+1} contains at least $\sharp\{q \in \mathbb{Z} : CH^q(M_{2k+1}) \neq 0\}$ critical points.*
2. *If $M > 2$ and $|\Gamma(M_{2k+1})| > M - 1$ then the set M_{2k+1} contains at least two critical points.*

Proof. In the case $M > 2$ conclusion follows directly from Lemma 4.8. Let $M = 0, 1, 2$ and let $C_M = \sharp M_{2k+1} \cap \mathrm{Crit}(\Phi)$ and $C_H = \sharp\{q \in \mathbb{Z}: CH^q(M_{2k+1}) \neq 0\}$. Suppose that $C_M < C_H$.

As before, the family $\{Q_1, \ldots, Q_{C_M}\} = M_{2k+1} \cap \mathrm{Crit}(\Phi)$ is a Morse decomposition of M_{2k+1}. By 2.17 and 2.18, renumbering the points if necessary, we have:

$$\begin{array}{c}
\cdots \xrightarrow{\delta^{q-1}} CH^q(Q_2) \longrightarrow CH^q(X_2) \longrightarrow CH^q(Q_1) \xrightarrow{\delta^q} \cdots \\
\vdots \\
\cdots \xrightarrow{\delta^{q-1}} CH^q(Q_n) \longrightarrow CH^q(X_n) \longrightarrow CH^q(X_{n-1}) \xrightarrow{\delta^q} \cdots \\
\cdots \xrightarrow{\delta^{q-1}} CH^q(Q_{n+1}) \longrightarrow CH^q(X_{n+1}) \longrightarrow CH^q(X_n) \xrightarrow{\delta^q} \cdots \\
\vdots \\
\cdots \xrightarrow{\delta^{q-1}} CH^q(Q_{C_M}) \longrightarrow CH^q(M_{2k+1}) \longrightarrow CH^q(X_{C_M-1}) \xrightarrow{\delta^q} \cdots
\end{array} \tag{2}$$

If $CH^q(M_{2k+1}) \neq 0$ then at least one of the groups $CH^q(Q_n)$ is nonzero. Otherwise, $CH^q(X_n) = 0$, for all n, and

$$\cdots \xrightarrow{\delta^{q-1}} 0 \to CH^q(M_{2k+1}) \to 0 \xrightarrow{\delta^q} \cdots$$

This contradicts the exactness of (2). Therefore, we have at least C_H nonzero groups

$$CH^{q_1}(Q_{n_1}), \ldots, CH^{q_{C_H}}(Q_{n_{C_H}}).$$

Since $|\Gamma(Q_n)| \leq 1$, $C_M \geq C_H$. A contradiction. $\square$

In the case $M > 2$, when nontrivial cohomology groups of M_{2k+1} are "far enough", we can obtain better estimations for the number of critical points in M_{2k+1}.

Lemma 4.11. *Let $M > 2$ and $1 \leq k \leq K$. Suppose that there exists $q_1 < \ldots < q_N$ such that $q_{i+1} - q_i > M - 1$ and $CH^{q_i}(M_{2k+1}) \neq 0$. Then the set M_{2k+1} contains at least N critical points.*

Proof. If $N = 1$ then the conclusion follows from the nontriviality of the $\mathcal{LS}$-index of M_{2k+1}. Let $C_M = \sharp M_{2k+1} \cap \mathrm{Crit}(\Phi)$. Suppose that $N > 1$ and $C_M < N$. As before, we have a Morse decomposition $\{P_1, \ldots, P_{C_M}\} = M_{2k+1} \cap \mathrm{Crit}(\Phi)$ of M_{2k+1}. Now, one can obtain a diagram similar to (2). From the exactness of the rows, every nontrivial group $CH^{q_i}(M_{2k+1})$ gives at least one nontrivial group $CH^{q_i}(P_{j_i})$. Since $C_M < N$ and $q_{i+1} - q_i > M - 1$, at least one of the lengths $|\Gamma(P_i)|$ is greater than $M - 1$. A contradiction. $\square$

Example 4.12. (Cf. p. 230, [17].) Suppose that $S_1, \ldots, S_K \subset S$ are all critical points of Φ such that $h_{\mathcal{LS}}(S_k) = \left[\bigvee_{a_k} S^{\gamma_k}\right]$ and $h_{\mathcal{LS}}(S) = \left[S^m\right]$. Theorem 2.19 implies $\sum_{k=1}^K a_k t^{\gamma_k} = t^m + (1 + t)Q(t)$. Setting $t = 1$ we get $\sum_{k=1}^K a_k = 1 + 2Q(1)$. Hence, $\sum_{k=1}^K a_k$ is odd.

In particular, if K is even and all of the S_k are nondegenerate (in this case $a_k = 1$) then Φ has additionally at least one critical point. Similar result for the classical Conley index was obtained in [17] under the assumption that all critical points of Φ are nondegenerate.

Remark 4.13. Note that our theory do not require any nondegeneracy conditions. The only assumption on known critical point is to be isolated.

4.2. The case $K = 1$

We first consider the case $K = 1$, i.e. there is exactly one known critical point. Since (Φ_6) reduces to

$$h_{\mathcal{LS}}(S, \nabla\Phi) \neq h_{\mathcal{LS}}(S_1, \nabla\Phi),$$

it follows from properties of $\mathcal{LS}$-index that the set $S' = S \setminus S_1$ contains at least one critical point.

Since $K = 1$, we have three-element Morse decomposition $\{M_1, S_1, M_3\}$ of S. Thus, the diagram (1) reduces to the two-row diagram below

$$\begin{array}{ccccccccc} & & \cdots \xrightarrow{\delta^{q-1}} CH^q(S_1) \longrightarrow CH^q(X_2) \longrightarrow CH^q(M_1) \xrightarrow{\delta^q} \cdots \\ & & \| \\ \cdots \xrightarrow{\delta^{q-1}} CH^q(M_3) \longrightarrow CH^q(S) \longrightarrow CH^q(X_2) \xrightarrow{\delta^q} \cdots & & \end{array} \tag{3}$$

Lemma 4.14. *If for some $q \in \mathbb{Z}$, $CH^q(S) \not\simeq CH^q(S_1)$, then at least one of the groups $CH^{q-1}(M_1)$, $CH^q(M_1)$, $CH^q(M_3)$, $CH^{q+1}(M_3)$ is nontrivial. In particular:*

1. *if $CH^q(S_1) = 0$ then $CH^q(M_1) \neq 0$ or $CH^q(M_3) \neq 0$,*
2. *if $CH^q(S) = 0$ then $CH^{q-1}(M_1) \neq 0$ or $CH^{q+1}(M_3) \neq 0$.*

Proof. If $CH^{q-1}(M_1) = CH^q(M_1) = CH^q(M_3) = CH^{q+1}(M_1) = 0$ then from exactness of the rows in (3) we get $CH^q(S_1) \simeq CH^q(X_2)$ and $CH^q(S) \simeq CH^q(X_2)$. A contradiction.

Assume that $CH^q(S_1) = 0$. Then $CH^q(S) \neq 0$ and $CH^q(M_3) \neq 0$ or $CH^q(X_2) \neq 0$. It follows that $CH^q(X_2) \longrightarrow CH^q(M_1)$ is a monomorphism. Hence $CH^q(M_1) \neq 0$. Similar arguments apply in the case $CH^q(S) = 0$. $\square$

Now we give conditions under which the set S' contains at least two critical points. We first consider $M = 0, 1, 2$, i.e. $|\Gamma(S_1)| \leq 1$.

Theorem 4.15. *Assume that $M = 0, 1, 2$ and $\Gamma(S) \neq \emptyset$.*

1. *If $\Gamma(S_1) = \emptyset$ then the set S' contains at least $\sharp\{q \in \mathbb{Z}: CH^q(S) \neq 0\}$ critical points.*
2. *If $\Gamma(S_1) \neq \emptyset$ then the minimal number of critical points in the set S' is equal to:*

$$\begin{cases} \sharp\{q \in \mathbb{Z}: CH^q(S) \neq 0\} + 1 & \text{if } g(S_1, S) > 1 \text{ or } g(S, S_1) > 1, \\ \sharp\{q \in \mathbb{Z}: CH^q(S) \neq 0\} & \text{if } g(S_1, S) = 1 \text{ or } g(S, S_1) = 1, \\ \sharp\{q \in \mathbb{Z}: CH^q(S) \neq 0\} - 1 & \text{if } g(S_1, S) = 0 \text{ or } g(S, S_1) = 0. \end{cases}$$

Proof. Lemma 4.10 shows that the minimal number of critical points in the set S' is greater than or equal to the number of nonzero groups $CH^q(M_1)$ and $CH^q(M_3)$.

If $\Gamma(S_1) = \emptyset$ then our statement follows from Lemma 4.14, since each nonzero cohomology group of S gives at least one nonzero cohomology group of M_1 or M_3.

Suppose that $\Gamma(S_1) \neq \emptyset$. Since $M = 0, 1, 2$, $|\Gamma(S_1)| = 1$ and $\underline{\gamma}(S_1) = \overline{\gamma}(S_1)$. We prove only the case $\Gamma(S_1) \prec \Gamma(S)$. The other case is similar and the details are left to the reader.

Let $g(S_1, S) > 1$. By Lemma 4.14, if $q = \underline{\gamma}(S_1)$ then at least one of the groups $CH^{q-1}(M_1)$, $CH^{q+1}(M_3)$ are nonzero. Similarly, if $q \in [\underline{\gamma}(S), \overline{\gamma}(S)]$ and $CH^q(S) \neq 0$ then $CH^q(M_1) \neq 0$ or $CH^q(M_3) \neq 0$. Since $\overline{\gamma}(S_1) < \underline{\gamma}(S) - 1$, we have at least $\sharp\{q \in \mathbb{Z}: CH^q(S) \neq 0\} + 1$ nonzero groups.

The cases $g(S_1, S) = 1$ and $g(S_1, S) = 0$ follow directly by Lemma 4.14. $\square$

If in above theorem we assume that the gap between S_1 and S is equal to 0, $\Gamma(S_1) \neq \emptyset$ and the only nonzero group $CH^q(S_1)$ is not isomorphic to $CH^q(S)$, $q = \underline{\gamma}(S_1)$, then one can prove that the set S' contains at least $\sharp\{q \in \mathbb{Z}: CH^q(S) \neq 0\}$ critical points. This simple exercise is left to the reader.

Now, we turn to the case $M > 2$. If $\Gamma(S_1) = \emptyset$ and $|\Gamma(S)| > M - 1$ then by Lemma 4.14 we have two (non-exclusive) possibilities: the cohomology groups of M_1 and M_3 are simultaneously nonzero or one of the numbers $|\Gamma(M_1)|$, $|\Gamma(M_3)|$ is greater than $M - 1$.

In the first case, by the non-triviality of $\mathcal{LS}$-index of M_1, M_3 and in the latter by Lemma 4.10 we have the existence of at least two critical points in the set S'.

Theorem 4.16. *Assume that $M > 2$, $\Gamma(S_1) \neq \emptyset$, $\Gamma(S) \neq \emptyset$ and $|\Gamma(S)| \leq M - 1$. The set S' contains at least two critical points if one of the following cases occurs:*

1. $\Gamma(S) < \Gamma(S_1)$ *and* $|\Gamma(S)| + |\Gamma(S_1)| + g(S, S_1) - 2 > M - 1$;
2. $\Gamma(S_1) < \Gamma(S)$ *and* $|\Gamma(S)| + |\Gamma(S_1)| + g(S_1, S) - 2 > M - 1$.

Proof. We only prove the first statement. The latter follows by the similar argument. Since $\Gamma(S_1) \neq \Gamma(S)$, S' contains at least one critical point. If for some q_1, q_2 we have $CH^{q_1}(M_1) \neq 0$ and $CH^{q_2}(M_3) \neq 0$ then the proof is finished.

Assume that $\Gamma(S_1) < \Gamma(S)$. Since $CH^{\underline{\gamma}(S_1)}(S_1) \neq 0 = CH^{\underline{\gamma}(S_1)}(S)$, Lemma 4.14 shows that $CH^{\underline{\gamma}(S_1)-1}(M_1) \neq 0$ or $CH^{\underline{\gamma}(S_1)+1}(M_3) \neq 0$. Similarly, $CH^{\overline{\gamma}(S)}(S_1) = 0 \neq CH^{\overline{\gamma}(S)}(S)$ implies that $CH^{\overline{\gamma}(S)}(M_3) \neq 0$ or $CH^{\overline{\gamma}(S)}(M_1) \neq 0$.

Suppose that $CH^q(M_3) = 0$, $q \in \mathbb{Z}$. Then $\underline{\gamma}(M_1) \leq \underline{\gamma}(S_1) - 1 \leq \overline{\gamma}(S) \leq \overline{\gamma}(M_1)$. Hence,

$$\begin{aligned}
|\Gamma(M_1)| &\geq \overline{\gamma}(M_1) - \underline{\gamma}(M_1) + 1 \geq \overline{\gamma}(S) - \underline{\gamma}(S_1) + 1 + 1 \\
&\geq \overline{\gamma}(S) - \underline{\gamma}(S) + 1 + \overline{\gamma}(S_1) - \underline{\gamma}(S_1) + 1 + -\overline{\gamma}(S_1) + \underline{\gamma}(S) \\
&\geq |\Gamma(S)| + |\Gamma(S_1)| + g(S_1, S) > M + 1.
\end{aligned}$$

Therefore, the set $M_1 \subset S'$ contains at least two critical points. Similarly, if $CH^q(M_1) = 0$, $q \in \mathbb{Z}$ then $\underline{\gamma}(M_3) \leq \underline{\gamma}(S_1) + 1 \leq \overline{\gamma}(S) \leq \overline{\gamma}(M_3)$. Hence,

$$\begin{aligned}
|\Gamma(M_3)| &\geq \overline{\gamma}(M_3) - \underline{\gamma}(M_3) + 1 \geq \overline{\gamma}(S) - \underline{\gamma}(S_1) - 1 + 1 \\
&\geq \overline{\gamma}(S) - \underline{\gamma}(S) + 1 + \overline{\gamma}(S_1) - \underline{\gamma}(S_1) + 1 + -\overline{\gamma}(S_1) + \underline{\gamma}(S) - 2 \\
&\geq |\Gamma(S)| + |\Gamma(S_1)| + g(S_1, S) - 2 > M - 1.
\end{aligned}$$

Therefore the set $M_3 \subset S'$ contains at least two critical points. $\square$

If in above theorem we assume that $|\Gamma(S)| > M - 1$ and the gap between S_1 and S is "big enough" then one can show that the set S' contains at least three critical points.

Theorem 4.17. *Assume that $M > 2$, $\Gamma(S_1) \neq \emptyset$ and $|\Gamma(S)| > M - 1$. The set S' contains at least three critical points if one of the following cases occurs:*

1. $\Gamma(S) < \Gamma(S_1)$ *and* $g(S, S_1) > M - 1$*;*
2. $\Gamma(S_1) < \Gamma(S)$ *and* $g(S_1, S) > M - 1$.

Proof. We give the proof only for the case $\Gamma(S) < \Gamma(S_1)$. Since $\Gamma(S) \neq \Gamma(S_1)$, there is at least one critical point in the set S'.

By Lemma 4.14 and the assumption $g(S, S_1) > M - 1$ we obtain that M_1 or M_3 has nonzero cohomology groups at dimensions

$$q \in \{\underline{\gamma}(S_1) - 1, \underline{\gamma}(S_1) + 1\},\ q \in \{\overline{\gamma}(S_1) - 1, \overline{\gamma}(S_1) + 1\},\ q = \underline{\gamma}(S) \text{ and } q = \overline{\gamma}(S).$$

If both M_1 and M_3 have nontrivial cohomology groups then the set S' contains at least two critical points. If the cohomology groups of M_1 (resp. M_3) are trivial then $|\Gamma(M_3)|$ (resp. $|\Gamma(M_1)|$) is greater than $M - 1$. Hence, by Lemma 4.10, the set S' contains at least two critical points.

Assume that $CH^q(M_3) = 0$, for all $q \in \mathbb{Z}$. Then we have at least three nonzero cohomology groups of M_1: $CH^{\underline{\gamma}(S)}(M_1)$, $CH^{\overline{\gamma}(S)}(M_1)$ and $CH^q(M_1)$, for some $q \in \{\underline{\gamma}(S_1) - 1, \overline{\gamma}(S_1) - 1\}$.

Suppose that M_1 contains only two critical points, say P_1, P_2. Then $\{P_1, P_2\}$ is a Morse decomposition of the set M_1 and $|\Gamma(P_1)|, |\Gamma(P_2)| \leq M - 1$. We have the exact sequence:

$$\cdots \xrightarrow{\delta^{q-1}} CH^q(P_2) \to CH^q(M_1) \to CH^q(P_1) \xrightarrow{\delta^q} \cdots$$

From non-triviality of $CH^q(M_1)$ it follows that $CH^q(P_1)$ or $CH^q(P_2)$ is nontrivial. Moreover, $|\Gamma(P_1)| > M - 1$ or $|\Gamma(P_2)| > M - 1$. Therefore, the set M_1 contains at least three critical points. The above remains true if we exchange the roles of M_1 and M_3.

Observe that if $\Gamma(M_1) \neq \emptyset$ and $\Gamma(M_3) \neq \emptyset$ then $|\Gamma(M_1)| \geq 2$ or $|\Gamma(M_3)| \geq 2$. Suppose that for some $q_1, q_2, q_3 \in \left[\underline{\gamma}(S_1) - 1, \overline{\gamma}(S_1) + 1\right] \cup \left\{\underline{\gamma}(S), \overline{\gamma}(S)\right\}$ the groups $CH^{q_1}(M_1)$, $CH^{q_2}(M_1)$ and $CH^{q_3}(M_3)$ are nonzero. We can assume that q_1, q_2, q_3 are pairwise different and any two of them are not simultaneously in $\left[\underline{\gamma}(S_1) - 1, \overline{\gamma}(S_1) + 1\right]$. Since $|\Gamma(M_1)| > M - 1$ and $\mathcal{LS}$-index of M_3 is nontrivial, the set S' contains at least three critical points. As before we can exchange the roles of M_1 and M_3. □

4.3. The case $K \geq 2$

Now, we turn to the case when more than one critical point is known. By (Φ_6), we have $S' = S \setminus (S_1 \cup \ldots \cup S_K) \neq \emptyset$. As opposed to the situation $K = 1$, it is possible that the set S' do not contain any critical points and consists of connecting orbits only.

We first give conditions which ensure the existence of at least one critical point in S'.

Lemma 4.18. *Suppose that for some $q \in \mathbb{Z}$ the following groups are trivial:*

$$CH^q(S),\ CH^{q+1}(S_k),\ \textit{for } k \neq 1$$

and $CH^q(S_1) \neq 0$. Then, at least one of the groups:

$$CH^{q-1}(M_1),\ CH^{q+1}(M_{2k+1}),\ \textit{for } k = 0, \ldots, K,$$

is nonzero.

Proof. On the contrary, assume that all of these groups are trivial. Diagram (1) takes the following form:

$$\begin{tikzcd}
 & & 0 \arrow[r, "\delta^{q-1}"] & CH^q(S_1) \arrow[r] & CH^q(X_2) \arrow[r] \arrow[d, dotted, no head] & CH^q(M_1) \arrow[r, "\delta^q"] & \cdots \\
 & & \cdots \arrow[r, "\delta^{q-1}"] & CH^q(S_k) \arrow[r] & CH^q(X_{2k}) \arrow[r] \arrow[d, equal] & CH^q(X_{2k-1}) \arrow[r, "\delta^q"] & 0 \\
 & \cdots \arrow[r, "\delta^{q-1}"] & CH^q(M_{2k+1}) \arrow[r] & CH^q(X_{2k+1}) \arrow[r] \arrow[d, dotted, no head] & CH^q(X_{2k}) \arrow[r, "\delta^q"] & 0 & \\
 & \cdots \arrow[r, "\delta^{q-1}"] & CH^q(S_K) \arrow[r] & CH^q(X_{2K}) \arrow[r] \arrow[d, equal] & CH^q(X_{2K-1}) \arrow[r, "\delta^q"] & 0 & \\
\cdots \arrow[r, "\delta^{q-1}"] & CH^q(M_{2K+1}) \arrow[r] & 0 \arrow[r] & CH^q(X_{2K}) \arrow[r, "\delta^q"] & 0 & &
\end{tikzcd}$$

Since in the above diagram all of the groups $CH^q(X_k)$ are trivial, $CH^q(S_1) = 0$. A contradiction. □

Lemma 4.19. *Assume that for some $q \in \mathbb{Z}$ the following groups are trivial:*

$$CH^{q+1}(S),\ CH^{q-1}(S_k),\ \textit{for } k < k_0 \textit{ and } CH^{q+1}(S_k),\ \textit{for } k > k_0$$

and $CH^q(S_{k_0}) \neq 0$, for $k_0 \neq 1$. Then, at least one of the groups:

$$CH^{q-1}(M_{2k+1}),\ \textit{for } k = 0, \ldots, k_0;\ CH^{q+1}(M_{2k+1}),\ \textit{for } k = k_0, \ldots, K,$$

is nonzero.

Proof. On the contrary, assume that all of these groups are zero. Since $CH^{q-1}(S_1)$ and $CH^{q-1}(M_1)$ are trivial. From diagram (1) we obtain that all groups $CH^{q-1}(X_k)$, $k = 2, \ldots, 2k_0 - 1$ are trivial. Similarly, since $CH^{q+1}(S)$ and $CH^{q+1}(M_{2K+1})$ are trivial, all groups $CH^{q+1}(X_k)$ are trivial, provided $k = 2k_0, \ldots, 2K + 1$. Thus, $CH^q(S_{k_0})$ is trivial. A contradiction. □

Lemma 4.20. *Assume that for some $q \in \mathbb{Z}$ the following groups are trivial:*

$$CH^q(S_k) = 0,\ k = 1, \ldots, K$$

and $CH^q(S) \neq 0$. Then, at least one of the groups:

$$CH^q(M_{2k+1}),\ k = 0, \ldots, K;$$

is nonzero.

Proof. On the contrary, assume that all of these groups are zero. Then, all groups $CH^q(X_k) = 0$, $k = 0, \dots, 2K$ are trivial. Hence $CH^q(X_{2K}) = 0$. A contradiction. □

Lemma 4.21. *Assume that for some $q \in \mathbb{Z}$ and $n \in \mathbb{N}$ the following groups are trivial:*

$$CH^{q-1}(S_k),\ CH^{q+n+1}(S_{k+1}),\ k = 1, \dots, K-1.$$

If $CH^{q-1}(M_{2k+1}) = 0$, $k = 0, \dots, K-1$ and all the cohomology groups of the sets M_{2k+1}, $k = 0, \dots, K$ are trivial in dimensions from $q-1$ to $q+n+1$ then each of the following sequences:

$$0 \xrightarrow{\delta^{q-1}} CH^q(S_2) \longrightarrow CH^q(X_4) \longrightarrow CH^q(S_1) \xrightarrow{\delta^q} \cdots \longrightarrow CH^{q+n}(S_1) \xrightarrow{\delta^{q+n}} 0$$
$$\vdots$$
$$0 \xrightarrow{\delta^{q-1}} CH^q(S_k) \longrightarrow CH^q(X_{2k}) \to CH^q(X_{2k-2}) \xrightarrow{\delta^q} \cdots \longrightarrow CH^{q+n}(X_{2k-2}) \xrightarrow{\delta^{q+n}} 0$$
$$\vdots$$
$$0 \xrightarrow{\delta^{q-1}} CH^q(S_K) \longrightarrow CH^q(S) \to CH^q(X_{2K-2}) \xrightarrow{\delta^q} \cdots \longrightarrow CH^{q+n}(X_{2K-2}) \xrightarrow{\delta^{q+n}} 0$$

is exact.

Proof. By exactness of the rows in the diagram (1) we get $CH^q(S_1) \simeq CH^q(X_2)$, $CH^q(X_{2k+1}) \simeq CH^q(X_{2k})$ and $CH^q(S) \simeq CH^q(X_{2K})$. Hence we can drop the odd rows from the diagram (1), which completes the proof. □

All of the above lemmas guarantee the existence of at least one critical point in the set S'. Repeated application of the above lemmas for different q may lead to the situation when some set M_{2k+1} possess more than one nonzero cohomology group. If $|\Gamma(M_{2k+1})|$ is big enough, say greater than $M-1$, then the set M_{2k+1} contains more than one critical point by Lemma 4.11.

Now we examine three situations when at least two critical points can be obtained. Set $S_0 = S$. Suppose that $\Gamma(S_k) \neq \emptyset$, $|\Gamma(S)| < M-1$ and

$$\Gamma\left(S_{k_0}\right) \prec \Gamma\left(S_{k_1}\right) \prec \ldots \prec \Gamma\left(S_{k_K}\right),$$

where $k_i = 0, 1, \dots, K$ and $k_i \neq k_j$, provided $i \neq j$.

Definition 4.22. We say that spans $\Gamma(S_k)$, $k = 0, 1, \dots K$, satisfy condition (A), if:

$$g(S_{k_i}, S_{k_{i+1}}) - 2 > M - 1, \text{ for all } k_i = 0, \dots, K-1.$$

Theorem 4.23. *Assume that the condition (A) holds. Then the set S' contains at least $K+1$ critical points.*

Proof. Fix $0 \le k_i \le K$. Let $q = \underline{\gamma}\left(S_{k_i}\right)$ and $n = \overline{\gamma}\left(S_{k_i}\right) - \underline{\gamma}\left(S_{k_i}\right) = \left|\Gamma\left(S_{k_i}\right)\right| - 1$. Since the cohomology groups of S_{k_i} are trivial in dimensions $q-1$ and $q+n+1$, the conditions of Lemma 4.21 are satisfied. For $k_j \neq k_i$ all the cohomology groups of S_{k_j} are trivial in dimensions from q to $q+n$. The set S_{k_i} possess at least one nonzero cohomology group in dimensions from

q to $q+n$. It is easy to check that sequences in Lemma 4.21 are not exact. Thus for all $0 \le i \le K$ there exist $0 \le k_i \le K$ and $\underline{\gamma}\left(S_{k_i}\right) - 1 \le q_i \le \overline{\gamma}\left(S_{k_i}\right) + 1$ such that $CH^{q_i}(M_{2k_i+1}) \ne 0$.

Note that if $i < j$ then $q_i < q_j$. For arbitrary $i < K$ we obtain:

$$q_{i+1} - q_i > \underline{\gamma}\left(S_{k_{i+1}}\right) - 1 - \overline{\gamma}\left(S_{k_i}\right) - 1 = g(S_{k_i}, S_{k_{i+1}}) - 2 > M - 1.$$

For each $k = 1, \dots, K$ we denote by h_{2k+1} the number of nonzero groups $CH^{q_i}(M_{2k_i+1})$ such that $k = k_i$. By Lemma 4.11 each set M_{2k+1} contains at least h_{2k+1} critical points. Hence, we have at least $\sum_{k=1}^{K} h_{2k+1} = K+1$ critical points in the set S'. $\square$

If in the above theorem we assume that $M = 0, 1, 2$ and $g(S_{k_{i-1}}, S_{k_i}) > 1$ then similar argument, using Lemmas 4.21 and 4.10, gives us at least $K+1$ critical points in the set S'.

Definition 4.24. We say that spans $\Gamma(S_k)$, $k = 0, 1, \dots K$, satisfy condition (B), if:

1. $\left|\Gamma\left(S_{k_i}\right)\right| + g(S_{k_{i-1}}, S_{k_i}) > 2;\ g(S_{k_i}, S_{k_{i+1}}) > 1;$
2. $\left|\Gamma\left(S_{k_j}\right)\right| + g(S_{k_j}, S_{k_{j+1}}) > 2;\ g(S_{k_{j-1}}, S_{k_j}) > 1;$
3. $g(S_{k_i}, S_{k_j}) - 2 > M - 1$, for some $0 \le i < j \le K$.

Note that if the condition (B) holds then there may exist sets S_{k_n} such that

$$\dots \prec \Gamma\left(S_{k_{i-1}}\right) \prec \Gamma\left(S_{k_i}\right) < \dots \prec \Gamma\left(S_{k_n}\right) \prec \dots < \Gamma\left(S_{k_j}\right) \prec \Gamma\left(S_{k_{j+1}}\right) \prec \dots$$

Theorem 4.25. *Assume that the condition (B) holds. Then the set S' contains at least two critical points.*

Proof. From (1), we have at least one nonzero cohomology group of some set M_{2k+1} in dimensions from $\overline{\gamma}\left(S_{k_i}\right) - 1$ to $\overline{\gamma}\left(S_{k_i}\right) + 1$. Similarly, from (1), we have at least one nonzero cohomology group in dimensions from $\underline{\gamma}\left(S_{k_j}\right) - 1$ to $\underline{\gamma}\left(S_{k_j}\right) + 1$.

If these nonzero groups are groups of different sets M_{2k+1} then there is at least two critical points in S'. If there are two nonzero groups of some set M_{2k+1} then

$$\begin{aligned} |\Gamma(M_{2k+1})| &= \overline{\gamma}(M_{2k+1}) - \underline{\gamma}(M_{2k+1}) + 1 \\ &\ge \underline{\gamma}\left(S_{k_j}\right) - 1 - \overline{\gamma}\left(S_{k_i}\right) - 1 + 1 = g(S_{k_i}, S_{k_j}) - 1 > M - 1. \end{aligned}$$

It follows that M_{2k+1} consists of at least two critical points. $\square$

Definition 4.26. We say that spans $\Gamma(S_k)$, $k = 0, 1, \dots K$, satisfy condition (C), if:

$$\left|\Gamma\left(S_{k_1}\right)\right| + g(S_{k_1}, S_{k_2}) > 2;\ g(S_{k_0}, S_{k_1}) - 2 > M - 1$$

or

$$\left|\Gamma\left(S_{k_{K-1}}\right)\right| + g(S_{k_{K-2}}, S_{k_{K-1}}) > 2;\ g(S_{k_{K-1}}, S_{k_K}) - 2 > M - 1.$$

Theorem 4.27. *Assume that the condition (C) holds. Then the set S' contains at least two critical points.*

We omit the proof, since it follows the similar line. One can find more similar conditions and, using above lemmas, prove analogical theorems, but we restrict ourselves to those given above.

Example 4.28. (Cf. Ex. 5.1, [9].) Let $N > 2$, $K = 2$ and suppose that

$$h_{\mathcal{LS}}(S_{k_1}) = [\mathcal{S}^4],\ h_{\mathcal{LS}}(S_{k_2}) = [\mathcal{S}^2 \wedge (S^1 \vee S^1)] \text{ and } h_{\mathcal{LS}}(S) = [\mathcal{S}^m].$$

If $m \neq 3$ then the equality $2t^3 + t^4 = t^m + (1+t)Q(t)$ does not hold for any t and for any Q. Thus, there exists a critical point $p \in S'$. If we set $P(t, \{p\}) = t^2 + t^m$ and $Q(t) = t^2 + t^3$ then the equality

$$t^2 + t^m + 2t^3 + t^4 = t^m + (1+t)Q(t)$$

holds for any t. In this case, if either $2 - m > M - 1$ or $m - 4 > M - 1$ then the set S' contains at least two critical points.

Suppose that $m = 3$ and $\Phi(S_{k_1}) < \Phi(S_{k_2})$. Then $M_2 = S_{k_1}$ and $M_4 = S_{k_2}$. The equality $2t^3 + t^4 = t^3 + (1+t)Q(t)$ holds if we set $Q(t) = t^3$.

Since $CH^5(S_2)$, $CH^5(S) = 0$ and $CH^3(S_1) \neq 0$, at least one of the groups: $CH^3(M_1)$, $CH^5(M_3)$, $CH^5(M_5)$ is nonzero, by Lemma 4.18. One can show that at least one of the groups: $CH^2(M_1)$, $CH^2(M_3)$, $CH^4(M_5)$ is nonzero. Thus, there exists at lest one critical point in the set S'.

If we assume that $M = 0, 1, 2$ instead of $M > 2$ then the set S' contains at least two critical points.

5. Applications to Hamiltonian systems

In this section we give some examples for which our abstract theorems can be applied. Namely, we prove the existence of periodic solution of certain Hamiltonian system. Given a Hamiltonian $H \in C^2(\mathbb{R}^{2N} \times \mathbb{R})$ which is 2π-periodic in t. Consider the Hamiltonian system

$$\dot{z} = J\nabla H(z, t) \tag{4}$$

where $J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$. Denote by $E = H^{1/2}(S^1, R^{2N})$ the Sobolev space of 2π-periodic $\mathbb{R}^{2N}$-valued functions. It is well known that $z \in E$ is 2π-periodic solution of (4) if and only if it is a critical point of the functional $\Phi \in C^2(E, \mathbb{R})$ defined by

$$\Phi(z) = \frac{1}{2}\langle Lz, z\rangle_E - \int_0^{2\pi} H(z, t)dt \tag{5}$$

where

$$\langle Lz, z\rangle_E = \int_0^{2\pi} \langle -J\dot{z}, z\rangle dt.$$

It is shown in [8,9] that (E, L) is a base pair and $\nabla\Phi$ is an $\mathcal{LS}$-field. Functional Φ is strongly indefinite and satisfies assumptions (Φ_1)–(Φ_6). The constant M in (Φ_5) is equal to $2N - 1$.

If A is a symmetric $2n \times 2n$ matrix and $\dot{z} = JAz$ is a linear Hamiltonian system then the vector field $\nabla\Phi_A$ corresponding to that system is linear. The E-Morse index i^- and nullity i^0 of $\nabla\Phi_A$ defined in Section 3 are equal to the generalized Morse index and nullity for linear Hamiltonian systems defined by Amann and Zehnder in [3] (see also [18]). We will use the symbol $i^-(A)$ (resp. $i^0(A)$) to denote E-Morse index (resp. nullity) corresponding to $\nabla\Phi_A$.

Throughout the rest of this section we assume that the Hamiltonian H is asymptotically linear at infinity, i.e. there is $R > 0$, $\|z\|_E > R$ implies $H(z,t) = \frac{1}{2}\langle A_\infty z, z\rangle + h_\infty(z,t)$, where A_∞ is linear, symmetric and $\nabla h_\infty(z,t)$ is bounded. It is clear that if X_∞ is a closed ball of a large radius then X_∞ is an isolating neighborhood.

Proposition 5.1. *(See [8].) If $i^0(A_\infty) = 0$ then*

$$h_{\mathcal{LS}}(X_\infty) = \left[\mathcal{S}^{i^-(A_\infty)}\right].$$

In [9,13] the existence of additional critical points follows from examining the Morse equation from Lemma 2.15. Now, we show that in some cases our method gives better results than those obtained by only examining the Morse equation.

Proposition 5.2. *Let $N \geq 2$. Assume that 0 is an isolated critical point of Φ. If $\Gamma(\{0\}) = (k+1, k+2)$ and*

$$i^-(A_\infty) \notin [k - (2N-3), k + 2N]$$

then (4) has at least two periodic solutions (in addition to the trivial one).

Proof. Note that

$$|\Gamma(\{0\})| + |\Gamma(X_\infty)| + g(\{0\}, X_\infty) - 2 > 2N - 1$$

and

$$|\Gamma(X_\infty)| + |\Gamma(\{0\})| + g(X_\infty, \{0\}) - 2 > 2N - 1.$$

Now, Theorem 4.16 implies that there exist at least two critical points in the set $S \setminus \{0\}$. □

Example 5.3. Let $N = 2$ and H be such that

$$H(x_1, x_2, y_1, y_2) = k_1 y_1^2 + k_2(x_2^2 + y_2^2) + x_1^3 - 3x_1(x_2^2 + y_2^2)$$

if $x_1^2 + x_2^2 + y_1^2 + y_2^2 < r$, $0 < r < R$, where $k_1, k_2 \in \mathbb{Z} \setminus \{0\}$. In this example $H(z) = \frac{1}{2}\langle Az, z\rangle + h(z)$, where $A = \operatorname{diag}\{0, k_2, k_1, k_2\}$, and $h(x_1, x_2, y_1, y_2) = x_1^3 - 3x_1(x_2^2 + y_2^2)$.

It is easy to show that $0 \in E$ is a critical point of Φ. Following the procedure given in [8] one can show that

$$h_{\mathcal{LS}}(\{0\}, \nabla\Phi) = \left[\mathcal{S}^{i^-(A)} \wedge (S^1 \vee S^2)\right].$$

Assume that

$$i^-(A_\infty) \notin [i^-(A) - (2N-3), i^-(A) + 2N]$$

In this case the equality $P(t, \{0\}) = P(t, X_\infty) + (1+t)Q(t)$ is not satisfied. Therefore, there exists at least one critical point, say p, different to 0. Letting

$$P(t, \{p\}) = t^{i^-(A_\infty)} \text{ and } Q(t) = t^{i^-(A)+1}$$

we obtain equality $P(t, \{0\}) + P(t, \{p\}) = P(t, X_\infty) + (1+t)Q(t)$, which is true for all t.

On the other hand, the above proposition implies that (4) has one more solution. This result cannot be obtained by examining Morse equation.

Example 5.4. Let $N = 2$ and assume that there is $K \geq 2$ known critical points of Φ with

$$h_{\mathcal{LS}}(S_k) = [\mathcal{S}^{6k} \wedge (S^1 \vee S^2)].$$

Moreover, assume that $h_{\mathcal{LS}}(S) = [\mathcal{S}^{6(K+1)+1}]$. Since $h_{\mathcal{LS}}(S) \neq \bigvee h_{\mathcal{LS}}(S_k)$, the set S' is nonempty. The equality

$$\sum_{k=1}^{K}(t^{6k+1} + t^{6k+2}) = t^{6(K+1)+1} + (1+t)Q(t)$$

does not hold, there exists a critical point $p \in S'$. Setting $P(t, p) = t^{6(K+1)+1}$ and $Q(t) = t^7 + \dots t^{6K+1}$ we obtain the equality

$$\sum_{k=1}^{K}\left(t^{6k+1} + t^{6k+2}\right) + P(t, p) = t^{6(K+1)+1} + (1+t)Q(t)$$

which holds for all t.

Since $\Gamma(S_k) = (6k+1, 6k+2)$ and $\Gamma(S) = (6(K+1)+1, 6(K+1)+1)$, gaps between spans of $S_1, \dots, S_K$ and S are greater than $2N - 1 = 3$. Hence, the condition (A) holds. Therefore, there are at least $K+1$ critical points in the set S'.

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