

# An upper bound for the double outer-independent domination number of a tree

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## Abstract

A vertex of a graph is said to dominate itself and all of its neighbors. A double outer-independent dominating set of a graph  $G$  is a set  $D$  of vertices of  $G$  such that every vertex of  $G$  is dominated by at least two vertices of  $D$ , and the set  $V(G) \setminus D$  is independent. The double outer-independent domination number of a graph  $G$ , denoted by  $\gamma_d^{oi}(G)$ , is the minimum cardinality of a double outer-independent dominating set of  $G$ . We prove that for every nontrivial tree  $T$  of order  $n$ , with  $l$  leaves and  $s$  support vertices we have  $\gamma_d^{oi}(T) \leq (2n + l + s)/3$ , and we characterize the trees attaining this upper bound.

**Keywords:** double outer-independent domination, double domination, tree.

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## 1 Introduction

Let  $G = (V, E)$  be a graph. By the neighborhood of a vertex  $v$  of  $G$  we mean the set  $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ . The degree of a vertex  $v$ , denoted by  $d_G(v)$ , is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). The path on  $n$  vertices we denote by  $P_n$ . We say that a subset of  $V(G)$  is independent if there is no edge between any two vertices of this set.

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A vertex of a graph is said to dominate itself and all of its neighbors. A subset  $D \subseteq V(G)$  is a dominating set of  $G$  if every vertex of  $G$  is dominated by at least one vertex of  $D$ , while it is a double dominating set of  $G$  if every vertex of  $G$  is dominated by at least two vertices of  $D$ . The domination (double domination, respectively) number of  $G$ , denoted by  $\gamma(G)$  ( $\gamma_d(G)$ , respectively), is the minimum cardinality of a dominating (double dominating, respectively) set of  $G$ . Double domination in graphs was introduced by Harary and Haynes [4], and further studied for example in [1, 3]. For a comprehensive survey of domination in graphs, see [5, 6].

A subset  $D \subseteq V(G)$  is a double outer-independent dominating set, abbreviated DOIDS, of  $G$  if every vertex of  $G$  is dominated by at least two vertices of  $D$ , and the set  $V(G) \setminus D$  is independent. The double outer-independent domination number of a graph  $G$ , denoted by  $\gamma_d^{oi}(G)$ , is the minimum cardinality of a double outer-independent dominating set of  $G$ . A double outer-independent dominating set of  $G$  of minimum cardinality is called a  $\gamma_d^{oi}(G)$ -set. The study of double outer-independent domination in graphs was initiated in [7].

A 2-dominating set of a graph  $G$  is a set  $D$  of vertices of  $G$  such that every vertex of  $V(G) \setminus D$  has at least two neighbors in  $D$ . The 2-domination number of  $G$ , denoted by  $\gamma_2(G)$ , is the minimum cardinality of a 2-dominating set of  $G$ . Blidia, Chellali, and Favaron [2] proved the following upper bound on the 2-domination number of a tree. For every nontrivial tree  $T$  of order  $n$  with  $l$  leaves we have  $\gamma_2(T) \leq (n + l)/2$ . They also characterized the extremal trees.

We prove the following upper bound on the double outer-independent domination number of a tree. For every nontrivial tree  $T$  of order  $n$ , with  $l$  leaves and  $s$  support vertices we have  $\gamma_d^{oi}(T) \leq (2n + l + s)/3$ . We also characterize the trees attaining this upper bound.

## 2 Results

Since the one-vertex graph does not have a double outer-independent dominating set, in this paper, by a tree we mean only a connected graph with no cycle, and which has at least two vertices.

We begin with the following two straightforward observations.

**Observation 1** *Every leaf of a graph  $G$  is in every  $\gamma_d(G)$ -set.*

**Observation 2** *Every support vertex of a graph  $G$  is in every  $\gamma_d(G)$ -set.*

We show that if  $T$  is a nontrivial tree of order  $n$ , with  $l$  leaves and  $s$  support vertices, then  $\gamma_d^{oi}(T)$  is bounded above by  $(2n + l + s)/3$ . For the purpose of characterizing the trees attaining this bound we introduce a family  $\mathcal{T}$  of trees  $T = T_k$  that can be obtained as follows. Let  $T_1$  be a path  $P_3$  with leaves labeled  $x$  and  $z$ , and the support vertex labeled  $y$ . Let  $A(T_1) = \{x, y, z\}$ . Let  $H_1$  be a path  $P_2$

with vertices labeled  $u$  and  $v$ . Let finally  $H_2$  be a path  $P_3$  with leaves labeled  $u$  and  $w$ , and the support vertex labeled  $v$ . If  $k$  is a positive integer, then  $T_{k+1}$  can be obtained recursively from  $T_k$  by one of the following operations.

- Operation  $\mathcal{O}_1$ : Attach a vertex, say  $z$ , by joining it to a support vertex of  $T_k$ . Let  $A(T_{k+1}) = A(T_k) \cup \{z\}$ .
- Operation  $\mathcal{O}_2$ : Attach a vertex, say  $z$ , by joining it to a leaf of  $T_k$  adjacent to a strong support vertex. Let  $A(T_{k+1}) = A(T_k) \cup \{z\}$ .
- Operation  $\mathcal{O}_3$ : Attach a copy of  $H_1$  by joining the vertex  $u$  to a vertex of  $T_k$  which is not a leaf and is adjacent to a support vertex. Let  $A(T_{k+1}) = A(T_k) \cup \{u, v\}$ .
- Operation  $\mathcal{O}_4$ : Attach a copy of  $H_2$  by joining the vertex  $u$  to a leaf of  $T_k$  adjacent to a weak support vertex. Let  $A(T_{k+1}) = A(T_k) \cup \{v, w\}$ .

We now prove that for every tree  $T$  of the family  $\mathcal{T}$ , the set  $A(T)$  defined above is a DOIDS of minimum cardinality equal to  $(2n + l + s)/3$ .

**Lemma 3** *If  $T \in \mathcal{T}$ , then the set  $A(T)$  defined above is a  $\gamma_d^{oi}(T)$ -set of size  $(2n + l + s)/3$ .*

**Proof.** We use the terminology of the construction of the trees  $T = T_k$ , the set  $A(T)$ , and the graphs  $H_1$  and  $H_2$  defined above. To show that  $A(T)$  is a  $\gamma_d^{oi}(T)$ -set of cardinality  $(2n + l + s)/3$  we use the induction on the number  $k$  of operations performed to construct the tree  $T$ . If  $T = T_1 = P_3$ , then  $(2n + l + s)/3 = (6 + 2 + 1)/3 = 3 = |A(T)| = \gamma_d^{oi}(T)$ . Let  $k \geq 2$  be an integer. Assume that the result is true for every tree  $T' = T_k$  of the family  $\mathcal{T}$  constructed by  $k - 1$  operations. For a given tree  $T'$ , let  $n'$  denote its order,  $l'$  the number of its leaves, and  $s'$  the number of support vertices. Let  $T = T_{k+1}$  be a tree of the family  $\mathcal{T}$  constructed by  $k$  operations.

First assume that  $T$  is obtained from  $T'$  by operation  $\mathcal{O}_1$ . We have  $n = n' + 1$ ,  $l = l' + 1$  and  $s = s'$ . The vertex to which is attached  $z$  we denote by  $x$ . Let  $y$  be a leaf adjacent to  $x$  and different from  $z$ . By Observation 2 we have  $x \in A(T')$ . It is easy to see that  $A(T) = A(T') \cup \{z\}$  is a DOIDS of the tree  $T$ . Thus  $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 1$ . Now let  $D$  be any  $\gamma_d^{oi}(T)$ -set. By Observations 1 and 2 we have  $z, y, x \in D$ . It is easy to see that  $D \setminus \{z\}$  is a DOIDS of the tree  $T'$ . Therefore  $\gamma_d^{oi}(T') \leq \gamma_d^{oi}(T) - 1$ . We now conclude that  $\gamma_d^{oi}(T) = \gamma_d^{oi}(T') + 1$ . We get  $\gamma_d^{oi}(T) = |A(T)| = |A(T')| + 1 = (2n' + l' + s')/3 + 1 = (2n - 2 + l - 1 + s)/3 + 1 = (2n + l + s)/3$ .

Now suppose that  $T$  is obtained from  $T'$  by operation  $\mathcal{O}_2$ . We have  $n = n' + 1$ ,  $l = l'$  and  $s = s' + 1$ . The leaf to which is attached  $z$  we denote by  $x$ . By  $y$  we denote the neighbor of  $x$  other than  $z$ . By Observation 1 we have  $x \in A(T')$ .

It is easy to see that  $A(T) = A(T') \cup \{z\}$  is a DOIDS of the tree  $T$ . Thus  $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 1$ . Now let  $D$  be any  $\gamma_d^{oi}(T)$ -set. By Observations 1 and 2 we have  $z, x, y \in D$ . It is easy to see that  $D \setminus \{z\}$  is a DOIDS of the tree  $T'$ . Therefore  $\gamma_d^{oi}(T') \leq \gamma_d^{oi}(T) - 1$ . We now conclude that  $\gamma_d^{oi}(T) = \gamma_d^{oi}(T') + 1$ . We get  $\gamma_d^{oi}(T) = |A(T)| = |A(T')| + 1 = (2n' + l' + s')/3 + 1 = (2n - 2 + l + s - 1)/3 + 1 = (2n + l + s)/3$ .

Now assume that  $T$  is obtained from  $T'$  by operation  $\mathcal{O}_3$ . We have  $n = n' + 2$ ,  $l = l' + 1$  and  $s = s' + 1$ . The vertex to which is attached  $P_2$  we denote by  $x$ . Let  $y$  be a support vertex adjacent to  $x$ , and let  $z$  be a leaf adjacent to  $y$ . Obviously,  $A(T) = A(T') \cup \{u, v\}$  is a DOIDS of the tree  $T$ . Thus  $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 2$ . Now let  $D$  be any  $\gamma_d^{oi}(T)$ -set. By Observations 1 and 2 we have  $v, z, u, y \in D$ . If  $x \in D$ , then it is easy to see that  $D \setminus \{u, v\}$  is a DOIDS of the tree  $T'$ . Now suppose that  $x \notin D$ . Let  $a$  denote a neighbor of  $x$  other than  $u$  and  $y$ . The set  $V(T) \setminus D$  is independent, thus  $a \in D$ . Let us observe that now also  $D \setminus \{u, v\}$  is a DOIDS of the tree  $T'$  as the vertex  $x$  is still dominated at least twice. Therefore  $\gamma_d^{oi}(T') \leq \gamma_d^{oi}(T) - 2$ . We now conclude that  $\gamma_d^{oi}(T) = \gamma_d^{oi}(T') + 2$ . We get  $\gamma_d^{oi}(T) = |A(T)| = |A(T')| + 2 = (2n' + l' + s')/3 + 2 = (2n - 4 + l - 1 + s - 1)/3 + 2 = (2n + l + s)/3$ .

Now assume that  $T$  is obtained from  $T'$  by operation  $\mathcal{O}_4$ . We have  $n = n' + 3$ ,  $l = l'$  and  $s = s'$ . The leaf to which is attached  $P_3$  we denote by  $x$ . By Observation 1 we have  $x \in A(T')$ . It is easy to see that  $D' \cup \{v, w\}$  is a DOIDS of the tree  $T$ . Thus  $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 2$ . Now let us observe that there exists a  $\gamma_d^{oi}(T)$ -set that does not contain the vertex  $u$ . Let  $D$  be such a set. By Observations 1 and 2 we have  $w, v \in D$ . Observe that  $D \setminus \{v, w\}$  is a DOIDS of the tree  $T'$ . Therefore  $\gamma_d^{oi}(T') \leq \gamma_d^{oi}(T) - 2$ . We now conclude that  $\gamma_d^{oi}(T) = \gamma_d^{oi}(T') + 2$ . We get  $\gamma_d^{oi}(T) = |A(T)| = |A(T')| + 2 = (2n' + l' + s')/3 + 2 = (2n - 6 + l + s)/3 + 2 = (2n + l + s)/3$ . ■

We now establish the main result, an upper bound on the double outer-independent domination number of a tree together with the characterization of the extremal trees.

**Theorem 4** *If  $T$  is a tree of order  $n$ , with  $l$  leaves and  $s$  support vertices, then  $\gamma_d^{oi}(T) \leq (2n + l + s)/3$  with equality if and only if  $T \in \mathcal{T}$ .*

**Proof.** If  $\text{diam}(T) = 1$ , then  $T = P_2$ . We have  $\gamma_d^{oi}(T) = 2 < (4 + 2 + 2)/3 = (2n + l + s)/3$ . Now suppose that  $\text{diam}(T) \geq 2$ . Thus the order  $n$  of the tree  $T$  is at least three. The result we obtain by the induction on the number  $n$ . Assume that the theorem is true for every tree  $T'$  of order  $n' < n$ , with  $l'$  leaves and  $s'$  support vertices.

First suppose that some support vertex of  $T$ , say  $x$ , is strong. Let  $y$  and  $z$  be leaves adjacent to  $x$ . Let  $T' = T - y$ . We have  $n' = n - 1$ ,  $l' = l - 1$  and  $s' = s$ . Let  $D'$  be any  $\gamma_d^{oi}(T')$ -set. By Observation 2 we have  $x \in D'$ . It is easy to see

that  $D' \cup \{y\}$  is a DOIDS of the tree  $T$ . Thus  $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 1$ . We now get  $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 1 \leq (2n' + l' + s')/3 + 1 = (2n - 2 + l - 1 + s)/3 + 1 = (2n + l + s)/3$ . If  $\gamma_d^{oi}(T) = (2n + l + s)/3$ , then obviously  $\gamma_d^{oi}(T') = (2n' + l' + s')/3$ . By the inductive hypothesis we have  $T' \in \mathcal{T}$ . The tree  $T$  can be obtained from  $T'$  by operation  $\mathcal{O}_1$ . Thus  $T \in \mathcal{T}$ . Henceforth, we can assume that every support vertex of  $T$  is weak.

We now root  $T$  at a vertex  $r$  of maximum eccentricity  $\text{diam}(T)$ . Let  $t$  be a leaf at maximum distance from  $r$ , and let  $v$  be the parent of  $t$  in the rooted tree. If  $\text{diam}(T) \geq 3$ , then let  $u$  be the parent of  $v$ . If  $\text{diam}(T) \geq 4$ , then let  $w$  be the parent of  $u$ . If  $\text{diam}(T) \geq 5$ , then let  $d$  be the parent of  $w$ . By  $T_x$  let us denote the subtree induced by a vertex  $x$  and its descendants in the rooted tree  $T$ .

First suppose that  $d_T(u) \geq 3$ . Assume that among the children of  $u$  there is a support vertex, say  $x$ , different from  $v$ . The leaf adjacent to  $x$  we denote by  $y$ . Let  $T' = T - T_v$ . We have  $n' = n - 2$ ,  $l' = l - 1$  and  $s' = s - 1$ . Let  $D'$  be any  $\gamma_d^{oi}(T')$ -set. Obviously,  $D' \cup \{v, t\}$  is a DOIDS of the tree  $T$ . Thus  $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 2$ . We now get  $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 2 \leq (2n' + l' + s')/3 + 2 = (2n - 4 + l - 1 + s - 1)/3 + 2 = (2n + l + s)/3$ . If  $\gamma_d^{oi}(T) = (2n + l + s)/3$ , then  $\gamma_d^{oi}(T') = (2n' + l' + s')/3$ . By the inductive hypothesis we have  $T' \in \mathcal{T}$ . The tree  $T$  can be obtained from  $T'$  by operation  $\mathcal{O}_3$ . Thus  $T \in \mathcal{T}$ .

Now assume that some child of  $u$ , say  $x$ , is a leaf. Let  $T' = T - t$ . We have  $n' = n - 1$ ,  $l' = l$  and  $s' = s - 1$ . Let  $D'$  be any  $\gamma_d^{oi}(T')$ -set. By Observation 1 we have  $v \in D'$ . It is easy to see that  $D' \cup \{t\}$  is a DOIDS of the tree  $T$ . Thus  $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 1$ . We now get  $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 1 \leq (2n' + l' + s')/3 + 1 = (2n - 2 + l + s - 1)/3 + 1 = (2n + l + s)/3$ . If  $\gamma_d^{oi}(T) = (2n + l + s)/3$ , then  $\gamma_d^{oi}(T') = (2n' + l' + s')/3$ . By the inductive hypothesis we have  $T' \in \mathcal{T}$ . The tree  $T$  can be obtained from  $T'$  by operation  $\mathcal{O}_2$ . Thus  $T \in \mathcal{T}$ .

If  $d_T(u) = 1$ , then  $T = P_3 = T_1 \in \mathcal{T}$ . By Lemma 3 we have  $\gamma_d^{oi}(T) = (2n + l + s)/3$ . Now consider the case when  $d_T(u) = 2$ . First assume that there is a child of  $w$  other than  $u$ , say  $k$ , such that the distance of  $w$  to the most distant vertex of  $T_k$  is three. It suffices to consider only the possibility when  $T_k$  is a path  $P_3$ . Let  $T' = T - T_u$ . We have  $n' = n - 3$ ,  $l' = l - 1$  and  $s' = s - 1$ . Let us observe that there exists a  $\gamma_d^{oi}(T')$ -set that does not contain the vertex  $k$ . Let  $D'$  be such a set. The set  $V(T') \setminus D'$  is independent, thus  $w \in D'$ . It is easy to observe that  $D' \cup \{v, t\}$  is a DOIDS of the tree  $T$ . Thus  $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 2$ . We now get  $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 2 \leq (2n' + l' + s')/3 + 2 = (2n - 6 + l - 1 + s - 1)/3 + 2 = (2n + l + s)/3 - 2/3 < (2n + l + s)/3$ .

Now suppose that  $w$  is adjacent to a leaf. Let  $T' = T - T_u$ . We have  $n' = n - 3$ ,  $l' = l - 1$  and  $s' = s - 1$ . Let  $D'$  be any  $\gamma_d^{oi}(T')$ -set. By Observation 2 we have  $w \in D'$ . It is easy to observe that  $D' \cup \{v, t\}$  is a DOIDS of the tree  $T$ . Thus  $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 2$ . We now get  $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 2 \leq (2n' + l' + s')/3 + 2 = (2n - 6 + l - 1 + s - 1)/3 + 2 = (2n + l + s)/3 - 2/3 < (2n + l + s)/3$ . Henceforth, we can assume that  $w$  is not adjacent to any leaf.

Now suppose that there is a child of  $w$ , say  $k$ , such that the distance of  $w$  to the most distant vertex of  $T_k$  is two. It suffices to consider only the possibility when  $k$  is a support vertex of degree two. The leaf adjacent to  $k$  we denote by  $l$ . Let  $T' = T - T_u - l$ . We have  $n' = n - 4$ ,  $l' = l - 1$  and  $s' = s - 1$ . Let  $D'$  be any  $\gamma_d^{oi}(T')$ -set. By Observations 1 and 2 we have  $k, w \in D'$ . It is easy to observe that  $D' \cup \{v, t, l\}$  is a DOIDS of the tree  $T$ . Thus  $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 3$ . We now get  $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 3 \leq (2n' + l' + s')/3 + 3 = (2n - 8 + l - 1 + s - 1)/3 + 3 = (2n + l + s)/3 - 1/3 < (2n + l + s)/3$ .

If  $d_T(w) = 1$ , then  $T = P_4$ . We have  $T \in \mathcal{T}$  as it can be obtained from  $P_3$  by operation  $\mathcal{O}_2$ . By Lemma 3 we have  $\gamma_d^{oi}(T) = (2n + l + s)/3$ . Now consider the case when  $d_T(w) = 2$ . Let  $T' = T - T_u$ . Let  $D'$  be any  $\gamma_d^{oi}(T')$ -set. By Observation 1 we have  $w \in D'$ . It is easy to see that  $D' \cup \{v, t\}$  is a DOIDS of the tree  $T$ . Thus  $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 2$ . First suppose that  $d$  is adjacent to a leaf. We have  $n' = n - 3$ ,  $l' = l$  and  $s' = s - 1$ . We now get  $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 2 \leq (2n' + l' + s')/3 + 2 = (2n - 6 + l + s - 1)/3 + 2 = (2n + l + s)/3 - 1/3 < (2n + l + s)/3$ .

Now assume that no neighbor of  $d$  is a leaf. Let  $T' = T - T_u$ . We have  $n' = n - 3$ ,  $l' = l$  and  $s' = s$ . We now get  $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 2 \leq (2n' + l' + s')/3 + 2 = (2n - 6 + l + s)/3 + 2 = (2n + l + s)/3$ . If  $\gamma_d^{oi}(T) = (2n + l + s)/3$ , then  $\gamma_d^{oi}(T') = (2n' + l' + s')/3$ . By the inductive hypothesis we have  $T' \in \mathcal{T}$ . The tree  $T$  can be obtained from  $T'$  by operation  $\mathcal{O}_4$ . Thus  $T \in \mathcal{T}$ . ■

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