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The Hopf theorem for gradient local vector fields on manifolds

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Abstract. We prove the Hopf theorem for gradient local vector fields on manifolds, i.e., we show that there is a natural bijection between the set of gradient otopy classes of gradient local vector fields and the integers if the manifold is connected Riemannian without boundary.

Contents

Introduction

The definition of otopy was introduced by Becker and Gottlieb [7, 8], and independently by Dancer, Gęba and Rybicki [10] as a generalization of the notion of homotopy. The essential difference between otopies and homotopies is that the domain of a map may change along otopy. What is important is that the topological degree is otopy invariant and otopy classes appear naturally in many classification results ([4, 5, 6, 9, 13, 16]), also in the equivariant case ([1, 2, 3, 12]).

In our paper [4] we studied otopy classes of gradient maps and proved that the usual topological degree establishes a bijection from the set of gradient otopy classes of gradient local maps with domains in the Euclidean space to the integers. This result was inspired by the Hopf type theorem proved by Parusiński in [17].

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The main purpose of this article is to generalize the result presented in [4] to an arbitrary connected Riemannian manifold M without boundary. Namely, let $\mathcal{F}^\nabla[M]$ be the set of gradient otopy classes of gradient local vector fields. We show a bijection $\mathcal{F}^\nabla[M] \approx \mathbb{Z}$. It may be worth pointing out that we do not assume orientability of the manifold M.

The important advantage of the above result is that it can be applied to examine the gradient equivariant case. Namely, let V be an orthogonal representation of a compact Lie group G and Ω be an open invariant subset of V. In [12] authors show a decomposition of the set of gradient equivariant otopy classes $\mathcal{F}_G^\nabla[\Omega]$ into factors indexed by orbit types (H) appearing in Ω. For each such factor the action of the Weyl group WH on the respective subset of Ω is free. So to obtain the complete information on this decomposition it remains to give an algebraic characterization of $\mathcal{F}_G^\nabla[\Omega]$, where G acts freely on Ω. On the other hand, in this case there is a one-to-one correspondence $\mathcal{F}_G^\nabla[\Omega] \approx \mathcal{F}^\nabla[M]$, where $M = \Omega/G$, so the description of $\mathcal{F}_G^\nabla[\Omega]$ follows from our main result.

The paper is arranged as follows. Section 1 presents some preliminaries. Our main result is stated and proved in Section 2. Sections 3 and 4 contain proofs of key lemmas needed in Section 2. Finally, in Section 5 we use our main result to study gradient equivariant local maps.

1. Preliminaries

The notation $A \Subset B$ means that A is a compact subset of B. For a topological space X, let $\tau(X)$ denote the topology on X. Recall that if A, B are topological spaces, then $\operatorname{Map}(A,B)$ denotes the set of all continuous maps of A into B equipped with the usual compact-open topology, i.e., having as subbasis all the sets $\Gamma(C,U) = \{\, f \in \operatorname{Map}(A,B) \mid f(C) \subset U \,\}$ for $C \Subset A$ and U open in B.

For any topological spaces X and Y, let $\mathcal{M}(X,Y)$ be the set of all continuous maps $f\colon D_f \to Y$ such that D_f is an open subset of X. Let $\mathcal{R}$ be a family of subsets of Y. We define

$$\operatorname{Loc}(X,Y,\mathcal{R}) := \{\, f \in \mathcal{M}(X,Y) \mid f^{-1}(R) \Subset D_f \text{ for all } R \in \mathcal{R} \,\}.$$

We introduce a topology in $\operatorname{Loc}(X,Y,\mathcal{R})$ generated by the subbasis consisting of all sets of the form

- $H(C,U) := \{\, f \in \operatorname{Loc}(X,Y,\mathcal{R}) \mid C \subset D_f,\ f(C) \subset U \,\}$ for $C \Subset X$ and $U \in \tau(Y)$,
- $M(V,R) := \{\, f \in \operatorname{Loc}(X,Y,\mathcal{R}) \mid f^{-1}(R) \subset V \,\}$ for $V \in \tau(X)$ and $R \in \mathcal{R}$.

Elements of $\operatorname{Loc}(X,Y,\mathcal{R})$ are called *local maps*. The natural base point of $\operatorname{Loc}(X,Y,\mathcal{R})$ is the empty map. The set-theoretic union of two local maps f and g with disjoint domains will be denoted by $f \sqcup g$. Moreover, in the case when $\mathcal{R} = \{\{y\}\}$ we will write $\operatorname{Loc}(X,Y,y)$ omitting double curly brackets.

Assume that M is a smooth (i.e., C^1) connected manifold without boundary. To simplify notation, we use the same letter M for the zero section of the tangent bundle TM. Let $\mathcal{F}(M) \subset \mathrm{Loc}\,(M, TM, \{M\})$ denote the space of local vector fields equipped with the induced topology.

Suppose, in addition, that M is Riemannian. Then a local vector field v is called *gradient* if there is a smooth function $f\colon D_v \to \mathbb{R}$ such that $v = \nabla f$. In that case $\mathcal{F}(M)$ contains the subspace $\mathcal{F}^{\nabla}(M)$ consisting of gradient local vector fields.

Let $I = [0,1]$. Suppose that Λ is an open subset of $I \times M$ and h is a continuous vector field on Λ. We say that h is an *otopy* if:

- h is tangent to the slices $(t \times M) \cap \Lambda$;
- the set $\{(t,x) \mid h(t,x) = 0\}$ is compact.

Given an otopy h we can define for each $t \in I$ sets $\Lambda_t = \{x \in M \mid (t,x) \in \Lambda\}$ and vector fields h_t on Λ_t with $h_t(x) = h(t,x)$. If h_t is a gradient vector field for each $t \in I$, then h is called a *gradient otopy*. The set of all gradient otopies on $I \times M$ will be denoted by $\mathcal{F}^{\nabla}(I \times M)$. It is easy to see that there is a one-to-one correspondence between (gradient) otopies and paths in $\mathcal{F}(M)$ ($\mathcal{F}^{\nabla}(M)$). Moreover, if h is a (gradient) otopy, we say that h_0 and h_1 are *(gradient) otopic*. If two gradient local fields v and v' are gradient otopic then we will write $v \sim v'$ for short. Of course, (gradient) otopy gives an equivalence relation on $\mathcal{F}(M)$ ($\mathcal{F}^{\nabla}(M)$). The sets of the respective equivalence classes will be denoted by $\mathcal{F}[M]$ and $\mathcal{F}^{\nabla}[M]$.

Observe that if v is a (gradient) local vector field and U is an open subset of D_v such that $v^{-1}(M) \subset U$, then v and $v|_U$ are (gradient) otopic. This property of (gradient) local vector fields will be called *localization*. In particular, if $v^{-1}(M) = \emptyset$ then v is (gradient) otopic to the empty map.

Let us denote by $\mathrm{I}(v)$ the oriented intersection number of a local vector field v with the zero section of the tangent bundle (see for instance [14]). It is evident that the intersection number is otopy invariant, i.e., if two local vector fields are otopic then they have the same intersection number. The converse is also true. Namely, the following result, which is a version of the well-known Hopf theorem, has been proved in [2, Rem. 2.3].

Theorem 1.1. *If M is smooth connected without boundary then*

$$\mathrm{I}\colon \mathcal{F}[M] \to \mathbb{Z}$$

is a bijection.

Now suppose again that M is Riemannian. Let us consider a smooth function $f\colon M \to \mathbb{R}$. Assume that $p \in M$ is a nondegenerate critical point of f. Let $H_p f$ denote the Hessian of f at p. In that situation, the Hessian is nondegenerate bilinear symmetric form and, in consequence, its matrix is nonsingular symmetric. The following obvious observation will be needed in Section 3.

Remark 1.2. Path-components of the space of nonsingular real symmetric $n \times n$ matrices are classified by the signature.

Let us introduce two types of vector fields in $\mathcal{F}^{\nabla}(M)$. A gradient local vector field is called *generic* if its potential is a Morse function.

Proposition 1.3. *Any gradient local vector field is gradient otopic (also homotopic) to generic one.*

Proof. Let v be a gradient local vector field. Since $v^{-1}(M)$ is compact, there is a set K such that

$$v^{-1}(M) \Subset \operatorname{int} K \subset K \Subset D_v.$$

Now by Theorem 1.2 in [15, Ch. 6], there exists a generic vector field v' defined on D_v such that $\langle v, v' \rangle > 0$ on $D_v \setminus K$. Consequently, the straight-line homotopy between v and v' gives the desired otopy. $\square$

A generic vector field s is called *standard centered at* p if $s^{-1}(M) = \{p\}$. The point p is called the *center* of s. If s is standard centered at p, then we will write $\mu(s)$ for the Morse index $\mu_p = (\dim M - \operatorname{sign} H_p f)/2$.

2. Main result

Let us formulate the main result of this paper.

Main Theorem. *Assume that M is a connected Riemannian manifold without boundary. Then*

$$\mathrm{I}\colon \mathcal{F}^{\nabla}[M] \to \mathbb{Z}$$

is a bijection.

Observe that the inclusion $\mathcal{F}^{\nabla}(M) \hookrightarrow \mathcal{F}(M)$ induces a well-defined map $\Phi\colon \mathcal{F}^{\nabla}[M] \to \mathcal{F}[M]$.

Corollary 2.1. *The map Φ is a bijection.*

The above result follows immediately from Main Theorem, Theorem 1.1 and the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{F}^{\nabla}[M] \arrow[rr, "\Phi"] \arrow[dr, "\mathrm{I}"'] & & \mathcal{F}[M] \arrow[dl, "\mathrm{I}"] \\
& \mathbb{Z}\,. &
\end{tikzcd}$$

Remark 2.2. Observe that Corollary 2.1 is an analogue of the Parusiński theorem ([17, Thm. 1]) and a generalization of our previous result ([4, Rem. 2.2]).

The proof of Main Theorem is based on the two following lemmas, which will be proved in the next two sections.

Lemma A. *If v is a gradient local vector field and $\mathrm{I}(v) = m$ then v is gradient otopic to a disjoint union of $|m|$ standard vector fields, each of them with the same intersection number equal to 1 (resp. -1) if $m \geq 0$ (resp. $m < 0$).*

Lemma B. *Consider two finite collections $\{s_i\}_{i=1}^m$ and $\{s_i'\}_{i=1}^m$ of standard vector fields such that for each collection the domains of local vector fields are pairwise disjoint and all $2m$ fields have the same sign of the intersection number. Then $\sqcup_{i=1}^m s_i$ is gradient otopic to $\sqcup_{i=1}^m s_i'$.*

Remark 2.3. Note that one can construct a gradient local vector field (in fact, standard) around any $p \in M$ with a given Morse index. Namely, let $f(x) = -\sum_{i=1}^{\mu} x_i^2 + \sum_{i=\mu+1}^{n} x_i^2$ be written in some chart containing p represented by 0. Then for $s = \nabla f$ we have $\mu(s) = \mu$ and $\mathrm{I}(s) = (-1)^{\mu}$.

Proof of Main Theorem. Injectivity follows immediately from Lemmas A and B. In turn, surjectivity follows easily from Remark 2.3. □

Remark 2.4. Let us note that in Main Theorem we do not assume the orientability of the manifold M. The reason is that the intersection number being an integer is well-defined for every local vector field on an arbitrary smooth manifold (orientable or non-orientable). Moreover, the assumption on the orientability of M is not needed in the proofs of Lemmas A and B. Namely, let s be a standard vector field defined on a small disc centered at p. Choose one of two possible orientations of that disc. If we move s along a closed path on a non-orientable manifold M in such a way that the orientation of the domain of s changes, then the orientation of T_pM will change as well. Consequently, the intersection number of s will remain the same. The more general case of vector bundles is discussed in [2].

We close this section with a remark concerning gradient proper vector fields. Recall that a local vector field v is called *proper* if for all $K \geq 0$ the set $\{x \in D_v \mid |v(x)| \leq K\}$ is compact. Using an approach similar to that in [5], one can obtain a result analogous to our Main Theorem, in which gradient local fields and their otopies are replaced by gradient proper ones.

3. Technical lemmas

We precede the proofs of Lemmas A and B by a number of technical results. Let us start with a lemma that would allow us to move a finite collection of standard vector fields with disjoint domains over the manifold M. Since we work with charts covering M that give the local Euclidean structure not necessarily coherent with the Riemannian structure of the manifold, we have to remember that the gradient of a real function on M depends on the latter. Note, however, that gradients in Euclidean and Riemannian structures have the same zeroes.

Lemma 3.1. *Assume that $\dim M > 1$ and s_i $(i = 1, \dots, k)$ are standard vector fields centered at p_i with disjoint domains. Let $p \neq p_i$ for all i. Then* *there are standard vector fields s'_i $(i = 1, \ldots, k)$ with disjoint domains such that:*

- *p is the center of s'_1.*
- *s'_i is a restriction of s_i for $i = 2, \ldots, k$.*
- *$\sqcup_{i=1}^k s_i$ is gradient otopic to $\sqcup_{i=1}^k s'_i$.*

Proof. Let $\omega \colon [0, r] \to M$ denote a path connecting p_1 with p such that $p_i \notin \omega([0, r])$ for $i \geq 2$ and $\omega([j, j+1])$ is contained in a chart φ_j for $j = 0, \ldots, r-1$. Let us define inductively open sets U_t and potentials $f_t \colon U_t \to \mathbb{R}$ for $t \in [0, r]$. Denote by U_0 an open subset of the domain of s_1 such that $p_1 \in U_0$ and by f_0 the potential of the field s_1 restricted to U_0. Assuming U_t and f_t to be defined for $t \in [0, j]$, we will define it for $t \in [j, j+1]$. Set

$$U_t = \{x + \omega(t) - \omega(j) \mid x \in U_j\} \quad \text{and} \quad f_t(x + \omega(t) - \omega(j)) = f_j(x)$$

written in the coordinates of the chart φ_j. We choose U_0 small enough so that:

- For $j = 0, \ldots, r-1$ and for all $t \in [j, j+1]$ the set U_t is contained in the chart φ_j.
- $p_i \notin \operatorname{cl}\left(\bigcup_{t \in [0,r]} U_t\right)$ for $i = 2, \ldots, k$.

This guarantees that the sets U_t are well-defined and disjoint with the domains of the restrictions s'_i for $i = 2, \ldots, k$.

Let $\Omega = \bigcup_{t \in [0,r]} t \times U_t$ and $F \colon \Omega \to \mathbb{R}$ is given by $F(t, x) = f_t(x)$. Since $\nabla_x F$ is a gradient otopy between s_1 and $s'_1 = \nabla f_r$, the required otopy from $\sqcup_{i=1}^k s_i$ to $\sqcup_{i=1}^k s'_i$ can be easily obtained by combining $\nabla_x F$ with the restriction of $\sqcup_{i=2}^k s_i$ to $\sqcup_{i=2}^k s'_i$ (see Figure 1). □

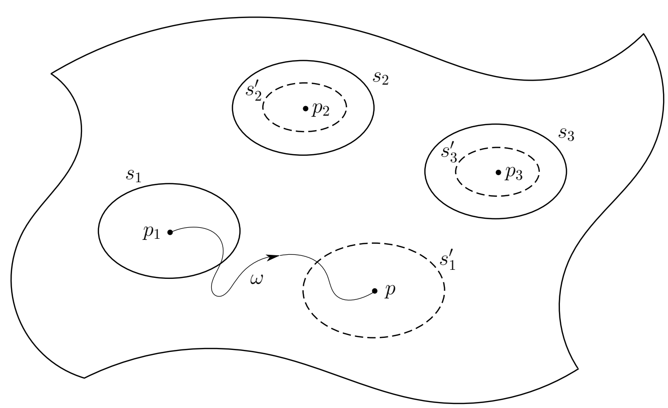

Figure 1. The idea of the otopy in Lemma 3.1.

In the remainder of this section we assume additionally that a manifold M can be covered by one chart. Thus M is diffeomorphic to an open subset of $\mathbb{R}^n$.

Let us introduce the following notation. Let $s = \nabla f$ be a standard vector field centered at p. Set $\widetilde{f}(x) = (x-p)^T(H_p f)(x-p)$ and $\widetilde{s} = \nabla\widetilde{f}$.

Remark 3.2. Observe that $\widetilde{s}$ is standard and gradient otopic to s (it is enough to consider the straight-line homotopy of the potentials on a sufficiently small neighborhood of p).

All vector fields considered in the following five lemmas are standard.

Lemma 3.3. *If s and s' have disjoint domains and $\mu(s') = \mu(s) + 1$ then $s \sqcup s'$ is gradient otopic to the empty map.*

Proof. Let $\mu = \mu(s)$. Applying Lemma 3.1, we may assume that s and s' are centered at $(0, \dots, -a)$ and $(0, \dots, 0, a)$. Note that $s \sqcup s'$ is gradient otopic to $\widetilde{s} \sqcup \widetilde{s}'$. Set $g(x) = -\sum_{i=1}^{\mu} x_i^2 + \sum_{i=\mu+1}^{n-1} x_i^2$. By Remark 1.2 we can assume that the potentials of $\widetilde{s}$ and $\widetilde{s}'$ have the form $g(x) + (x_n + a)^2$ and $g(x) - (x_n - a)^2$. Now we can perform "annihilation" by bringing closer the centers of $\widetilde{s}$ and $\widetilde{s}'$ along the n-th coordinate axis, glueing together both potentials and finally removing the zeroes of their gradient fields. More precisely, let

$$f_t(x) = \begin{cases} g(x) + (x_n + a(1-2t))^2 - a^2(1-2t)^2 & \text{if } x_n \in (-2a, 0) \text{ and } t \in I, \\ g(x) - (x_n - a(1-2t))^2 + a^2(1-2t)^2 & \text{if } x_n \in (0, 2a) \text{ and } t \in I, \\ 0 & \text{if } x_n = 0 \text{ and } t \neq 0. \end{cases}$$

Then $h_t(x) = \nabla f_t(x)$ is the desired "annihilation" otopy (see Figure 2). $\square$

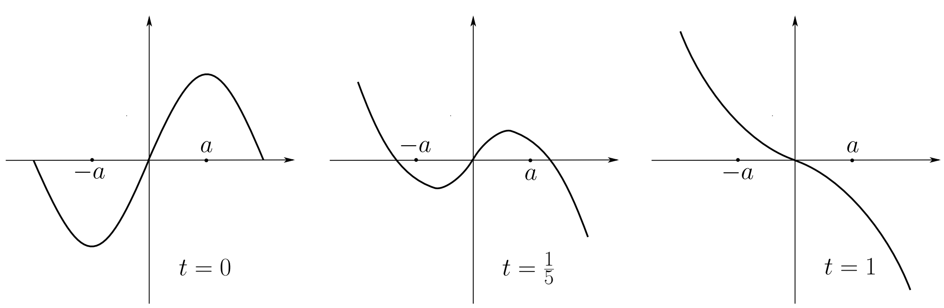

FIGURE 2. Annihilation — the graph of the n-th summand of the otopy potential.

Recall that below $\sim$ denotes the relation of gradient otopy.

Lemma 3.4. *If $\mu(s) \equiv \mu(s') \pmod 2$ then s is gradient otopic to s'.*

Proof. Without loss of generality we can assume that $\mu(s') \geq \mu(s)$. The proof is by induction on k, where $2k = \mu(s') - \mu(s)$. The lemma holds for $k = 0$ by Remarks 1.2 and 3.2. Assume the lemma is true if $\mu(s') = \mu(s)+2k$. We will prove it if $\mu(s') = \mu(s) + 2k + 2$. Let s'' and s''' be standard vector fields near s' (all three with pairwise disjoint domains) such that $\mu(s'') = \mu(s) + 2k + 1$ and $\mu(s''') = \mu(s) + 2k$. By inductive assumption, Lemma 3.3 and localization property,

$$s \sim s''' \sim (s' \sqcup s'') \sqcup s''' = s' \sqcup (s'' \sqcup s''') \sim s',$$

which is our assertion. □

Lemma 3.5. *If s and s' have disjoint domains and $\mu(s') \equiv \mu(s)+1 \pmod 2$ then $s \sqcup s'$ is gradient otopic to the empty map.*

Proof. Let s'' be a standard vector field with the same center as s' and such that $|\mu(s'') - \mu(s)| = 1$. Then $\mu(s'') \equiv \mu(s') \pmod 2$. By Lemma 3.3, $s \sqcup s'' \sim \emptyset$ and by Lemma 3.4, $s' \sim s''$. Therefore $s \sqcup s' \sim \emptyset$. □

Remarks 1.2, 2.3 and 3.2 imply that the formula $\mathrm{I}(s) = (-1)^{\mu(s)}$ holds for any standard vector field s. Thus the next two results follow immediately from Lemmas 3.4 and 3.5, respectively.

Lemma 3.6. *If $\mathrm{I}(s) = \mathrm{I}(s')$ then s is gradient otopic to s'.*

Lemma 3.7. *If s and s' have disjoint domains and $\mathrm{I}(s') = -\mathrm{I}(s)$ then $s \sqcup s'$ is gradient otopic to the empty map.*

4. Proofs of Lemmas A and B

Proof of Lemma A. By Proposition 1.3, any gradient local vector field is gradient otopic (also homotopic) to generic one and by localization property, any generic local vector field is gradient otopic to a finite disjoint union of standard ones. If all these standard vector fields have the same intersection number, we are done. Otherwise, without restriction of generality we can assume that $s = \sqcup_{i=1}^{k} s_i$ and $\mathrm{I}(s_1) = -\mathrm{I}(s_2)$. Now we need to consider the following two cases.

Case 1. $\dim M > 1$. By Lemma 3.1, we can assume the domains of s_1 and s_2 are contained in a chart of the manifold, which is disjoint with the remaining fields. Now by Lemma 3.7, $\sqcup_{i=1}^{k} s_i$ is gradient otopic to $\sqcup_{i=3}^{k} s_i$. Proceeding by induction, we obtain our claim.

Case 2. $\dim M = 1$. The reasoning is analogous as in the previous case, but instead of using Lemma 3.1 we choose s_1 and s_2 to be adjacent on the one dimensional manifold M. □

Proof of Lemma B. First, if the set of centers of the collection $\{s_i\}_{i=1}^{m}$ is the same as that of $\{s_i'\}_{i=1}^{m}$, then it is enough to use Lemma 3.6. Otherwise, we can achieve this situation by applying the following inductive procedure.

Without loss of generality assume that the center of s_1 is not equal to any center of s'_i. Then in the case of $\dim M > 1$, by Lemma 3.1, we can translate the center of s_1 to the position of the center of s'_1. Similarly in the case of $\dim M = 1$, instead of using Lemma 3.1, we can just move the collection $\{s_i\}_{i=1}^m$ to the position of the collection $\{s'_i\}_{i=1}^m$, where the order is irrelevant. $\square$

5. Gradient equivariant local maps

Assume that V is a real finite dimensional orthogonal representation of a compact Lie group G. Let X be an arbitrary G-space. We say that $f\colon X \to V$ is *equivariant*, if $f(gx) = gf(x)$ for all $x \in X$ and $g \in G$. We will denote by $\mathcal{F}_G(X)$ the space $\{f \in \operatorname{Loc}(X, V, 0) \mid f \text{ is equivariant}\}$ with the induced topology.

Assume that Ω is an open invariant subset of V and the action of G on I is trivial. Elements of $\mathcal{F}_G(\Omega)$ are called *equivariant local maps* and elements of $\mathcal{F}_G(I \times \Omega)$ are called *otopies*. Otopies give an equivalence relation on $\mathcal{F}_G(\Omega)$: f and k are otopic iff there is an otopy h such $h_0 = f$ and $h_1 = k$. Let $\mathcal{F}_G[\Omega]$ denote the set of equivalence classes of this relation. Since there is a natural bijection between $\mathcal{F}_G(I \times \Omega)$ and $\operatorname{Map}(I, \mathcal{F}_G(\Omega))$ (it is even a homeomorphism — see [1, Cor. 2.2 and Rem. 2.3]), we may identify $\mathcal{F}_G[\Omega]$ with the set of path-connected components of $\mathcal{F}_G(\Omega)$.

Let $\mathcal{F}_G^\nabla(\Omega)$ denote the subspace of $\mathcal{F}_G(\Omega)$ (with the relative topology) consisting of those maps f for which there is an invariant C^1-function $\varphi\colon D_f \to \mathbb{R}$ such that $f = \nabla\varphi$. Similarly, we write $\mathcal{F}_G^\nabla(I \times \Omega)$ for the subspace of $\mathcal{F}_G(I \times \Omega)$ consisting of such otopies h that $h_t \in \mathcal{F}_G^\nabla(\Omega)$ for each $t \in I$. These otopies are called *gradient*. Let us denote by $\mathcal{F}_G^\nabla[\Omega]$ the set of the equivalence classes of the gradient otopy relation. Alternatively, $\mathcal{F}_G^\nabla[\Omega]$ may be viewed as the set of path-connected components of $\mathcal{F}_G^\nabla(\Omega)$.

In the remainder of this section we assume that V is a real finite dimensional orthogonal representation of a compact Lie group G, Ω is an open invariant subset of V, G acts freely on Ω and $M := \Omega/G$. It is well known that M is a Riemannian manifold equipped with the so-called quotient Riemannian metric (see for instance [11, Prop. 2.28]).

If U is an open invariant subset of Ω and $\varphi\colon U \to \mathbb{R}$ is an invariant function then $\widetilde{\varphi}$ stands for the quotient function $\widetilde{\varphi}\colon U/G \to \mathbb{R}$. Let the function $\Psi\colon \mathcal{F}_G^\nabla(\Omega) \to \mathcal{F}^\nabla(M)$ be given by $\Psi(\nabla\varphi) = \nabla\widetilde{\varphi}$. Since $\nabla\widetilde{\varphi}$ does not depend on the choice of φ, the function Ψ is well-defined. We can now formulate the main result of this section.

Theorem 5.1. *The function Ψ is a bijection. Moreover, Ψ induces a bijection between $\mathcal{F}_G^\nabla[\Omega]$ and $\mathcal{F}^\nabla[M]$.*

Proof. We call a potential admissible if its gradient is a local map in the respective function space. Since the assignment $\varphi \mapsto \widetilde{\varphi}$ is a bijection between the sets of admissible potentials on U and U/G, the function

$$\Psi^{-1}(\nabla\widetilde{\varphi}) = \nabla\varphi$$

is well-defined, and so Ψ is a bijection. Similar arguments establish a bijection between $\mathcal{F}_G^{\nabla}(I \times \Omega)$ and $\mathcal{F}^{\nabla}(I \times M)$, which shows that Ψ induces a bijection from $\mathcal{F}_G^{\nabla}[\Omega]$ to $\mathcal{F}^{\nabla}[M]$. □

The following result is an immediate consequence of Main Theorem and Theorem 5.1.

Corollary 5.2. *There is a natural bijection*

$$\mathcal{F}_G^{\nabla}[\Omega] \approx \sum_{\alpha} \mathbb{Z},$$

where the direct sum is taken over the set of all connected components α of M.

Remark 5.3. The important point to note here is the difference between the sets of gradient equivariant and equivariant otopy classes. Namely, in [2] we proved that there is a bijection

$$\mathcal{F}_G[\Omega] \approx \sum_{\alpha} \mathbb{Z},$$

with the direct sum taken over all connected components of M, but only if $\dim G = 0$. If $\dim G > 0$, then the set $\mathcal{F}_G[\Omega]$ is trivial, i.e., consists of one element. Consequently, the map $\mathcal{F}_G^{\nabla}[\Omega] \to \mathcal{F}_G[\Omega]$ induced by the inclusion is a bijection for $\dim G = 0$, but the sets $\mathcal{F}_G^{\nabla}[\Omega]$ and $\mathcal{F}_G[\Omega]$ are essentially different for $\dim G > 0$. Therefore the analogy with the Parusiński result ([17]) occurs only if $\dim G = 0$.

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